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CONVERGENCE OF TRIGONOMETRIC AND WALSH-FOURIER SERIES

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Dedicated to Professor Ferenc Schipp on the occasion of his 75th birthday, to Professor William Wade on the occasion of his 70th birthday and to Professor Péter Simon on the occasion of his 65th birthday.

Abstract. In this paper we present some results on convergence and summability of one- and multi-dimensional trigonometric and Walsh-Fourier series. The Fejér and Cesàro summability methods are investigated. We will prove that the maximal operator of the summability means is bounded from the corresponding classical or martingale Hardy space H_p to L_p for some $p > p_0$. For $p = 1$ we obtain a weak type inequality by interpolation, which ensures the almost everywhere convergence of the summability means.

1. Introduction

In this survey paper we will consider summation methods for one- and multi-dimensional trigonometric and Walsh-Fourier series. Two types of summability methods will be investigated, the Fejér and Cesàro or (C, α) methods. The Fejér summation is a special case of the Cesàro method, $(C, 1)$ is exactly the Fejér method. In the multi-dimensional case two types of convergence and maximal operators will be considered, the restricted (convergence over the diagonal or over a cone or over a cone-like set), and the unrestricted (convergence over $\mathbb{N}^d$ in Pringsheim's sense). We introduce three types of classical and martingale Hardy spaces H_p and prove that the maximal operators of the summability means are bounded from the corresponding H_p to L_p whenever

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$p > p_0$ for some $p_0 < 1$. For $p = 1$ we obtain a weak type inequality by interpolation, which implies the almost everywhere convergence of the summability means to the original function. The almost everywhere convergence and the weak type inequality are proved usually with the help of a Calderon-Zygmund type decomposition lemma. However, this lemma does not work in higher dimensions. Our method, that can be applied in higher dimension, too, can be regarded as a new method to prove the almost everywhere convergence and weak type inequalities. In this survey paper we summarize the results appeared in this topic in the last 10–20 years. This paper was the base of my talk given at the Conference on Dyadic Analysis and Related Fields with Applications, June 2014, in Nyíregyháza (Hungary).

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2. Trigonometric and Walsh system

We consider either the torus $\mathbb{X} = \mathbb{T}$ or the unit interval $\mathbb{X} = [0, 1)$, both with the Lebesgue measure λ. We briefly write $L_p(\mathbb{X})$ instead of the real $L_p(\mathbb{X}, \lambda)$ space equipped with the norm (or quasinorm)

$$\|f\|_p := \left(\int_{\mathbb{X}} |f|^p \, d\lambda \right)^{1/p} \qquad (0 < p \le \infty),$$

where λ is the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I. The weak $L_p(\mathbb{X})$ space $L_{p,\infty}(\mathbb{X})$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{p,\infty} := \sup_{\rho > 0} \rho \lambda(|f| > \rho)^{1/p} < \infty.$$

Note that $L_{p,\infty}$ is a quasi-normed space. It is easy to see that

$$L_p(\mathbb{X}) \subset L_{p,\infty}(\mathbb{X}) \quad \text{and} \quad \|\cdot\|_{p,\infty} \le \|\cdot\|_p$$

for each $0 < p < \infty$.

The *Rademacher functions* are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \quad (x \in [0, 1), n \in \mathbb{N}).$$

The product system generated by the Rademacher functions is the *one-dimensional Walsh system*:

$$w_n := \prod_{k=0}^{\infty} {r_k}^{n_k} \quad (n \in \mathbb{N}),$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad (0 \le n_k < 2)$$

(see Figure (1)). In what follows let $\phi_n(x)$ denote the *trigonometric system*

