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ON VARIOUS TYPES OF CONTINUITY OF MULTIPLE DYADIC INTEGRALS

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Dedicated to Professor Ferenc Schipp on the occasion of his 75th birthday, to Professor William Wade on the occasion of his 70th birthday and to Professor Péter Simon on the occasion of his 65th birthday.

Abstract. The paper presents a survey of results related to continuity properties of dyadic integrals used in solving the problem of recovering, by generalized Fourier formulas, the coefficients of series with respect to multiple Haar and Walsh systems.

1. Introduction

This paper can be considered as a supplement and a continuation of the surveys [27] and [22]. We concentrate here first of all on continuity properties of dyadic integrals used in the problem of recovering, by generalized Fourier formulas, the coefficients of series with respect to Haar and Walsh systems.

For many classical orthogonal systems the uniqueness problem and the more general problem of recovering the coefficients can be reduced to the problem of recovering a function from its derivative with respect to a suitable derivation basis. In particular to solve the coefficient problem for Haar and Walsh series it is enough to recover a function (the so-called quasi-measure, defined by the series) from its derivative with respect to the appropriate dyadic derivation basis, and this in turn can be done by the choice of a suitable integration process. The choice of a derivation basis depends on the type of convergence. The difficulties which should be overcome in applying this method are related to the fact that the primitive we want to recover is differentiable not everywhere but outside an exceptional set and one have to impose on the primitive some continuity assumptions at the points of the exceptional set to guarantee its

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uniqueness. Moreover the type of continuity we are choosing should be implied by a corresponding assumption on the behaviour of the series.

The Walsh series can be considered on the unit interval or on the dyadic Cantor group. In the case of the interval, exceptional sets appear unavoidably even in the case of convergence everywhere. Namely the convergence at a dyadic rational point does not imply differentiability of the quasi-measure. So if we want to recover, by our method, the coefficients of everywhere convergent Walsh series, we have to recover the primitive by the dyadic derivative defined everywhere outside the set of the dyadic-rational points. In the case of the one-dimensional interval, a usual continuity with respect to the dyadic basis is enough to obtain the uniqueness of the primitive. But in a dimension greater than one the set of points with at least one dyadic-rational coordinate is not countable anymore and the continuity with respect to the multidimensional dyadic bases does not supply the uniqueness. Besides, in the multidimensional case various types of convergence and various types of corresponding dyadic bases enter into play, and this fact also affects choice of the type of continuity.

If Walsh and Haar series are considered on the dyadic group then the relation between convergence of the series and dyadic differentiability of the quasi-measure is more close. Now there is no exceptional points of the type mentioned above. But an exceptional set can appear, both in the case of group and interval, if we consider the problem of recovering the coefficient for series which are convergent not everywhere but outside some set of uniqueness. Besides, some continuity assumptions can be required to justify correctness of Perron type integrals which are used to solve the coefficient problem. Here again the type of continuity is implied by the derivation basis we choose.

In this paper we consider unrestricted rectangular convergence (Pringsheim convergence), regular rectangular convergence and its particular case — square (or cubic) convergence for multiple series. To these modes of convergence there correspond the dyadic basis, the regular dyadic basis and the cubic basis, respectively.

The rectangular convergence of a Walsh series is rather strong assumption and it implies strong enough type of continuity (so called Saks continuity), which allows to recover the quasi-measure that is differentiable only outside some comparatively large exceptional sets. In the case of regular convergence of multiple Haar and Walsh series such a continuity is not available and more delicate types of continuity are needed. These are so called "chessboard" types of continuity (see [12, 15, 18, 17, 19, 20]). While in the case of "usual" types of continuity of quasi-measures (such as continuity with respect of basis or Saks continuity) we consider the values of the quasi-measure on dyadic intervals and take the limit as the measure or diameter of dyadic intervals tends to 0 in some sense, the "chessboard" continuity involves sums of the values of the quasi-measure on adjacent dyadic intervals, taken with $\pm$ signs, and the signs alternate in the chessboard pattern.

In Section 2 we introduce basic notation. In Section 3 we consider types of convergence of multiple Walsh and Haar series and discuss a method of application of the dyadic derivatives and the dyadic integrals to the theory of Walsh and Haar series which is based on the notion of a quasi-measure, generated by the series. Section 4 is concerned with rectangular convergent series and with the role which is plaid by the local Saks continuity in defining a Perron type integral which solves the coefficient problem in the case of this convergence. In the next sections we deal with uniqueness problems for regular rectangular convergent multiple Haar and Walsh series and present multidimensional generalized integrals based on various notions of continuity. We discuss and apply a local (point-wise) type of "chessboard" continuity in Section 5, a nonlocal type of "chessboard" continuity in Section 6, and a "chessboard" smoothness in Section 7.

2. Preliminaries

Here we introduce some notation having in mind both the group setting and real interval setting for defining Walsh and Haar systems.

We denote the set of *non-negative integers* by $\mathbb{N}$, the set of *positive integers* by $\mathbb{N}_+$, the set of all *real numbers* by $\mathbb{R}$, the set of positive *real numbers* by $\mathbb{R}_+$.

Let Q_d be the set of all *dyadic-rational numbers in* $[0,1]$, i.e., the numbers of the form $\frac{j}{2^n}$ with $0 \le j \le 2^n$, $n = 0, 1, 2, \ldots$. The points $[0,1] \setminus Q_d$ constitute the set of *dyadic-irrational numbers* in $[0,1]$.

We denote one-dimensional *dyadic intervals* by

$$I_j^{(n)} := \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right], \quad 0 \le j \le 2^n - 1,$$

where $n = 0, 1, 2, \ldots$ is the *rank* of the interval.

In what follows $q \in \mathbb{N}_+$ is usually stands for the dimension. If $\mathbf{k} = (k_1, \ldots, k_q) \in \mathbb{N}^q$, then we agree $2^{\mathbf{k}}$ to denote the vector $(2^{k_1}, \ldots, 2^{k_q})$. The symbol $\mathbf{1}$ denotes the q-dimensional vector $(1, \ldots, 1)$, and the symbol $\mathbf{0}$ the q-dimensional vector $(0, \ldots, 0)$. Let $\mathbf{a} = (a_1, \ldots, a_q) \in \mathbb{N}^q$ and $\mathbf{b} = (b_1, \ldots, b_q) \in \mathbb{N}^q$. We say that $\mathbf{a} \le \mathbf{b}$ if $a_i \le b_i$ for all $i = 1, \ldots, q$. We set

$$\|\mathbf{k}\| := k_1 + \cdots + k_q$$

for every $\mathbf{k} = (k_1, \ldots, k_q) \in \mathbb{N}^q$.

We write $[x]$ for the integer part of $x \in \mathbb{R}$. We denote by $\operatorname{int}(E)$ the interior of a set E and by $|E|$ the Lebesgue measure of E.

By K we denote the unit cube $[0,1]^q$. An important role in this paper, starting with Section 4, will be played by the set Z of points having at least one dyadic-rational coordinate, i.e.,

$$Z := \bigcup_{i=1}^{q} ([0,1]^{i-1} \times Q_d \times [0,1]^{q-i}). \tag{2.1}$$

We shall use also a more general set

$$Y := \bigcup_{i=1}^{q}([0,1]^{i-1} \times Y_i \times [0,1]^{q-i}) \tag{2.2}$$

where Y_i, $i = 1, 2, \dots, q$, is any countable set *containing* Q_d.

Let $\mathcal{I}$ be the family of all q-dimensional dyadic intervals

$$I_{\mathbf{j}}^{(\mathbf{n})} := I_{j_1}^{(n_1)} \times \cdots \times I_{j_q}^{(n_q)} \tag{2.3}$$

in K, where $\mathbf{n} = (n_1, \dots, n_q)$ is the rank of $I_{\mathbf{j}}^{(\mathbf{n})}$. We denote by $I^{(\mathbf{n})}$ an arbitrary interval of rank $\mathbf{n}$ and by $I^{(\mathbf{n})}(\mathbf{x})$, where $\mathbf{x} = (x_1, \dots, x_q) \in K$, an interval of rank $\mathbf{n}$ containing $\mathbf{x}$.

The *parameter of regularity* of a dyadic interval (2.3) (or of a vector $\mathbf{a} = (a_1, \dots, a_q)$) is the number reg $I^{(\mathbf{n})}$ (resp. reg $\mathbf{a}$) which is equal to $\min\limits_{i,j}\{2^{n_i}/2^{n_j}\}$ (resp. $\min\limits_{i,j}\{a_i/a_j\}$).

The dyadic intervals are used in the theory of dyadic integrals to construct the so-called dyadic derivation basis $\mathcal{B}$. Because of this it will be convenient to refer to elements of $\mathcal{I}$ as *$\mathcal{B}$-intervals.* We need not here a rather general notion of derivation basis as it is usually understood in the Henstock theory of integration (see [11] or [22]). For us it will be a collection of dyadic intervals such that for each $\mathbf{x}$, from the unit cube where the basis is defined, there is at least a sequence $\{A_j\}$ of sets from this collection with $\mathbf{x} \in A_j$ for every j and diameter of A_j tending to 0 (see [3]).

So our dyadic derivation basis $\mathcal{B}$ is the union $\cup_{\mathbf{x}\in K}\mathcal{B}(\mathbf{x})$ where $\mathcal{B}(\mathbf{x})$ is, for each fixed $\mathbf{x} \in K$, a sequence (or subsequence) of $\mathcal{B}$-intervals $\{I^{(\mathbf{n})}(\mathbf{x})\}$ such that $\cap_{\mathbf{n}} I^{(\mathbf{n})}(\mathbf{x}) = \{\mathbf{x}\}$. Note that if $\mathbf{x}$ is an interior point of K, the sequence $\{I^{(\mathbf{n})}(\mathbf{x})\}$ is constituted by 2^s subsequences of pair-wise overlapping $\mathcal{B}$-intervals with nested projections to coordinate axis, where s is the number of dyadic-rational coordinates of the point $\mathbf{x}$. In particular, if $\mathbf{x} \in K \setminus Z$, the sequence $\{I^{(\mathbf{n})}(\mathbf{x})\}$ cannot be split into non-overlapping subsequences and $\mathbf{x}$ is an interior point for any interval of this sequence.

We denote by $\mathcal{B}_\rho$ the *ρ-regular dyadic basis* constituted by the collection of those dyadic intervals whose parameter of regularity is $\geq \rho$.

Now we pass to the terminology in the group setting.

The *dyadic group* $\mathbb{G}$ is the set of all 0–1 sequences $t = (t^0, t^1, t^2, \dots) = (t^i,\ i \in \mathbb{N})$ with the sequence $0 := (t^i = 0,\ i \in \mathbb{N})$ as zero element of $\mathbb{G}$ and with the group operation $\oplus$ given by

$$x \oplus y = (|x^i - y^i|,\ i \in \mathbb{N})$$

for every $x = (x^i,\ i \in \mathbb{N}) \in \mathbb{G}$, $y = (y^i,\ i \in \mathbb{N}) \in \mathbb{G}$.

The map

$$\lambda\colon t \mapsto x = \sum_{i=1}^{\infty} \frac{t^i}{2^{i+1}} \tag{2.4}$$

is one-to-one correspondence between the group G and the interval [0,1], up to a countable set. Indeed, each $x \in Q_d$ has two dyadic expansions, a finite one and an infinite one. If we exclude from G the elements corresponding to one type of expansion, for example to the infinite one, then the correspondence (2.4) is one-to-one and the converse mapping λ^{-1} is defined on [0,1).