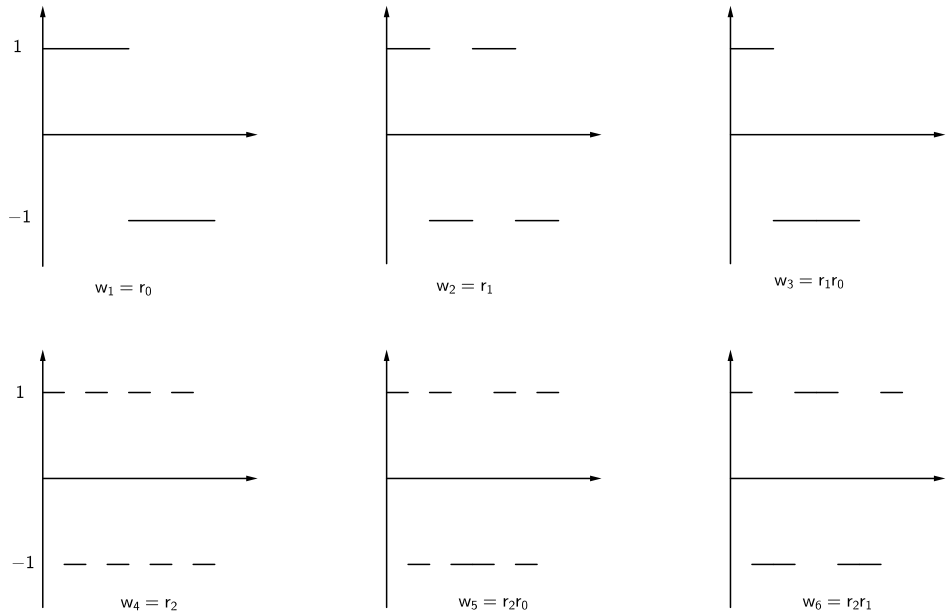


FIGURE 1. Walsh system.

$e^{2\pi\imath n\cdot x}$ ($n \in \mathbb{Z}$) defined on $\mathbb{T}$ or the *Walsh system* $\phi_n(x) := w_n(x)$ ($n \in \mathbb{N}$) defined on the unit interval. For the Walsh system let $\phi_n = 0$ if $n \in \mathbb{Z} \setminus \mathbb{N}$.

In this paper the constants C_p depend only on p and may denote different constants in different contexts.

3. Partial sums of one-dimensional Fourier series

For an integrable function $f \in L_1(\mathbb{X})$ ($\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = [0,1)$) its kth *trigonometric* or *Walsh-Fourier coefficient* is defined by

$$\widehat{f}(k) := \int_{\mathbb{X}} f\overline{\phi_k}\, d\lambda \quad (k \in \mathbb{Z}).$$

The definition of the Fourier coefficients can be extended easily to distributions in case of the trigonometric system and to martingales in case of the Walsh system (see Weisz [77, 82]).

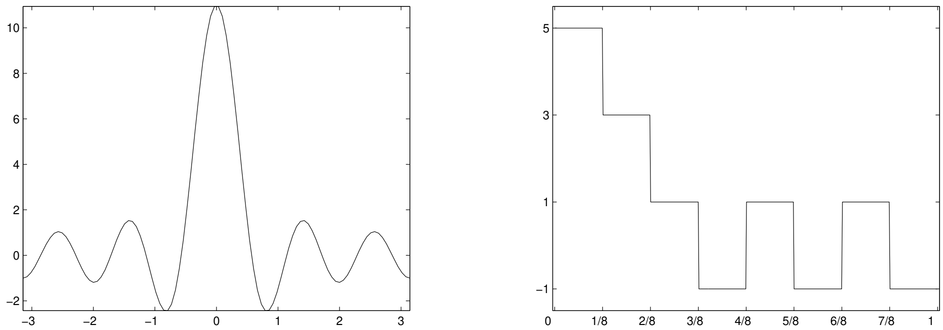
For $f \in L_1(\mathbb{X})$ the nth *partial sum* $s_n f$ of the Fourier series of f is introduced by

$$s_n f(x) := \sum_{|k|\le n} \widehat{f}(k)\phi_k(x) = \int_{\mathbb{X}} f(x-u) D_n(u)\, du \quad (n \in \mathbb{N}),$$

where

$$D_n(u) := \sum_{|k|\le n} \phi_k(u)$$

is the nth *trigonometric* or *Walsh-Dirichlet kernel* (see Figure 2). In case of the Walsh system we use dyadic addition instead of addition.



(a) The trigonometric Dirichlet kernel (b) The Walsh-Dirichlet kernel

FIGURE 2. The Dirichlet kernels D_n with $n = 5$.

It is a basic question as to whether the function f can be reconstructed from the partial sums of its Fourier series. It can be found in most books about Fourier series (e.g., Zygmund [85], Bary [1], Torchinsky [62], Grafakos [28], Schipp, Wade, Simon and Pál [50]), that the partial sums converge to f in the L_p-norm if $1 < p < \infty$.

Theorem 1. *If $f \in L_p(\mathbb{T})$ for some $1 < p < \infty$, then*

$$\|s_n f\|_p \leq C_p \|f\|_p \quad (n \in \mathbb{N})$$

and

$$\lim_{n\to\infty} s_n f = f \quad \text{in the } L_p\text{-norm.}$$

This theorem is due to Riesz [44] for trigonometric series and to Paley [43] for Walsh-Fourier series.

One of the deepest results in harmonic analysis is Carleson's result, i.e., the partial sums $s_n f$ of the Fourier series converge almost everywhere to $f \in L_p(\mathbb{X})$ $(1 < p < \infty)$. This result is due to Carleson [8] for trigonometric Fourier series and for one-dimensional functions $f \in L_2(\mathbb{T})$. Later Hunt [30] extended this result to all $f \in L_p(\mathbb{T})$ spaces, $1 < p < \infty$. Billard [4], Sjölin [58] and Schipp [47, 51] generalized both results for Walsh-Fourier series.