We set $\Delta_0^0 := \mathbb{G}$. Suppose $n \in \mathbb{N}_+$,

$$j = \sum_{i=0}^{n-1} j_i\, 2^i, \quad j_i \in \{0,1\}, \quad 0 \le i \le n-1; \tag{2.5}$$

then the sets

$$\Delta_j^{(n)} := \{t = (t^i,\, i \in \mathbb{N}) \in G :\ t^i = j_{n-1-i},\ i = 0, \dots, n-1\} \tag{2.6}$$

are in fact cosets of the subgroup $\Delta_0^{(n)}$. We'll write $\Delta^{(n)}$ for an arbitrary coset of rank k. As the function λ maps each set $\Delta^{(n)}$ onto a dyadic interval $I^{(n)}$ we shall often keep for $\Delta^{(n)}$ the name *dyadic interval* of rank n of $\mathbb{G}$.

We shall consider the *dyadic product group* $\mathbb{G}^q$. For $\mathbf{y} = (y_1, \dots, y_q) \in \mathbb{G}^q$ and $\mathbf{z} = (z_1, \dots, z_q) \in \mathbb{G}^q$ the sum $\mathbf{y} \oplus \mathbf{z} \in \mathbb{G}^q$ is defined by

$$\mathbf{y} \oplus \mathbf{z} = (y_1 \oplus z_1, \dots, y_q \oplus z_q).$$

Sets

$$\begin{aligned} \Delta_{\mathbf{j}}^{(\mathbf{n})} :=& \Delta_{j_1}^{(n_1)} \times \dots \times \Delta_{j_q}^{(n_q)}, \\ & \mathbf{n} = (n_1, \dots, n_q) \in \mathbb{N}^q, \quad \mathbf{j} = (j_1, \dots, j_q) \in \mathbb{N}^q, \quad \mathbf{0} \le \mathbf{j} \le 2^{\mathbf{n}} - \mathbf{1}, \end{aligned} \tag{2.7}$$

are called the $\mathcal{B}$-*intervals* of rank $\mathbf{n}$ of $\mathbb{G}^q$. We denote by $\Delta^{(\mathbf{n})}$ an arbitrary interval of rank $\mathbf{n}$ and by $\Delta^{(\mathbf{n})}(\mathbf{t})$, where $\mathbf{t} = (t_1, \dots, t_q) \in \mathbb{G}^q$, the unique interval of rank $\mathbf{n}$ containing $\mathbf{t}$. Dyadic intervals

$$\Delta_{\mathbf{j}}^{(n)} := \Delta_{\mathbf{j}}^{(n,\dots,n)} \tag{2.8}$$

are said to be the *dyadic cubes* of rank n.

The *topology* on $\mathbb{G}^q$ is generated by the collection of all dyadic intervals. Each dyadic interval is clopen in this topology. The dyadic group is metrizable (we omit details, see [23, Introduction]). We write $d(E)$ for the *diameter* of a set $E \subset \mathbb{G}^q$ in this metric. We have [23, Introduction]

$$d(\Delta^{(\mathbf{n})}) = \sqrt{2^{-2n_1} + \dots + 2^{-2n_q}}. \tag{2.9}$$

The *parameter of regularity* for $\mathcal{B}$-interval $\Delta^{(\mathbf{n})}$ is defined as for the interval $I^{(\mathbf{n})}$, i.e., as $\min_{i,j}\{2^{n_i}/2^{n_j}\}$.

The dyadic group $(\mathbb{G}^q, \oplus)$ is a compact Abelian group. Denote by μ the normalized Haar measure on $\mathbb{G}^q$. Thus μ is a translation invariant Borel measure on $\mathbb{G}^q$ such that $\mu(\mathbb{G}^q) = 1$. It is clear that

$$\mu(\Delta^{(\mathbf{n})}) = 2^{-\|\mathbf{n}\|}. \tag{2.10}$$

$\mathcal{B}$-intervals $\Delta^{(\mathbf{n})}$ form a basis in $\mathbb{G}^q$. There is a close relation between bases in $\mathbb{G}^q$ and in K. But as we shall see below, the fact that $\mathbb{G}^q$ is a zero-dimensional space while $[0,1]^q$ is connected, implies an essential difference in the properties of the integrals defined with respect to those bases.

Set functions $\tau\colon \mathcal{I} \to \mathbb{R}$ are called *$\mathcal{B}$-interval functions*. We define the derivative of a $\mathcal{B}$-interval function with respect to our dyadic bases.

Definition 2.1. Given a $\mathcal{B}$-interval function F, the *upper* and the *lower* *$\mathcal{B}$-derivatives* of F at a point $\mathbf{x}$, with respect to the basis $\mathcal{B}$, are defined as

$$\overline{D}_{\mathcal{B}}F(\mathbf{x}) := \inf_{\delta>0} \sup_{d(I^{(\mathbf{n})}(\mathbf{x}))\le\delta} \frac{F(I^{(\mathbf{n})}(\mathbf{x}))}{|I^{(\mathbf{n})}(\mathbf{x})|} \tag{2.11}$$
$$\text{and}\quad \underline{D}_{\mathcal{B}}F(\mathbf{x}) := \sup_{\delta>0} \inf_{d(I^{(\mathbf{n})}(\mathbf{x}))\le\delta} \frac{F(I^{(\mathbf{n})}(\mathbf{x}))}{|I^{(\mathbf{n})}(\mathbf{x})|},$$

respectively. If $\overline{D}_{\mathcal{B}}F(\mathbf{x}) = \underline{D}_{\mathcal{B}}F(\mathbf{x})$ we call this common value the *$\mathcal{B}$-derivative* $D_{\mathcal{B}}F(\mathbf{x})$ at $\mathbf{x}$. We say that F is *$\mathcal{B}$-differentiable* at $\mathbf{x}$ if the $\mathcal{B}$-derivative at this point exists and is finite.

In the same way the $\mathcal{B}_\rho$-derivatives with respect to the basis $\mathcal{B}_\rho$ and the derivatives with respect to bases on group $\mathbb{G}^q$ are defined.

We also define the continuity with respect to the basis.

Definition 2.2. We say a $\mathcal{B}$-interval function F is *$\mathcal{B}$-continuous* (resp. *$\mathcal{B}_\rho$-continuous*) at a point $\mathbf{x}$, if

$$\lim_{\mathbf{n}\to\infty} F(I^{(\mathbf{n})}(\mathbf{x})) = 0 \quad \left(\text{resp.} \quad \lim_{\mathbf{n}\to\infty,\ \text{reg}\ \mathbf{n}\ge\rho} F(I^{(\mathbf{n})}(\mathbf{x})) = 0\right). \tag{2.12}$$

3. Modes of convergence of multiple Walsh and Haar series. Quasi-measure

We recall the definitions (see [5] and [23]).

The *Walsh functions* (in Paley numeration) on $\mathbb{G}$ are defined by

$$w_n(t) := (-1)^{\sum\limits_{i=0}^{\infty} t^i n_i}$$

where

$$t = \{t^i\} \in \mathbb{G}, \quad n = \sum_{i=0}^{\infty} n_i 2^i \quad (n_i \in \{0,1\}).$$

Using mapping converse to $\lambda\colon \mathbb{G} \to [0,1]$ given by (2.4) we can define Walsh system on the unit interval as $w(\lambda^{-1}(x))$. For these functions we shall use the same notation: $w(x)$.

The *Haar functions* on $\mathbb{G}$ are defined as follows. $\chi_0 \equiv 1$. If $n = 2^k + j$, $k = 0, 1, \dots$, $j = 0, \dots, 2^k - 1$, then

$$\chi_n(t) := \begin{cases} 2^{k/2}, & \text{if } t \in \Delta_{2j}^{(k+1)}, \\ -2^{k/2}, & \text{if } t \in \Delta_{2j+1}^{(k+1)}, \\ 0, & \text{if } t \in \mathbb{G} \setminus \Delta_j^{(k)}. \end{cases}$$

The *Haar functions* on $[0,1]$ are defined similarly. $\chi_0(x) \equiv 1$. If $n = 2^k + j$, $k = 0, 1, \ldots$, $j = 0, \ldots, 2^k - 1$, we put

$$\chi_n(x) := \begin{cases} 2^{k/2}, & \text{if } x \in \left(\frac{2j}{2^{k+1}}, \frac{2j+1}{2^{k+1}}\right), \\ -2^{k/2}, & \text{if } x \in \left(\frac{2j+1}{2^{k+1}}, \frac{2i+2}{2^{k+1}}\right), \\ 0, & \text{if } x \in (0,1) \setminus \left[\frac{2j}{2^{k+1}}, \frac{2j+2}{2^{k+1}}\right], \end{cases}$$

and we agree that at each point of discontinuity $\chi_n(x) = \frac{1}{2}(\chi_n(x+0)+\chi_n(x-0))$ and that at $x = 0$ and $x = 1$ Haar functions are continuous from the right and from the left, respectively.

A *q-dimensional Walsh* and *Haar series* are defined by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \ldots \sum_{n_q=0}^{\infty} a_{n_1,\ldots,n_q} \prod_{i=1}^{q} w_{n_i}(x_i) \tag{3.1}$$

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \ldots \sum_{n_q=0}^{\infty} b_{n_1,\ldots,n_q} \prod_{i=1}^{q} \chi_{n_i}(x_i) \tag{3.2}$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are real numbers. It follows from the above definitions that for $\mathbf{n} = (n_1, \ldots, n_q)$ with $2^{k_j-1} \le n_j < 2^{k_j}$, $j = 1, \ldots, q$, the functions $\chi_{\mathbf{n}}$ and $w_{\mathbf{n}}$ are constant in the interior of each dyadic interval of rank $\mathbf{k} = (k_1, \ldots, k_q)$. Moreover, with the same notation, the functions $\chi_{\mathbf{n}}$ are supported by some intervals of rank $\mathbf{k} - \mathbf{1} = (k_1 - 1, \ldots, k_q - 1)$.

If $\mathbf{N} = (N_1, \ldots, N_q)$, then the $\mathbf{N}$*th rectangular partial sum* $S_{\mathbf{N}}$ of series (3.1) (resp., (3.2)) at a point $\mathbf{x} = (x_1, \ldots, x_q)$ is

$$S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_q=0}^{N_q-1} a_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{x}) \quad \left(\text{resp.,} \quad S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_q=0}^{N_q-1} b_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) \right).$$

The series (3.1) (or (3.2)) *rectangularly converges* to sum $S(\mathbf{x})$ at a point $\mathbf{x}$ and we write $\lim_{\mathbf{N}\to\infty} S_{\mathbf{N}}(\mathbf{x}) = S(\mathbf{x})$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \quad \text{as} \quad \min_i \{N_i\} \to \infty.$$

We say that the series (3.1) (or (3.2)) *ρ-regular rectangularly converges* to sum $S(\mathbf{x})$ if in the above definition the limit is taken under the additional assumption that $\operatorname{reg} \mathbf{N} \ge \rho$. 1-regular rectangular convergence is called *cubic convergence.*

Similarly, we can define an *q-dimensional Walsh* and *Haar series* on $\mathbb{G}^q$ by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{t}) := \sum_{n_1=0}^{\infty} \ldots \sum_{n_q=0}^{\infty} a_{n_1,\ldots,n_q} \prod_{i=1}^{q} w_{n_i}(t_i) \tag{3.3}$$

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{t}) := \sum_{n_1=0}^{\infty} \ldots \sum_{n_q=0}^{\infty} b_{n_1,\ldots,n_q} \prod_{i=1}^{q} \chi_{n_i}(t_i), \tag{3.4}$$

and the appropriate types of convergence of series (3.3) and (3.4).

A standard method (see [27]) of application of the dyadic derivative and the dyadic integral to the theory of Walsh and Haar series is based on the fact that for the partial sums $S_{2^{\mathbf{k}}}$ of those series, the integral $\int_{I_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k}}}$ defines an additive $\mathcal{B}$-interval function $\psi(I)$ on the family $\mathcal{I}$ of all dyadic intervals (in fact it can be extended as an additive function to the algebra generated by $\mathcal{I}$, but we need not this). In dyadic analysis the function ψ is referred to as the *quasi-measure generated by the series* (see [23], [33]). Since the sum $S_{2^{\mathbf{k}}}$ is constant on interior of each $I_{\mathbf{j}}^{(\mathbf{k})}$ we get

$$S_{2^{\mathbf{k}}}(\mathbf{x}) = \frac{1}{|I_{\mathbf{j}}^{(\mathbf{k})}|} \int_{I_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k}}} = \frac{\psi(I_{\mathbf{j}}^{(\mathbf{k})})}{|I_{\mathbf{j}}^{(\mathbf{k})}|} \tag{3.5}$$

for any point $\mathbf{x} \in \operatorname{int}(I_{\mathbf{j}}^{(\mathbf{k})})$.

In fact any additive $\mathcal{B}$-interval function ψ defines Walsh or Haar series for which it is a quasi-measure and (3.5) holds. So we have one-to-one correspondence between family of additive $\mathcal{B}$-interval functions and family of Walsh or Haar series.

The equality (3.5) obviously gives a relation between $\mathcal{B}$-differentiability of ψ at $\mathbf{x}$ and convergence of the series. In particular, at least at the points $\mathbf{x} \in K \setminus Z$, we get

$$\lim_{\mathbf{k}\to\infty} S_{2^{\mathbf{k}}}(\mathbf{x}) = D_{\mathcal{B}}\psi(\mathbf{x}) \quad \text{and} \quad \lim_{\mathbf{k}\to\infty,\,\mathrm{reg}\ \mathbf{k}\geq\rho} S_{2^{\mathbf{k}}}(\mathbf{x}) = D_{\mathcal{B}_\rho}\psi(\mathbf{x}) \tag{3.6}$$

and therefore the convergence of the series (3.1) (or (3.2)) at such points $\mathbf{x}$ to a sum $f(\mathbf{x})$ implies $\mathcal{B}$-differentiability (or $\mathcal{B}_\rho$-differentiability) of the function ψ at $\mathbf{x}$ with $f(\mathbf{x})$ being the value of $\mathcal{B}$-derivative (or $\mathcal{B}_\rho$-derivative).

In the case of the group we rewrite (3.5) in the form

$$S_{2^{\mathbf{k}}}(\mathbf{t}) = \frac{1}{|\Delta_{\mathbf{j}}^{(\mathbf{k})}|} \int_{\Delta_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k}}} = \frac{\psi(\Delta_{\mathbf{j}}^{(\mathbf{k})})}{|\Delta_{\mathbf{j}}^{(\mathbf{k})}|}, \tag{3.7}$$

and this time it is true for each $\mathbf{t} \in \Delta_{\mathbf{j}}^{(\mathbf{k})}$. So in this case analogue of (3.6) holds at each point of $\mathbb{G}$ as soon as at least one side of this equality exists. Here is an advantage of considering Walsh series on the group.

The following statement is essential for establishing that a given Walsh or Haar series is the Fourier series in the sense of some general integral (see for example [27]); a proof, in the one-dimensional version, can be found in [5, Th. 3.1.8]).

Proposition 3.1. *Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}$. Assume a series of the form* (3.1) *or* (3.2) *is given. Let a $\mathcal{B}$-interval function ψ be the quasi-measure generated by this series and* (3.5) *holds. Then this series is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\psi(I) = (\mathcal{A}) \int_I f$ for any $\mathcal{B}$-interval I.*

In view of (3.6) and the above proposition, in order to solve the coefficient problem it is enough to show that the quasi-measure ψ generated by Haar or Walsh series is the indefinite integral of its $\mathcal{B}$-derivative which exists at those points of $\mathbb{G}$ (or at points of $K \setminus Z$ in case of unit cube as a domain) at which the sequence of partial sums $S_{2^{\mathbf{k}}}$ of the series is convergent. By this we reduce the problem of recovering the coefficients to the corresponding theorem on recovering the primitive with appropriate continuity assumptions which can be obtained either from a convergence condition or from some additional growth assumptions imposed on the coefficients of the series.

4. Local Saks continuity and coefficient problem for rectangular convergent multiple Walsh and Haar series

Here we consider the coefficient problem for multiple Walsh and Haar series which are rectangular convergent everywhere outside some sets of uniqueness or U-sets. We recall that a set E is said to be U-set for a system of functions if the convergence of a series with respect to this system to zero outside the set E implies that all coefficients of the series are zero. For references to a large body of literature on the theory of uniqueness of Walsh, Haar and Vilenkin series, including subtle theory of sets of uniqueness, see [1], [5], [23], [31], [32] whereas the classical trigonometric case is treated for example in [6].

While looking for continuity assumptions which should be imposed on the primitive at the points of these exceptional sets to guarantee its uniqueness, it turns out that usual continuity with respect to the dyadic basis (see Definition 2.2) is not enough for this purpose in the multidimensional case, and we introduce a stronger notion of continuity, which we call local Saks continuity with respect to the basis.

We recall that an interval function F is said to be *continuous in the sense of Saks* if $\lim_{|I|\to 0,} F(I) = 0$. We define a local version of this type of continuity adjusted to $\mathcal{B}$-interval functions.

Definition 4.1. We say that a $\mathcal{B}$-interval function F is *locally $\mathcal{B}$-continuous in the sense of Saks,* or briefly *$\mathcal{B}S$-continuous, at a point* $\mathbf{x}$ if

$$\lim_{|I^{(\mathbf{n})}(\mathbf{x})|\to \mathbf{0}} F(I^{(\mathbf{n})}(\mathbf{x})) = \mathbf{0}. \tag{4.1}$$

In the two-dimensional case the last equality can be rewritten in terms of ranks of $\mathcal{B}$-intervals in the following way:

$$\lim_{k+l\to\infty} F(I^{(k,l)}(\mathbf{x})) = \mathbf{0}. \tag{4.2}$$

The most natural integration process to recover primitives is Kurzweil-Henstock integral (see [29]). We are not going to give here the definition of the multidimensional dyadic Kurzweil-Henstock integral ($H_{\mathcal{B}}$-integral, see [22]), because we shall use here first of all Perron-type integrals. We just note that although the $H_{\mathcal{B}}$-integral can be shown to have the local Saks continuity (see [28]), it solves the problem of recovering a primitive only in the case of rather "thin" exceptional sets. For example we have

Theorem 4.2 (See [28]). *If an additive $\mathcal{B}$-interval function F is $\mathcal{B}$-differentiable with $D_{\mathcal{B}}F(\mathbf{x}) = f(\mathbf{x})$ everywhere on K outside a countable set where F is $\mathcal{B}$-continuous, then the function f is $H_{\mathcal{B}}$-integrable on K and F is its indefinite $H_{\mathcal{B}}$-integral.*

If instead of a countable set we use here the set (2.1) or (2.2) then this type of theorem is not true anymore even with $\mathcal{B}$-continuity being replaced with $\mathcal{BS}$-continuity. An example is given in [28].

In the one-dimensional case $Z = Q_d$, that is the exceptional set Z (and Y) is in fact countable. Moreover $\mathcal{B}$-continuity everywhere on $[0, 1]$ follows from the condition $\lim_{n\to\infty} a_n = 0$ (which in turn is a consequence of the convergence of the series at least at one dyadic-irrational point). So we can apply Theorem 4.2 to get the following result:

Theorem 4.3. *If the series* (3.1) (*in one dimension*) *is convergent to a sum f outside a countable set, then f is $H_{\mathcal{B}}$-integrable and* (3.1) *is the Fourier–Walsh series of f, i.e.,*

$$a_n = (H_{\mathcal{B}}) \int_{[0,1]} f w_n.$$

The Kurzweil-Henstock integral with respect to a basis is known to be equivalent to the Perron integral with respect to the same basis (see [11]). In particular it is true for the dyadic basis. Moreover this Perron dyadic integral, $P_{\mathcal{B}}$-integral, can be defined by $\mathcal{B}$-continuous major and minor functions (see [2] for the case of full interval basis, a proof for the dyadic case is similar). We need not recall here the definition of $P_{\mathcal{B}}$-integral and we pass directly to constructing another Perron-type integral defined by $\mathcal{BS}$-continuous major and minor functions, which will be used to solve the coefficient problem.

Definition 4.4. Let f be a point function defined at least on $K \setminus Z$. An additive $\mathcal{BS}$-continuous on K $\mathcal{B}$-interval function M (resp., m) is called a *$\mathcal{BS}$-major* (resp., *$\mathcal{BS}$-minor*) function of f if the lower (resp., the upper) $\mathcal{B}$-derivative satisfies the inequality

$$\underline{D}_{\mathcal{B}}M(\mathbf{x}) \geq f(\mathbf{x}) \quad (\text{resp.} \quad \overline{D}_{\mathcal{B}}m(\mathbf{x}) \leq f(\mathbf{x})) \quad \text{for all} \quad \mathbf{x} \in K \setminus Z. \tag{4.3}$$

It can be shown (see [28]) that if M and m are a $\mathcal{BS}$-major and a $\mathcal{BS}$-minor function for a point-function f on K then for each $\mathcal{B}$-interval I we have $M(I) \geq m(I)$. This implies that for any function f we have

$$\inf_M \{M(K)\} \geq \sup_m \{m(K)\}$$

where "inf" and "sup" are taken over all $\mathcal{BS}$-major and $\mathcal{BS}$-minor function of f, respectively. This justifies the following definition.

Definition 4.5. A point function f defined at least on $K \setminus Z$ is said to be $P_{\mathcal{B}}S$*-integrable* on K, if there exist at least one $\mathcal{B}S$-major function and at least one $\mathcal{B}S$-minor function of f and

$$-\infty < \inf_{M}\{M(K)\} = \sup_{m}\{m(K)\} < +\infty$$

where "inf" and "sup" are taken as above. The common value is called $P_{\mathcal{B}}S$-*integral* of f on K and is denoted by $(P_{\mathcal{B}}S)\int_K f$.

In the same way we can define $P_{\mathcal{B}}S$-integral on any $\mathcal{B}$-interval I.

Directly from the definitions we get the following result which shows that the $P_{\mathcal{B}}S$-integral solves the problem of recovering the primitive from its $\mathcal{B}$-derivative in the form we need.

Theorem 4.6 (See [28])**.** *If an additive $\mathcal{B}S$-continuous $\mathcal{B}$-interval function F is $\mathcal{B}$-differentiable with $D_{\mathcal{B}}F(\mathbf{x}) = f(\mathbf{x})$ everywhere on $K \setminus Z$ then the function f is $P_{\mathcal{B}}S$-integrable on K and F is its indefinite $P_{\mathcal{B}}S$-integral.*

We can extend the previous definition of $P_{\mathcal{B}}S$-integral to the case when the inequalities (4.3) related to major and minor function hold outside a fixed set Y defined by (2.2). Such an integral for a function f, defined at least on $K \setminus Y$, depends on the chosen exceptional set Y and we call it $P_{\mathcal{B}}^{Y}S$-integral. As Y contains Z, $P_{\mathcal{B}}^{Y}S$-integral includes $P_{\mathcal{B}}S$-integral. Theorem 4.6, with Z replaced by Y, holds true for this integral.

It follows from an example given in [28] that the assumption of $\mathcal{B}S$-continuity of F in the above theorem cannot be weakened to the one of $\mathcal{B}$-continuity.

The following propositions were proved in [25] and [26], respectively.