Theorem 2. *If $f \in L_p(\mathbb{X})$ for some $1 < p < \infty$, then*

$$\left\| \sup_{n\in\mathbb{N}} |s_n f| \right\|_p \leq C_p \|f\|_p$$

and

$$\lim_{n\to\infty} s_n f = f \quad a.e.$$

The inequalities of Theorems 1 and 2 do not hold if $p = 1$ or $p = \infty$, and the almost everywhere convergence does not hold if $p = 1$. du Bois Reymond proved the existence of a continuous function $f \in C(\mathbb{T})$ and a point $x_0 \in \mathbb{T}$ such that the partial sums $s_n f(x_0)$ diverge as $n \to \infty$. Kolmogorov gave an integrable function $f \in L_1(\mathbb{T})$, whose Fourier series diverges almost everywhere or even everywhere (see Kolgomorov [32, 33], Zygmund [85] or Grafakos [28]). The analogous results for Walsh-Fourier series can be found in Schipp [45] and Simon [53].

4. Hardy spaces $H_p(\mathbb{X})$

To prove almost everywhere convergence of the summability means introduced in the next section, we will need the concept of Hardy spaces and their atomic decomposition. First we consider the classical Hardy spaces for the trigonometric system and then the dyadic Hardy spaces for the Walsh system.

4.1. The $H_p(\mathbb{T})$ classical Hardy spaces.

A distribution f is in the *classical Hardy space* $H_p(\mathbb{T})$ $(0 < p \leq \infty)$ if

$$\|f\|_{H_p} := \left\| \sup_{0<t} |f * P_t| \right\|_p < \infty,$$

where

$$P_t(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi \imath k x} = \frac{1 - r^2}{1 + r^2 - 2r\cos 2\pi x} \quad (r := e^{-t}, x \in \mathbb{T})$$

is the periodic Poisson kernel. Since $P_t \in L_1(\mathbb{T})$, the convolution in the definition of the norms are well defined.

4.2. The $H_p[0,1)$ dyadic Hardy spaces.

By a *dyadic interval* we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$, $0 \leq k < 2^n$. Given $n \in \mathbb{N}$ and $x \in [0,1)$ let $I_n(x)$ be the dyadic interval of length 2^{-n} which contains x. The σ-algebra generated by the dyadic intervals $\{I_n(x) : x \in [0,1)\}$ will be denoted by $\mathcal{F}_n$ $(n \in \mathbb{N})$. It is easy to show that for a martingale $f = (f_n, n \in \mathbb{N})$ we have $s_{2^n} f = f_n$.

We investigate the class of *martingales* $f = (f_n, n \in \mathbb{N})$ with respect to $(\mathcal{F}_n, n \in \mathbb{N})$. For $0 < p \leq \infty$ the *dyadic Hardy space* $H_p[0,1)$ consists of all martingales for which

$$\|f\|_{H_p} := \left\| \sup_{n\in\mathbb{N}} |f_n| \right\|_p < \infty.$$

4.3. Atomic decomposition of $H_p(\mathbb{X})$.

The results of this subsection hold for both the classical and the dyadic Hardy spaces. It is known (see e.g. Stein [59] or Weisz [77]) that

$$H_p(\mathbb{X}) \sim L_p(\mathbb{X}) \quad (1 < p \leq \infty)$$

and $H_1(\mathbb{X}) \subset L_1(\mathbb{X})$, where $\sim$ denotes the equivalence of spaces and norms.

The atomic decomposition provides a useful characterization of Hardy spaces. A function $a \in L_\infty(\mathbb{T})$ is a *classical p-atom* if there exists an interval $I \subset \mathbb{T}$ such that

(1) $\operatorname{supp} a \subset I$,
(2) $\|a\|_\infty \le |I|^{-1/p}$,
(3) $\int_I a(x) x^k \, dx = 0$ for all $k \in \mathbb{N}$ with $k \le \lfloor 1/p - 1 \rfloor$,

where $\lfloor \cdot \rfloor$ denotes the integer part. While a function $a \in L_\infty[0,1)$ is called a *dyadic p-atom* if there exists a dyadic interval $I \subset [0,1)$ such that (i), (ii) and (iii) with $k = 0$ hold.

The Hardy space $H_p(\mathbb{X})$ has an atomic decomposition. In other words, every function (more exactly, distribution resp. martingale) from the Hardy space can be decomposed into the sum of atoms. A first version of the atomic decomposition was introduced by Coifman and Weiss [10] in the classical case and by Herz [29] in the martingale case. The proof of the next theorem can be found in Latter [34], Lu [36], Wilson [83, 84], Stein [59] and Weisz [66, 77].