Proposition 4.7. *If a two-dimensional series* (3.1) *is rectangular convergent everywhere on the "cross" $\{a \times [0,1]\} \cap \{[0,1] \times b\}$, where $(a,b) \in K$, $a, b \notin Q_d$, except a countable set then for this series*

$$\lim_{i+j\to\infty} a_{i,j} = 0. \tag{4.4}$$

Proposition 4.8. *If a two-dimensional series* (3.2) *is rectangular convergent on the "cross" $\{a \times [0,1]\} \cap \{[0,1] \times b\}$, $(a,b) \in K$, everywhere except a countable set E and at each point of E we have*

$$\lim_{k,l\to\infty} \frac{b_{n_k,m_l}\chi_{n_k,m_l}(x,y)}{2^k 2^l} = 0, \tag{4.5}$$

then for this series

$$\lim_{k+l\to\infty} \frac{b_{n_k,m_l}\chi_{n_k,m_l}(a,b)}{2^k 2^l} = 0 \tag{4.6}$$

where $2^{k-1} \le n_k < 2^k$, $2^{l-1} \le m_l < 2^l$.

Note that (4.6) and (4.5) are in fact meaningful only for those indexes n_k, m_l for which the support of function χ_{n_k,m_l} contains the point (x,y).

Similar propositions can be formulated for any dimension.

On the basis of these propositions it was proved in fact in [25] that Z (and also Y) is U-set for rectangular convergent multiple Walsh series (see also [7]). So it makes sense to state a problem of recovering the coefficients of those series from their sums defined outside of these U-sets. As for Haar series, non-empty U-sets exist only under additional assumptions of the type (4.5) or (4.6). Namely, Z is U-sets for Haar series under condition, that (4.6) holds everywhere. Under weaker assumption (4.5) on the exceptional set only countable sets are U-sets for rectangular convergent Haar series. Note that for ρ-regular convergent Haar series, with ρ close to 1, even the empty set is not U set (see [13, 16]).

Now we consider continuity properties of the quasi-measure (see, for example, [28]).

Lemma 4.9. *If the coefficients of two-dimensional series* (3.1) *satisfy the condition* (4.4)*, then at each point $(x, y) \in K$ the quasi-measure ψ is $\mathcal{B}S$-continuous, i.e.,* (4.2) *holds everywhere on K.*

Lemma 4.10. *If the coefficients of two-dimensional series* (3.2) *satisfy the condition* (4.6) *at a point $(x, y) \in K$, then at this point the quasi-measure ψ is $\mathcal{B}S$-continuous, i.e.,* (4.2) *holds at* (x, y).

Note that the above statement is not true for Walsh series which are convergent with respect to regular rectangles, for example with respect to cubes, even under assumption of convergence everywhere on K (see [19]).

In view of (3.6) and the Proposition 3.1, we can solve now the coefficient problem. It is enough to show that the quasi-measure ψ generated by Haar or Walsh series is the indefinite integral of its $\mathcal{B}$-derivative which exists at least on $K \setminus Z$. To this end we use the corresponding theorem on recovering the primitive with appropriate continuity assumptions.

Using Theorem 4.6 we get

Theorem 4.11. *If a series* (3.1) *is rectangular convergent to a sum f everywhere in $K \setminus Z$, then f is $P_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}S$-Fourier coefficients of f.*

We can enlarge the exceptional set Z here by replacing it by the set Y defined in (2.2). Then we get

Theorem 4.12 (See [28]). *If the series* (3.1) *is rectangular convergent to a sum f everywhere in $K \setminus Y$, then f is $P^Y_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P^Y_{\mathcal{B}}S$-Fourier coefficients of f.*

In the same way using Proposition 4.8 and Lemma 4.10 we obtain

Theorem 4.13 (See [28]). *If a two-dimensional series* (3.2) *is rectangular convergent to a sum f everywhere in K outside a countable set E and* (4.5) *holds everywhere on E then f is $P_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}S$-Fourier coefficients of f.*

Note that in the above theorem we can omit condition (4.5) if we assume that the series (3.2) is convergent everywhere on K.

Analyzing the proof of the above theorem and the one of Lemma 4.9 we note that the convergence everywhere of the series has been used in order to obtain the condition (4.6) on coefficients of the series which in turn implies $\mathcal{B}S$-continuity everywhere. So we can weaken the assumption of convergence in the formulation of Theorem 4.13 by supposing a priori that the condition (4.6) are fulfilled. In this way we can obtain the following version of Theorem 4.13.

Theorem 4.14 (See [28]). *If the series* (3.2) *is rectangular convergent to a sum f everywhere in $K \setminus Z$ and the coefficients of the series satisfy everywhere the condition* (4.6)*, then f is $P_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}S$-Fourier coefficients of f.*

5. Σ-continuity and uniqueness problems for regular convergent multiple Haar and Walsh series on the dyadic product group

5.1. **Σ-continuity.** Σ-continuity was introduced in [15] and applied in [19, 18, 15] for constructing generalized integrals which solve the problem of recovering the coefficients of regular convergent multiple Haar and Walsh series from their sums, by generalized Fourier formulas. In those papers the regular convergence everywhere outside some at most countable exceptional sets is considered.

The choice of this continuity has a double reason. First, Σ-continuity being imposed on the primitive, guarantee its uniqueness. Secondly, Σ-continuity of quasi-measures is provided by regular convergence of appropriate multiple Haar and Walsh series, while, as it is shown in [19], regular convergence of those series to a finite function, even everywhere, does not guarantee that the corresponding quasi-measure is $\mathcal{B}S$-continuous or $\mathcal{B}_\rho$-continuous at some points.

We write Σ for the set $\{0,1\}^q$ of 0–1 q-dimensional vectors. If $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_q) \in \Sigma$, let $|\boldsymbol{\sigma}|$ denote the sum $\sigma_1 + \dots + \sigma_q$.

For a fixed point $\mathbf{t}_0 \in \mathbb{G}^q$ and $\mathbf{k} = (k_1, \dots, k_q) \in \mathbb{N}^q$ consider the interval

$$\Delta^{(\mathbf{k})}(\mathbf{t}_0) = \Delta^{(k_1)} \times \dots \times \Delta^{(k_q)} \tag{5.1}$$

and for each $i = 1, 2, \dots, q$ denote

$$\Delta^{(k_i)}_{(0)} = \Delta^{(k_i)}, \quad \Delta^{(k_i)}_{(1)} = \Delta^{(k_i - 1)} \setminus \Delta^{(k_i)}. \tag{5.2}$$

If $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_q) \in \Sigma$ then we put

$$\Delta^{(\mathbf{k})}_{(\boldsymbol{\sigma})} = \Delta^{(k_1)}_{(\sigma_1)} \times \dots \times \Delta^{(k_q)}_{(\sigma_q)}. \tag{5.3}$$

Definition 5.1. We say a $\mathcal{B}$-interval function τ is Σ-*continuous* (is Σ-*bounded*) at the point $\mathbf{t}_0 \in \mathbb{G}^q$ if it satisfies

$$\lim_{k_1 = \dots = k_q \to \infty} \sum_{\boldsymbol{\sigma} \in \Sigma} (-1)^{|\boldsymbol{\sigma}|} \tau\left(\Delta^{(\mathbf{k})}_{(\boldsymbol{\sigma})}\right) = 0$$

$$\left(\text{resp.}, \sum_{\boldsymbol{\sigma}\in\Sigma}(-1)^{|\boldsymbol{\sigma}|}\tau\left(\Delta^{(\mathbf{k})}_{(\boldsymbol{\sigma})}\right) = \mathrm{O}(1) \quad \text{as } k_1 = \ldots = k_q \to \infty\right).$$

The definition of Σ-continuity for quasi-measures may be reformulated in an equivalent form.

Definition 5.2. We say a $\mathcal{B}$-interval function τ is Σ^*-*continuous* at some point $\mathbf{t}_0 \in \mathbb{G}^q$ if

$$\lim_{k_1=\ldots=k_q\to\infty}\sum_{\boldsymbol{\sigma}\in\Sigma}\left(-\frac{1}{2}\right)^{|\boldsymbol{\sigma}|}\tau\left(\Delta^{(\mathbf{k}-\boldsymbol{\sigma})}(\mathbf{t}_0)\right) = 0. \tag{5.4}$$

Theorem 5.3 (See [15])**.** *Suppose a $\mathcal{B}$-interval function τ is a quasi-measure. Then τ is Σ-continuous at some point $\mathbf{t}_0 \in \mathbb{G}^q$ if and only if τ is Σ^*-continuous at $\mathbf{t}_0$.*

5.2. **Relation between Σ-continuity and other types of continuity.** Here we consider a relation between Σ-continuity and both $\mathcal{B}_\rho$-continuity and $\mathcal{B}S$-continuity.

The next result follows from Theorem 5.3 and formula (5.4).

Theorem 5.4 (See [15])**.** *If a quasi-measure τ is $\mathcal{B}_\rho$-continuous with $\rho = 1/2$ at a point $\mathbf{t}_0 \in \mathbb{G}^q$, then τ is Σ-continuous at $\mathbf{t}_0$.*

Corollary 5.5. *In the one-dimensional case, any $\mathcal{B}$-continuous at a point $t_0 \in \mathbb{G}$ quasi-measure is Σ-continuous at t_0.*

The following example shows that ρ-continuity and $\mathcal{B}S$-continuity are not more general than Σ-continuity.

Example 1. Assume that $q \geq 2$. We consider dyadic cubes (2.8) and set

$$\tau(\Delta^{(k)}_{\mathbf{0}}) = 1, \quad k = 0, 1, 2, \ldots. \tag{5.5}$$

Further, let Δ be a dyadic cube such that $\Delta \neq \Delta^{(k)}_{\mathbf{0}}$ for any $k = 0, 1, 2, \ldots$. Clearly,

$$\Delta \subset \Delta^{(k)}_{\boldsymbol{\sigma}} \tag{5.6}$$

holds for the uniquely determined $\boldsymbol{\sigma} \in \Sigma$, $\boldsymbol{\sigma} \neq \mathbf{0}$, and $k \in \mathbb{N}_+$. We set

$$\tau(\Delta) = \begin{cases} -\dfrac{\mu(\Delta)}{(2^q-2)\,\mu(\Delta^{(k)}_{\boldsymbol{\sigma}})}, & \text{if } \sigma_1 + \cdots + \sigma_q = 0 \pmod 2, \\ \dfrac{\mu(\Delta)}{2^q\,\mu(\Delta^{(k)}_{\boldsymbol{\sigma}})}, & \text{if } \sigma_1 + \cdots + \sigma_q = 1 \pmod 2. \end{cases} \tag{5.7}$$

So, the set function τ is defined on $\mathcal{B}_1$. It is not difficult to check that the equality

$$\tau(\Delta^{(k)}_{\mathbf{j}}) = \sum_{\boldsymbol{\sigma}\in\Sigma}\tau(\Delta^{(k+1)}_{\mathbf{2j}+\boldsymbol{\sigma}}) \tag{5.8}$$

holds for each dyadic cube $\Delta_{\mathbf{j}}^{(k)}$ of the form (2.8). Therefore, τ can be extended to a quasi-measure.

It follows from (5.7) that the quasi-measure τ is absolutely continuous, with respect to the Haar measure, on each dyadic cube $\Delta \subset \mathbb{G}^q \setminus \{\mathbf{0}\}$. This observation implies that τ is Σ-continuous at each point $\mathbf{t} \in \mathbb{G}^q \setminus \{\mathbf{0}\}$. After elementary calculations it can be proved that also τ is Σ-continuous at the point $\mathbf{0}$.

Finally, $\tau(\Delta_{\mathbf{0}}^{(k)}) = 1$ for all $k = 0, 1, 2, \dots$ by (5.5). Consequently, the quasi-measure τ is not $\mathcal{B}_1$-continuous at the point $\mathbf{0}$. This implies that τ is not $\mathcal{B}_\rho$-continuous at the point $\mathbf{0}$ for any $\rho \in (0, 1]$ and is not $\mathcal{B}S$-continuous.

Now we construct a quasi-measure τ such that τ is $\mathcal{B}_1$-continuous everywhere on $\mathbb{G}^q$, but not Σ-continuous at some point.

Example 2. Assume that $q \geq 2$. We consider dyadic cubes (2.8) and set

$$\tau(\Delta_{\mathbf{0}}^{(k)}) = \frac{1}{2^k}, \quad k = 0, 1, 2, \dots. \tag{5.9}$$

Further, let us fix two arbitrary non-zero vectors $\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1 \in \Sigma$ satisfying

$$|\boldsymbol{\sigma}_0| = 0 \pmod 2, \quad |\boldsymbol{\sigma}_1| = 1 \pmod 2. \tag{5.10}$$

Let Δ be a dyadic cube such that $\Delta \neq \Delta_{\mathbf{0}}^{(k)}$ for any $k = 0, 1, 2, \dots$. Then (5.6) holds for the uniquely determined $\boldsymbol{\sigma} \in \Sigma$, $\boldsymbol{\sigma} \neq \mathbf{0}$, and $k \in \mathbb{N}_+$ (see Example 1). We set

$$\tau(\Delta) = \begin{cases} \dfrac{\left(1 + \frac{1}{2^k}\right) \mu(\Delta)}{\mu(\Delta_{\boldsymbol{\sigma}}^{(k)})}, & \text{if } \boldsymbol{\sigma} = \boldsymbol{\sigma}_0, \\ -\dfrac{\mu(\Delta)}{\mu(\Delta_{\boldsymbol{\sigma}}^{(k)})}, & \text{if } \boldsymbol{\sigma} = \boldsymbol{\sigma}_1, \\ 0, & \text{if } \boldsymbol{\sigma} \neq \boldsymbol{\sigma}_0 \text{ and } \boldsymbol{\sigma} \neq \boldsymbol{\sigma}_1. \end{cases} \tag{5.11}$$

So, the set function τ is defined on $\mathcal{B}_1$. It is not difficult to check that (5.8) holds for each dyadic cube $\Delta_{\mathbf{j}}^{(k)}$ of the form (2.8). Therefore, τ can be extended to a quasi-measure.

It follows from (5.11) that the quasi-measure τ is locally absolutely continuous, with respect to the Haar measure, on $\mathbb{G}^q \setminus \{\mathbf{0}\}$. This observation implies that τ is $\mathcal{B}_1$-continuous at each point $\mathbf{t} \in \mathbb{G}^q \setminus \{\mathbf{0}\}$. The formula (5.9) yields that τ is $\mathcal{B}_1$-continuous at the point $\mathbf{0}$.

Finally, for every $k \in \mathbb{N}_+$ we have

$$\sum_{\boldsymbol{\sigma} \in \Sigma} (-1)^{|\boldsymbol{\sigma}|} \tau(\Delta_{(\boldsymbol{\sigma})}^{(k)}) \overset{(5.6), (5.9), (5.10), (5.11)}{=} \frac{1}{2^k} + \left(1 + \frac{1}{2^k}\right) - (-1) = 2 + \frac{1}{2^{k-1}}.$$

Therefore the quasi-measure τ is not Σ-continuous at the point $\mathbf{0}$.

In the one-dimensional case the notion of continuity which involves the differences of the values of the quasi-measure on adjacent dyadic intervals was considered in [24, 34, 35]. But the following statement shows that Σ-continuity gives a new notion only in a multidimensional case.

Theorem 5.6. *A quasi-measure τ is Σ-continuous at a point $t \in \mathbb{G}$ if and only if τ is $\mathcal{B}$-continuous at t.*

Proof. By Corollary 5.5, it is sufficient to prove that if the quasi-measure τ is Σ-continuous at a point $t \in \mathbb{G}$, then τ is continuous at t.

Let the quasi-measure τ is Σ^*-continuous at the point t_0. We choose and fix any $\varepsilon > 0$. Then there exists $k_0 = k_0(\varepsilon)$ such that

$$\left|\tau(\Delta_{k+1}) - \frac{1}{2}\,\tau(\Delta_k)\right| < \varepsilon, \quad \text{for each} \quad k \geq k_0. \tag{5.12}$$

We shall prove by induction with respect to $k = k_0, k_0+1, \dots$ that for all such k the next inequality holds:

$$|\tau(\Delta^{(k+1)}| < \frac{1}{2^{k+1-k_0}}\,|\tau(\Delta^{(k)})| + \varepsilon\left(2 - \frac{1}{2^{k-k_0}}\right). \tag{5.13}$$

If $k = k_0$, then (5.13) immediately follows from (5.12). Assume inductively that (5.13) is proved for each $k \leq k_1 - 1$ and prove (5.13) for $k = k_1$. The formula (5.12) implies

$$\left|\tau(\Delta^{(k_1+1)})\right| < \frac{1}{2}\,\left|\tau(\Delta^{(k_1+1)})\right| + \varepsilon. \tag{5.14}$$

Then by the inductive assumption

$$\left|\tau(\Delta^{(k+1)})\right| < \frac{1}{2^{k_1-k_0}}\,\left|\tau(\Delta^{(k_0)})\right| + \varepsilon\left(2 - \frac{1}{2^{k_1-1-k_0}}\right). \tag{5.15}$$

Combining (5.14) and (5.15), we obtain:

$$\begin{aligned} \left|\tau(\Delta^{(k_1+1)})\right| &< \frac{1}{2}\left|\tau(\Delta^{(k_1)})\right| + \varepsilon \\ &< \frac{1}{2}\left(\frac{1}{2^{k_1-k_0}}\,\left|\tau(\Delta^{(k_0)})\right| + \varepsilon\Big(2 - \frac{1}{2^{k_1-1-k_0}}\Big)\right) + \varepsilon \\ &= \frac{1}{2^{k_1+1-k_0}}\left|\tau(\Delta^{(k_0)})\right| + \varepsilon\left(2 - \frac{1}{2^{k_1-k_0}}\right). \end{aligned} \tag{5.16}$$

It follows from (5.16) that the formula (5.13) is true if $k = k_1$. Consequently, (5.13) holds for all $k \geq k_0$. Since $\varepsilon > 0$ is arbitrary, (5.13) yields the continuity of the quasi-measure τ at the point t_0. The theorem is proved. □

Summing up the results of this subsection, we get the following conclusion.

1. Σ-continuity is strictly more general than $\mathcal{B}S$-continuity.
2. If $q \geq 2$ and $\rho \in (0, 1/2]$, then Σ-continuity is strictly more general than $\mathcal{B}_\rho$-continuity.
3. If $q \geq 2$, then Σ-continuity and $\mathcal{B}_1$-continuity are incomparable.
4. If $q = 1$, then Σ-continuity is equal to $\mathcal{B}$-continuity.

5.3. **Σ-continuity, generalized integral, and uniqueness problems for multiple Haar and Walsh series.** In [17, 19] Σ-continuity was applied for constructing some dyadic Perron-type integral and for solving the uniqueness problem and the coefficients problem for multiple Haar and Walsh series.

Definition 5.7. Let f be a point function defined on $\mathbb{G}^q$ except possibly on some at most countable set E. A Σ-continuous quasi-measure M (resp., m) is called a Σ-*major* (resp., Σ-*minor*) function of f if the lower (resp., the upper) $\mathcal{B}_1$-derivative satisfies the inequality

$$\underline{D}_{\mathcal{B}_1} M(\mathbf{t}) \geq f(\mathbf{t}) \quad (\text{resp.} \quad \overline{D}_{\mathcal{B}_1} m(\mathbf{t}) \leq f(\mathbf{t})) \tag{5.17}$$

at each point $\mathbf{t} \in \mathbb{G}^q \setminus E$.

It can be shown (see [19]) that if M and m are a Σ-major and a Σ-minor function for a point function f on $\mathbb{G}^q$ then for each $\mathcal{B}$-interval Δ we have $M(\Delta) \geq m(\Delta)$. This implies that for any function f and for each $\mathcal{B}$-interval Δ we have

$$\inf_M \{M(\Delta)\} \geq \sup_m \{m(\Delta)\} \tag{5.18}$$

where "inf" and "sup" run over all Σ-major and Σ-minor function of f, respectively. This justifies the following definition.

Definition 5.8. Suppose a finite-valued point function f is defined everywhere on $\mathbb{G}^q$ except possibly on some at most countable set E. We say the function f is P_Σ-*integrable* if there exists at least one Σ-major function and at least one Σ-minor function of f and

$$-\infty < \inf_M \{M(\mathbb{G}^q)\} = \sup_m \{m(\mathbb{G}^q)\} < +\infty$$

where "inf" and "sup" are taken as above. The common value is called P_Σ-*integral* of f on $\mathbb{G}^q$ and is denoted by $(P_\Sigma)\int_{\mathbb{G}^q} f$.

In the same way we can define P_Σ-integral on any $\mathcal{B}$-interval Δ. It is easy to see that the value of P_Σ-*integral* does not depend on the choice of an exceptional at most countable set E.

The following result which follows directly from the definitions, shows that the $P_\Sigma S$-integral solves the problem of recovering the primitive from its $\mathcal{B}$-derivative in the form we need.

Theorem 5.9 (See [19])**.** *Suppose an additive Σ-continuous $\mathcal{B}$-interval function F is $\mathcal{B}_1$-differentiable with $D_{\mathcal{B}_1} F(\mathbf{t}) = f(\mathbf{t})$ nearly everywhere on $\mathbb{G}^q$; then the function f is P_Σ-integrable on $\mathbb{G}^q$ and F is its indefinite P_Σ-integral.*

The next example shows that Σ-continuity can't be replaced by Σ-boundedness at no point $\mathbf{t}_0 \in \mathbb{G}^q$.

Example 3. Choose an arbitrary point $\mathbf{t}_0 \in \mathbb{G}^q$ and consider any non-trivial quasi-measure τ supported by the one-point set $\{\mathbf{t}_0\}$:

$$\tau(\Delta) = \begin{cases} C \neq 0, & \text{if } \mathbf{t}_0 \in \Delta, \\ 0, & \text{if } \mathbf{t}_0 \notin \Delta. \end{cases}$$

Then clearly τ is Σ-bounded everywhere on $\mathbb{G}^q$ and $D_{\mathcal{B}_1}\tau(\mathbf{t}) = 0$ on $\mathbb{G}^q \setminus \{\mathbf{t}_0\}$ (moreover, $D_{\mathcal{B}}\tau(\mathbf{t}) = 0$ on $\mathbb{G}^q \setminus \{\mathbf{t}_0\}$) but τ is not identical zero.

Lemmas 5.10–5.13 proved in [17, 19] establish the connections between series (3.3) or (3.4) and quasi-measures generated by these series. Notice (see Lemma 5.12) that for multiple Haar series Σ-continuity of quasi-measure generated by it means exactly that the series satisfies some weak multidimensional analogue of well-known Arutyunyan–Talalyan condition.

Lemma 5.10. *Suppose a series* (3.3) *satisfy*

$$\lim a_{n_1,\dots,n_q} = 0 \quad \textit{as} \quad 2^k \le n_1, \dots, n_q \le 2^{k+1} - 1 \quad \textit{and} \quad k \to \infty; \tag{5.19}$$

then the quasi-measure ψ generated by this series is Σ-continuous at each point $\mathbf{t} \in \mathbb{G}^q$.

Lemma 5.11. *Suppose a series* (3.3) 1/2*-regular rectangularly converges to a finite some at least in one point $\mathbf{t}_0 \in \mathbb{G}^q$, then the quasi-measure ψ generated by this series is Σ-continuous everywhere on $\mathbb{G}^q$.*

Lemma 5.12. *A series* (3.4) *satisfies the condition*

$$\begin{aligned} &b_{n_1,\dots,n_q}\chi_{n_1,\dots,n_q}(\mathbf{t}_0) = \mathrm{o}(n_1 \cdot \dots \cdot n_q) \\ &\qquad\qquad \textit{as} \quad 2^k \le n_1, \dots, n_q \le 2^{k+1} - 1 \quad \textit{and} \quad k \to \infty \end{aligned} \tag{5.20}$$

(weak multidimensional analogue of Arutyunyan–Talalyan condition*) if and only if the quasi-measure ψ generated by this series is Σ-continuous at the point $\mathbf{t}_0$.*

Lemma 5.13. *If the rectangular partial sums $S_{\mathbf{N}} = S_{N_1,\dots,N_q}$ of a series* (3.4) *at a point $\mathbf{t}_0 \in \mathbb{G}^q$ satisfy the condition*

$$\begin{aligned} &S_{N_1,\dots,N_q}(\mathbf{t}_0) = \mathrm{o}(N_1 \cdot \dots \cdot N_q) \\ &\qquad\qquad \textit{as} \quad 2^k \le N_1, \dots, N_q \le 2^{k+1} - 1 \quad \textit{and} \quad k \to \infty, \end{aligned} \tag{5.21}$$

then the quasi-measure ψ generated by this series is Σ-continuous at the point $\mathbf{t}_0$.

Combining (3.6), Proposition 3.1, and the above lemmas we can get now the solution of coefficient problem for multiple Haar and Walsh series on the dyadic product group under regular convergence.

Theorem 5.14 (See [19])**.** *If a series* (3.3) *satisfies* (5.19) *and is cubic convergent to a finite-valued function f nearly everywhere on $\mathbb{G}^q$, then f is P_Σ-integrable on $\mathbb{G}^q$ and the coefficients of this series are P_Σ-Fourier–Walsh coefficients of f.*

Theorem 5.15 (See [19])**.** *If a series* (3.3) *is cubic convergent to a finite-valued function f nearly everywhere on $\mathbb{G}^q$ and* $1/2$*-regular rectangular convergent to a finite sum at some point $\mathbf{t}_0 \in \mathbb{G}^q$, then f is P_Σ-integrable on $\mathbb{G}^q$ and the coefficients of the series are P_Σ-Fourier–Walsh coefficients of f.*

Corollary 5.16. *If a series* (3.3) $1/2$*-regular rectangularly converges to a finite-valued sum f everywhere on $\mathbb{G}^q$, then f is P_Σ-integrable on $\mathbb{G}^q$ and the coefficients of the series are P_Σ-Fourier–Walsh coefficients of f.*

Theorem 5.17 (See [19])**.** *If a series* (3.4) *satisfies* (5.20) *or* (5.21) *everywhere on $\mathbb{G}^q$ and is cubic convergent to a finite-valued function f nearly everywhere on $\mathbb{G}^q$, then f is P_Σ-integrable on $\mathbb{G}^q$ and the coefficients of the series are P_Σ-Fourier–Haar coefficients of f.*

Corollary 5.18. *Every at most countable set $E \subset \mathbb{G}^q$ is a U-set for cubic convergent series* (3.4) *under one of the conditions* (5.20) *or* (5.21)*.*

Notice that the condition (5.20) or (5.21) can not be relaxed even at a single point. Having fixed any point $\mathbf{t}_0 \in \mathbb{G}^q$ it is not difficult to construct a non-trivial series (3.4) satisfying (5.20) or (5.21) at the point $\mathbf{t}_0$ and rectangular convergent to zero everywhere on $\mathbb{G}^q \setminus \{\mathbf{t}_0\}$.

Note also that 1/2-regular rectangular convergence to a finite sum at some point $\mathbf{t}_0 \in \mathbb{G}^q$ of a series (3.4) implies that both the conditions (5.20) and (5.21) hold. This fact with Theorem 5.17 yields the following statement.

Theorem 5.19 (See [19])**.** *If a series* (3.4) $1/2$*-regular rectangularly converges to a finite-valued function f everywhere on $\mathbb{G}^q$, then f is P_Σ-integrable on $\mathbb{G}^q$ and the coefficients of the series are P_Σ-Fourier–Haar coefficients of f.*

6. (Σ, Δ)-CONTINUITY OF QUASI-MEASURES ON THE DYADIC PRODUCT GROUP

A non-local analog of Σ-continuity was suggested in [17, 20].

Definition 6.1. Let $\Delta \subset \mathbb{G}^q$ be a dyadic cube. We say a $\mathcal{B}$-interval function τ is (Σ, Δ)-*continuous* if

$$\lim_{k\to\infty} \sum (-1)^{j_1+\cdots+j_q} \tau(\Delta_{\mathbf{j}}^{(k)}) = 0 \tag{6.1}$$

where sum runs over all dyadic cubes $\Delta_{\mathbf{j}}^{(k)}$ of the form (2.8) such that $\Delta_{\mathbf{j}}^{(k)} \subset \Delta$.

In [20] it is shown that every quasi-measure τ being absolutely continuous with respect to the Haar measure is (Σ, Δ)-continuous for every dyadic cube Δ.

In [20] (Σ, Δ)-continuity was applied for constructing some dyadic Perron-type integral and for solving the uniqueness problem and the coefficient one for multiple Haar and Walsh series.

Definition 6.2. We say a set $F \subset \mathbb{G}^q$ is a *RD_q-set* (a *Rademacher–Dirichlet set*) if there exists an increasing sequence $\{k_s\}_{s=1}^{\infty}$ of non-negative integers, such that

$$F \subset \bigcap_{s=1}^{\infty} F_{k_s},$$

where F_k denotes the "kth chessboard set" defined as the union of all dyadic cubes (2.8) of rank k, satisfying

$$j_1 + \cdots + j_q = 0 \qquad (\mathrm{mod}\ 2).$$

Proposition 6.3 (See [21]). 1) *$\mu(F) = 0$ for all RD_q-sets F.*

2) *For every positive integer q there exist perfect RD_q-sets, the Hausdorff dimension of whose is equal q.*

Definition 6.4. Let f be a point function defined on $\mathbb{G}^q$ except possibly on some set $F \cup E \subset \mathbb{G}^q$ where F is RD_q-set and E is at most countable set. We say a quasi-measure M (resp., m) is (Σ, Δ)-*major* (resp., (Σ, Δ)-*minor*) function of f if it is (Σ, Δ)-continuous for any dyadic cube $\Delta \subset \mathbb{G}^q$ and satisfies the condition (5.17) everywhere on $\mathbb{G}^q \setminus (F \cup E)$.

It can be shown (see [20]) that (5.18) holds for each $\mathcal{B}$-interval Δ where "inf" and "sup" run over all (Σ, Δ)-major and (Σ, Δ)-minor function of f, respectively. This justifies the following definition.

Definition 6.5. Suppose a finite-valued point function f is defined everywhere on $\mathbb{G}^q$ except possibly on some set $F \cup E \subset \mathbb{G}^q$ where F is RD_q-set and E is at most countable set. We say the function f is $P_{\Sigma,\Delta}$-*integrable* if there exists at least one (Σ, Δ)-major function and at least one (Σ, Δ)-minor function of f and

$$-\infty < \inf_M \{M(\mathbb{G}^q)\} = \sup_m \{m(\mathbb{G}^q)\} < +\infty$$

where "inf" and "sup" are taken as above. The common value is called $P_{\Sigma,\Delta}$-*integral* of f on $\mathbb{G}^q$ and is denoted by $(P_{\Sigma,\Delta}) \int_{\mathbb{G}^q} f$.

In the same way we can define $P_{\Sigma,\Delta}$-integral on any $\mathcal{B}$-interval Δ. In [20] it is proved that the value of $P_{\Sigma,\Delta}$-*integral* does not depend on the choice of an exceptional set $F \cup E$.

Theorem 6.6 (See [20]). *Let f be a finite-valued point function defined everywhere on $\mathbb{G}^q$ except possibly on some set $F \cup E$ where F is RD_q-set and E is at most countable set. If f is summable, then f is $P_{\Sigma,\Delta}$-integrable and*

$$(L) \int_{\Delta} f = (P_{\Sigma,\Delta}) \int_{\Delta} f$$

for each $\mathcal{B}$-interval Δ.

It follows from the definitions above that the $P_{\Sigma,\Delta}S$-integral solves the problem of recovering the primitive from its $\mathcal{B}_1$-derivative in the form we need.

Theorem 6.7 (See [20]). *Suppose a (Σ,Δ)-continuous quasi-measure F is $\mathcal{B}_1$-differentiable with $D_{\mathcal{B}_1}F(\mathbf{t}) = f(\mathbf{t})$ everywhere on $\mathbb{G}^q$ except possibly on some set $F \cup E$ where F is RD_q-set and E is at most countable set. Then the function f is $P_{\Sigma,\Delta}$-integrable on $\mathbb{G}^q$ and F is its indefinite $P_{\Sigma,\Delta}$-integral.*

The following result shows the relation between convergence of multiple Walsh series (3.3) and (Σ,Δ)-continuity of quasi-measures.

Lemma 6.8 (See [20]). *Suppose a series* (3.3) *is cubic convergent to a finite sum on some Borel set A. Then the quasi-measure ψ generated by this series is (Σ,Δ)-continuous for each dyadic cube $\Delta \subset \mathbb{G}^q$ such that $\mu(\Delta \cap A) > 0$.*

Corollary 6.9. *If a series* (3.3) *is cubic convergent to a finite sum almost everywhere on $\mathbb{G}^q$, then the quasi-measure ψ generated by this series is (Σ,Δ)-continuous for every dyadic cube $\Delta \subset \mathbb{G}^q$.*

Combining (3.6), Propositions 3.1 and 6.3, and Corollary 6.9 we get a solution of the coefficient problem for cubic convergent multiple Walsh series on the dyadic product group.

Theorem 6.10 (See [20]). *Suppose a series* (3.3) *is cubic convergent to a finite-valued function f everywhere on $\mathbb{G}^q$ except possibly on some set $F \cup E$ where F is RD_q-set and E is at most countable set. Then f is $P_{\Sigma,\Delta}$-integrable on $\mathbb{G}^q$ and this series is $P_{\Sigma,\Delta}$-Fourier–Walsh series of f.*

Taking into account Theorems 6.6 and 6.10 we get the solution of the coefficient problem for multiple Walsh series which is cubic convergent to a summable function.

Theorem 6.11 (See [20]). *Suppose a series* (3.3) *is cubic convergent to a finite-valued summable function f everywhere on $\mathbb{G}^q$ except possibly on some set $F \cup E$ where F is RD_q-set and E is at most countable set. Then the coefficients of the series are Fourier–Walsh coefficients of f.*

Each of the Theorems 6.10 and 6.11 implies the following important corollary.

Theorem 6.12 (See [20]). *Any set $F \cup E \subset \mathbb{G}^q$, where F is RD_q-set and E is at most countable set, is a U-set for cubic convergent series* (3.3).

7. (Σ,α)-SMOOTHNESS OF QUASI-MEASURES ON THE UNIT CUBE

7.1. (Σ,α)**-smoothness in the two-dimensional case.** (Σ,α)-smoothness was introduced in the two-dimensional case in [12].

Let K be the two-dimensional unit square $[0,1]^2$. For a given point $\mathbf{x}_0 \in Z$ there exist 2^s sequences $\{I^{(k_1,k_2)}(\mathbf{x}_0),\ k_1,\, k_2 \in \mathbb{N}\}$ of dyadic intervals containing the point $\mathbf{x}_0$ and such that

$$
\begin{aligned}
I^{(k_1+1,k_2)}(\mathbf{x}_0) \supset \Delta^{(k_1,k_2)}(\mathbf{x}_0), I^{(k_1,k_2+1)}(\mathbf{x}_0) \supset \Delta^{(k_1,k_2)}(\mathbf{x}_0),\\
\operatorname{rank}\ I^{(k_1,k_2)}(\mathbf{x}_0) = (k_1,k_2), \quad k_1,\, k_2 \in \mathbb{N},
\end{aligned}
\tag{7.1}
$$

where s is the number of dyadic-rational coordinates of the point $\mathbf{x}_0$, different from 0 or 1. We fix one of this sequences and set

$$I^{(k,k)}_{(0,0)} = I^{(k,k)}(\mathbf{x}_0)$$

$$I^{(k,k)}_{(1,0)} = I^{(k-1,k)}(\mathbf{x}_0) \setminus I^{(k,k)}(\mathbf{x}_0), \quad I^{(k,k)}_{(0,1)} = I^{(k,k-1)}(\mathbf{x}_0) \setminus I^{(k,k)}(\mathbf{x}_0),$$

$$I^{(k,k)}_{(1,1)} = I^{(k-1,k-1)}(\mathbf{x}_0) \setminus \left(I^{(k,k-1)}(\mathbf{x}_0) \cup I^{(k-1,k)}(\mathbf{x}_0)\right), \quad k \in \mathbb{N}_+.$$

Supposing $\mathbf{x}_0 \in Q_d \times I_d$, we say a $\mathcal{B}$-interval function τ is (Σ,α)-*smooth* at the point $\mathbf{x}_0$ if for any sequence $\{I^{(k_1,k_2)}(\mathbf{x}_0),\ k_1,\, k_2 \in \mathbb{N}\}$ of $\mathcal{B}$-intervals containing the point $\mathbf{x}_0$ and satisfying (7.1) the following condition is true:

$$\lim_{k\to\infty} \frac{1}{\left|I^{(k,k)}_{(0,0)}\right|^{\alpha}} \left(\tau\left(I^{(k,k)}_{(0,0)}\right) - \tau\left(I^{(k,k)}_{(1,0)}\right)\right) = 0.$$

In the case $\mathbf{x}_0 \in I_d \times Q_d$, we say a $\mathcal{B}$-interval function τ is (Σ,α)-*smooth* at the point $\mathbf{x}_0$ if

$$\lim_{k\to\infty} \frac{1}{\left|I^{(k,k)}_{(0,0)}\right|^{\alpha}} \left(\tau\left(I^{(k,k)}_{(0,0)}\right) - \tau\left(I^{(k,k)}_{(0,1)}\right)\right) = 0.$$

for any sequences $\{I^{(k_1,k_2)}(\mathbf{x}_0),\ k_1,\, k_2 \in \mathbb{N}\}$ of $\mathcal{B}$-intervals containing the point $\mathbf{x}_0$ and satisfying (7.1).

Finally, in the case $\mathbf{x}_0 \in Q_d \times Q_d$ we say a $\mathcal{B}$-interval function τ is (Σ,α)-*smooth* at the point $\mathbf{x}_0$ if

$$\lim_{k\to\infty} \frac{1}{\left|I^{(k,k)}_{(0,0)}\right|^{\alpha}} \left(\tau\left(I^{(k,k)}_{(0,0)}\right) - \tau\left(I^{(k,k)}_{(0,1)}\right) - \tau\left(I^{(k,k)}_{(0,1)}\right) + \tau\left(I^{(k,k)}_{(1,1)}\right)\right) = 0$$

for any sequence $\{I^{(k_1,k_2)}(\mathbf{x}_0),\ k_1,\, k_2 \in \mathbb{N}\}$ of $\mathcal{B}$-intervals containing the point $\mathbf{x}_0$ and satisfying (7.1).

Definition 7.1. Let f be a point function defined on $K \setminus Z$. A quasi-measure M (resp., m) is called a $(\Sigma,1)$-*major* (resp., $(\Sigma,1)$-*minor*) function of f if it is $(\Sigma,1)$-smooth on Z and satisfies

$$\underline{D}_{\mathcal{B}_1} M(\mathbf{x}) \geq f(\mathbf{x}) \quad (\text{resp.} \quad \overline{D}_{\mathcal{B}_1} m(\mathbf{x}) \leq f(\mathbf{x})) \tag{7.2}$$

at each point $\mathbf{x} \in K \setminus Z$.

In [12] it was proved that if M and m are a $(\Sigma,1)$-major and a $(\Sigma,1)$-minor function for a point function f on K, then for each $\mathcal{B}$-interval I the inequality (4.3) holds where "inf" and "sup" in (4.3) run over all $(\Sigma,1)$-major and $(\Sigma,1)$-minor function of f, respectively. This justifies the following definition.

Definition 7.2. We say a finite-valued point function f, defined everywhere on $K \setminus Z$, is $P_{\Sigma,1}$-*integrable* on K if there exists at least one $(\Sigma,1)$-major function and at least one $(\Sigma,1)$-minor function of f and

$$-\infty < \inf_M\{M(K)\} = \sup_m\{m(K)\} < +\infty$$

where "inf" and "sup" are taken as above. The common value is called $P_{\Sigma,1}$-*integral* of f on K and is denoted by $(P_{\Sigma,1}) \int_K f$. Similarly we can define $P_{\Sigma,1}$-integral on any $\mathcal{B}$-interval I.

An interesting fact is that the $(P_{\Sigma,1})$-integral and the Lebesgue one are incomparable [14]. But it follows from results of [18] that if a finite-valued function f both summable and $(P_{\Sigma,1})$-integrable then we have

$$(L) \int\limits_I f = (P_{\Sigma,1}) \int\limits_I f$$

for every dyadic interval I.

Theorem 7.3 (See [12])**.** *Suppose an additive* $(\Sigma, 1)$*-smooth* $\mathcal{B}$*-interval function* F *is* $\mathcal{B}_1$*-differentiable with* $D_{\mathcal{B}_1} F(\mathbf{x}) = f(\mathbf{x})$ *everywhere on* $K \setminus Z$*. Then the function* f *is* $P_{\Sigma,1}$*-integrable on* K *and* F *is its indefinite* $P_{\Sigma,1}$*-integral.*

Now we consider (see [12]) the relationship between convergence of two-dimensional series (3.2) and $(\Sigma, 1)$-smoothness of quasi-measures.

Lemma 7.4. *Suppose a two-dimensional series* (3.2) *is* $1/2$*-regular rectangular convergent to a finite sum at some point* $\mathbf{x} \in Z$*. Then the quasi-measure* ψ *generated by this series is* $(\Sigma, 1)$*-smooth at* $\mathbf{x}$*.*

Theorem 7.3 and Lemma 6.8 yield the solution of coefficient problem for regular converging multiple Haar series on K.

Theorem 7.5 (See [12])**.** *Let a two-dimensional series* (3.2) *is cubic convergent to a finite-valued function* f *on* $K \setminus Z$ *and* $1/2$*-regular rectangular convergent to a finite sum on* Z*. Then* f *is* $P_{\Sigma,1}$*-integrable and this series is the* $P_{\Sigma,1}$*-Fourier–Haar series of* f*.*

Corollary 7.6. *Let a two-dimensional series* (3.2) *everywhere on* K *is* $1/2$*-regular rectangular convergent to a finite-valued function* f*. Then* f *is* $P_{\Sigma,1}$*-integrable and this series is the* $P_{\Sigma,1}$*-Fourier–Haar series of* f*.*

Corollary 7.7. *The empty set is a* U*-set for everywhere* $1/2$*-regular rectangular convergent two-dimensional series* (3.2)*. In other words, if a two-dimensional series* (3.2) *everywhere on* K $1/2$*-regular rectangular converges to zero, then all of its coefficients are equal to zero.*

Corollary 7.7 remains true if $1/2$-regular rectangular convergence is replaced by ρ-regular rectangular convergence where $\rho \in (0, \sqrt{2}/2)$ [16]. The constant $\sqrt{2}/2$ is sharp: *for each* $\rho \in (\sqrt{2}/2, 1]$ *there exists a non-trivial two-dimensional series* (3.2) $1/2$*-regular rectangular convergent to zero everywhere on* K [13]. In particular, *there exists a non-trivial two-dimensional series* (3.2) *which is square convergent to zero everywhere on* K*.*

7.2. (Σ,α)**-smoothness in general case.** We return to a general case. Let K be again the unit square $[0,1]^q$.

We recall (see Subsection 5.1) that Σ denote the set $\{0,1\}^q$ of 0–1 q-dimensional vectors and $|\boldsymbol{\sigma}| \stackrel{\text{def}}{=} \sigma_1+\cdots+\sigma_q$ if $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_q)\in\Sigma$. Let s be the number of dyadic-rational coordinates of the point $\mathbf{x}\in Z$. We write $\Sigma_{\mathbf{x}}$ for the set of 0–1 q-dimensional vectors $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_q)$ such that $\sigma_i=0$ if ith coordinate of the point $\mathbf{x}$ is dyadic-irrational. Obviously, $\Sigma_{\mathbf{x}}\subset\Sigma$ and the set $\Sigma_{\mathbf{x}}$ contains 2^s elements.

Clearly, there exist 2^s sequences $\{I^{(\mathbf{k})}(\mathbf{x}),\ \mathbf{k}\in\mathbb{N}^q\}$ of $\mathcal{B}$-intervals such that

$$\mathbf{x}\in I^{(\mathbf{k})}(\mathbf{x}),\quad I^{(\mathbf{k})}(\mathbf{x})\supset\Delta^{(\mathbf{k}+\boldsymbol{\sigma})}(\mathbf{x}),\quad \operatorname{rank} I^{(\mathbf{k})}(\mathbf{x})=\mathbf{k},\qquad \text{for all}\quad \mathbf{k}\in\mathbb{N}^q\quad\text{and}\quad\boldsymbol{\sigma}\in\Sigma. \tag{7.3}$$

We fix the point $\mathbf{x}$ and one of these sequences.

Let

$$I^{(\mathbf{k})}(\mathbf{x})=I^{(k_1)}\times\cdots\times I^{(k_q)}. \tag{7.4}$$

If $k_i=1,2,\ldots$, then we set

$$I^{(k_i)}_{(0)}=I^{(k_i)},\quad I^{(k_i)}_{(1)}=I^{(k_i-1)}\setminus I^{(k_i)}. \tag{7.5}$$

For $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_q)\in\Sigma$ we put

$$I^{(\mathbf{k})}_{(\boldsymbol{\sigma})}=I^{(k_1)}_{(\sigma_1)}\times\cdots\times I^{(k_q)}_{(\sigma_q)}. \tag{7.6}$$

Definition 7.8. Given $\alpha\in\mathbb{R}_+$ we say a $\mathcal{B}$-interval function τ is (Σ,α)-*smooth* at the point $\mathbf{x}$ if

$$\frac{1}{\left|I^{(\mathbf{k})}_{(\boldsymbol{\sigma})}\right|^{\alpha}}\sum_{\boldsymbol{\sigma}\in\Sigma_{\mathbf{x}}}(-1)^{|\boldsymbol{\sigma}|}\tau\left(I^{(\mathbf{k})}_{(\boldsymbol{\sigma})}\right)=\mathrm{o}(1)\quad\text{as}\quad k_1=\ldots=k_q\to\infty, \tag{7.7}$$

holds for any sequence $\{I^{(\mathbf{k})}(\mathbf{x}),\ \mathbf{k}\in\mathbb{N}^q\}$ of $\mathcal{B}$-intervals satisfying (7.3).

Theorem 7.9 (See [15])**.** *Let $E\subset K$ be some at most countable set. Suppose a quasi-measure τ has the following properties:*

(1) *if a point $\mathbf{x}\in Z\setminus E$ has exactly s dyadic-rational coordinates, then τ is $(\Sigma,1-s/q)$-smooth at the point $\mathbf{x}$;*
(2) *$D_{\mathcal{B}_1}\tau(\mathbf{x})=0$ at each point $\mathbf{x}\in K\setminus(Z\cup E)$;*
(3) *if $\mathbf{x}\in K$, then*

$$\sum_{\boldsymbol{\sigma}\in\Sigma}(-1)^{|\boldsymbol{\sigma}|}\tau\left(I^{(\mathbf{k})}_{(\boldsymbol{\sigma})}\right)=\mathrm{o}(1)\quad\textit{as}\quad k_1=\ldots=k_q\to\infty, \tag{7.8}$$

for any sequence $\{I^{(\mathbf{k})}(\mathbf{x}),\ \mathbf{k}\in\mathbb{N}^q\}$ of $\mathcal{B}$-intervals satisfying (7.3).

Then $\tau(I)=0$ for every $I\in\mathcal{B}$.

Theorem 7.9 is sharp in the following sense: "o" in the condition (7.8) can not be replaced by "O" even at a single point $\mathbf{x}\in K$. Moreover, $(\Sigma,1-s/q)$-smoothness in the condition (1) can not be replaced by the condition (7.7) with "O" instead "o" at all points $\mathbf{x} \in Z \setminus E$ having exactly s dyadic-rational coordinates. The last fact is implied by the following theorem.

Theorem 7.10 (See [21]). *There exists a non-trivial quasi-measure τ such that*

(1) *if a point $\mathbf{x} \in Z \setminus E$ has exactly s dyadic-rational coordinates, then we have*
$$\frac{1}{\left|I^{(\mathbf{k})}_{(\boldsymbol{\sigma})}\right|^{1-s/q}} \sum_{\boldsymbol{\sigma}\in\Sigma} (-1)^{|\boldsymbol{\sigma}|} \tau\left(I^{(\mathbf{k})}_{(\boldsymbol{\sigma})}\right) = \mathrm{O}(1), \quad \text{as} \quad k_1 = \ldots = k_q \to \infty;$$
(2) $D_{\mathcal{B}_1}\tau(\mathbf{x}) = 0$ *at each point* $\mathbf{x} \in K \setminus Z$;
(3) (7.8) *holds for any sequences* $\{I^{(\mathbf{k})}(\mathbf{x}),\ \mathbf{k} \in \mathbb{N}^q\}$ *of* $\mathcal{B}$*-intervals satisfying* (7.3) *for* $\mathbf{x} \in K$.

Definition 7.11. Let f be a finite-valued function defined at least on a set $K \setminus (Z \cup E)$ where E is some at most countable set. We say a function f is $P_{\Sigma,*}$*-integrable* if there exists at least one quasi-measure M ($(\Sigma,*)$*-major* function of f) and at least one quasi-measure m ($(\Sigma,*)$*-minor* function of f) satisfying the following conditions:

(A) if a point $\mathbf{x} \in Z \setminus E$ has exactly s dyadic-rational coordinates, then M and m are $(\Sigma, 1 - s/q)$-smooth at this point;
(B) $\underline{D}_{\mathcal{B}_1} M(\mathbf{x}) \geq f(\mathbf{x}) \geq \overline{D}_{\mathcal{B}_1} m(\mathbf{x})$ at each point $\mathbf{x} \in K \setminus (Z \cup E)$;
(C) M and m satisfy (7.8) at each point $\mathbf{x} \in K$;
(D) the inequalities
$$-\infty < \inf_M\{M(K)\} = \sup_m\{m(K)\} < +\infty \tag{7.9}$$
are true, where "inf" and "sup" in (7.9) run over all $(\Sigma,*)$-major and $(\Sigma,*)$-minor function of f, respectively.

The common value in (7.9) is called $P_{\Sigma,*}$*-integral* of f on K and is denoted by $(P_{\Sigma,*})\int_K f$. Similarly we can define $P_{\Sigma,*}$-integral on any $\mathcal{B}$-interval I.

Correctness of this definition was justified in [15] by proving the inequality $\inf_M\{M(K)\} \geq \sup_m\{m(K)\}$ whenever the conditions (A), (B), (C) of Definition 7.11 hold.

The next lemmas [15] establish the relationships between a rate of growth of the partial sums of multiple Haar series (3.2) and behaviour of the quasi-measures.

Lemma 7.12. *Consider any $\alpha \in \mathbb{R}_+$ and a multiple Haar series of the form* (3.2). *Suppose*
$$S_{\mathbf{N}}(\mathbf{x}) = \mathrm{o}\left((N_1 \cdot \cdots \cdot N_q)^\alpha\right) \quad \text{as} \quad \min_i\{N_i\} \to \infty \quad \text{and} \quad \operatorname{reg} \mathbf{N} \geq 1/2 \tag{7.10}$$
holds for $\mathbf{N}$*th rectangular partial sums* $S_{\mathbf{N}}(\mathbf{x}) = S_{N_1,\ldots,N_q}(\mathbf{x})$ *of this series at some point* $\mathbf{x} \in Z$. *Then the quasi-measure* ψ *generated by this series is* $(\Sigma, 1-\alpha)$*-smooth at* $\mathbf{x}$.

Lemma 7.13. *Consider any multiple Haar series of the form* (3.2)*. Suppose the series satisfies at a point* $\mathbf{x} \in K$ *at least one of the following conditions*:

$$S_{\mathbf{N}}(\mathbf{x}) = \mathrm{o}\left((N_1 \cdot \ldots \cdot N_q)\right) \quad \textit{as} \quad \min_i\{N_i\} \to \infty \ \textit{ and } \ \operatorname{reg} \mathbf{N} \geq 1/2; \tag{7.11}$$

$$\begin{aligned} &b_{n_1,\ldots,n_q}\chi_{n_1,\ldots,n_q}(\mathbf{x}_0) = \mathrm{o}(n_1 \cdot \cdots \cdot n_q) \\ &\qquad\qquad \textit{as} \quad 2^k \leq n_1, \ldots, n_q \leq 2^{k+1} - 1 \quad \textit{and} \quad k \to \infty. \end{aligned} \tag{7.12}$$

Then the quasi-measure ψ *generated by this series is* $(\Sigma, 0)$*-smooth at* $\mathbf{x}$.

Combining the analogue of Theorem 4.6 for $P_{\Sigma,*}$-*integral* and Lemmas 7.12 and 7.13, we get the solution of the coefficient problem for regular convergent multiple Haar series on K.

Theorem 7.14 (See [15])**.** *Suppose an* q*-multiple Haar series of the form* (3.2) *and a finite-valued function* f*, defined at least on a set* $K \setminus (Z \cup E)$ *where* E *is some at most countable set, satisfy the following conditions:*

(1) *if a point* $\mathbf{x} \in Z \setminus E$ *has exactly* s *dyadic-rational coordinates, then we have the equality*

$$S_{\mathbf{N}}(\mathbf{x}) = \mathrm{o}\left((N_1 \cdot \ldots \cdot N_q)^{s/q}\right) \quad \textit{as} \quad \min_i\{N_i\} \to \infty \ \textit{ and } \ \operatorname{reg} \mathbf{N} \geq 1/2; \tag{7.13}$$

(2) *the series rectangularly converges to* f *everywhere on* $K \setminus (Z \cup E)$*;*

(3) *everywhere on* K *we have at least one of the equalities* (7.11) *or* (7.12)*.*

Then the function f *is* $P_{\Sigma,*}$*-integrable and the series is the* $P_{\Sigma,*}$*-Fourier–Haar series of* f*.*

Theorem 7.14 yields the following corollaries.

Theorem 7.15 (See [15])**.** *Let* $E \subset K$ *be some at most countable set. Suppose a series of the form* (3.2) *satisfy the following conditions:*

(1) *if a point* $\mathbf{x} \in Z \setminus E$ *has exactly* s *dyadic-rational coordinates, then the rectangular partial sums of the series satisfies* (7.13) *at the point* $\mathbf{x}$*;*

(2) *the series* $1/2$*-regular rectangularly converges to zero everywhere on* $K \setminus (Z \cup E)$*;*

(3) *the series satisfies* (7.11) *or* (7.12) *everywhere on* K*.*

Then all the coefficients of the series are equal zero.

Theorem 7.16 (See [15])**.** *Let* E *be some at most countable set. Suppose a series* (3.2) $1/2$*-regular rectangularly converges to zero everywhere on* $K \setminus E$ *and satisfies* (7.11) *or* (7.12) *everywhere on* K*. Then all its coefficients are equal 0. In other words, any at most countable set is a* U*-set for* ρ*-regular rectangular convergent series* (3.2) *under condition* (7.11) *or* (7.12) *if* $\rho \leq 1/2$*.*

Theorem 7.17 (See [15])**.** *The trivial series is the only series of the form* (3.2)*, which is* $1/2$*-regular rectangular convergent to zero everywhere on* K*. It means that* $\emptyset$ *is a* U*-set for* ρ*-regular rectangular convergent series* (3.2) *if* $\rho \leq 1/2$*.*

The above results are sharp in the following sense. Theorem 7.15 becomes false if we replace "o" in (7.11) or (7.12) by "O" even at a single point $\mathbf{x} \in K$. Moreover, "o" in (7.13) can not be replaced by "O" at all points $\mathbf{x} \in Z \setminus E$ having exactly s dyadic-rational coordinates. The last fact is justified by the following theorem.

Theorem 7.18 (See [21]). *There exists a non-trivial series of the form* (3.2) *such that*

(1) *if a point* $\mathbf{x} \in Z \setminus E$ *has exactly* s *dyadic-rational coordinates, then the equality*

$$S_{\mathbf{N}}(\mathbf{x}) = \mathrm{O}\left((N_1 \cdot \dots \cdot N_q)^{s/q}\right) \quad \text{as} \quad \min_i \{N_i\} \to \infty \ \text{ and } \ \operatorname{reg} \mathbf{N} \geq 1/2$$

holds for the N*th rectangular partial sums of the series* (S);

(2) *the series* $1/2$*-regular rectangularly converges to zero everywhere on* $K \setminus Z$;

(3) *the series satisfies both the conditions* (7.11) *and* (7.12) *everywhere on* K.

Theorems 7.9, 7.10, 7.15, and 7.18 underline once again a deep duality between Haar series and the quasi-measures generated by these series. In contrast to the case of Walsh series where, according to (3.6), $\mathcal{B}$-differentiability at points $\mathbf{x} \in K \setminus Z$ of the quasi-measure generated by a Walsh series is equivalent to the convergence of the subsequence S_{2^n} of the partial sums, which is not equivalent to the convergence of the series, in the Haar case the convergence of the above subsequence and the convergence of the series are equivalent. This explains why the same number s/q appears both in the condition for uniqueness of the quasi-measures given by Theorems 7.9 and 7.10 and in the condition for uniqueness of series (3.2) given by Theorems 7.15 and 7.18. This method based on duality between Haar series and the quasi-measures is useful not only in the problem of uniqueness, but also in some other areas of the Haar series theory (see, for example, [4, 8, 9, 10, 30]).

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