Theorem 3. *A distribution (resp. martingale) f is in $H_p(\mathbb{X})$ $(0 < p \le 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of classical (resp. dyadic) p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} \mu_k a^k = f$$

in the sense of distributions (resp. martingales). Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.$$

The "only if" part of the theorem holds also for $0 < p < \infty$. The following result gives a sufficient condition for an operator to be bounded from $H_p(\mathbb{X})$ to $L_p(\mathbb{X})$ (see Weisz [77, 81] and, for $p_0 = 1$, Schipp, Wade, Simon and Pál [50] and Móricz, Schipp and Wade [40]). For $I \subset \mathbb{T}$ let I^r be the interval having the same center as the interval I and length $2^r|I|$. If $I \subset [0,1)$ is a dyadic interval then let I^r be a dyadic interval, for which $I \subset I^r$ and $|I^r| = 2^r|I|$ $(r \in \mathbb{N})$.

Theorem 4. *For each $n \in \mathbb{N}$, let $V_n \colon L_1(\mathbb{X}) \to L_1(\mathbb{X})$ be a bounded linear operator and let*

$$V_* f := \sup_{n \in \mathbb{N}} |V_n f|.$$

Suppose that

$$\int_{\mathbb{X} \setminus I^r} |V_* a|^{p_0} \, d\lambda \le C_{p_0}$$

for all classical (resp. dyadic) p_0-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$, where the interval I is the support of the atom. If V_* is bounded from

$L_{p_1}(\mathbb{X})$ to $L_{p_1}(\mathbb{X})$ for some $1 < p_1 \leq \infty$, then

$$\|V_* f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p(\mathbb{X})) \tag{1}$$

for all $p_0 \leq p \leq p_1$. Moreover, if $p_0 < 1$, then the operator V_* is of weak type $(1,1)$, i.e., if $f \in L_1(\mathbb{X})$ then

$$\sup_{\rho>0} \rho\, \lambda(|V_* f| > \rho) \leq C\|f\|_1. \tag{2}$$

Now we give a typical proof of this theorem. If, instead of V_*, the linear or sublinear operator V satisfies the condition

$$\int_{\mathbb{X}^d \setminus I^r} |Va|^{p_0}\, d\lambda \leq C_{p_0}$$

with $p_0 \leq 1$, then we can easily show that $\|Va\|_{p_0} \leq C$ for all p_0-atoms a. We take an atomic decomposition of f:

$$f = \sum_{k=0}^{\infty} \mu_k a^k,$$

where each a^k is a p_0-atom and

$$\left(\sum_{k=0}^{\infty} |\mu_k|^{p_0} \right)^{1/p_0} \leq C_{p_0} \|f\|_{H_{p_0}}.$$

Next

$$|Vf| \leq \sum_{k=0}^{\infty} |\mu_k| \left| V a^k \right| \tag{3}$$

and

$$\|Vf\|_{p_0}^{p_0} \leq \sum_{k=0}^{\infty} |\mu_k|^{p_0} \left\| V a^k \right\|_{p_0}^{p_0} \leq C_{p_0} \|f\|_{H_{p_0}}^{p_0}.$$

The problem is that this proof is falls because the inequality (3) does not necessarily hold. Indeed, Bownik [6] have given an operator V for which (3) does not hold. Moreover, though the L_{p_0}-norms of Va are uniformly bounded, V is not bounded from $H_{p_0}(\mathbb{R})$ to $L_{p_0}(\mathbb{R})$. The correct proof of Theorem 4 can be found in Weisz [81].

Note that (2) can be obtained from (1) by interpolation. For the basic definitions and theorems on interpolation theory see Bergh and Löfström [3] and Bennett and Sharpley [2] or Weisz [66, 77]. The interpolation of martingale Hardy spaces was worked out in [66]. Theorem 4 can be regarded also as an alternative tool to the Calderon-Zygmund decomposition lemma for proving weak type $(1,1)$ inequalities. In many cases this theorem can be applied better and more simply than the Calderon-Zygmund decomposition lemma.

5. Summability of one-dimensional Fourier series

Though Theorems 1 and 2 are not true for $p=1$ and $p=\infty$, with the help of some summability methods they can be generalized for these endpoint cases. Obviously, summability means have better convergence properties than the original Fourier series. Summability is intensively studied in the literature. We refer at this time only to the books Stein and Weiss [61], Butzer and Nessel [7], Trigub and Belinsky [63], Grafakos [28] and Weisz [77, 82] and the references therein.

The best known summability method is the Fejér method. In 1904 Fejér [14] investigated the arithmetic means of the partial sums, the so called Fejér means and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means converge to $(f(x-0)+f(x+0))/2$. One year later Lebesgue [35] extended this theorem and obtained that every integrable function is Fejér summable at each Lebesgue point, thus almost everywhere. Some years later M. Riesz [44] proved that the Cesàro means of a function $f \in L_1(\mathbb{T})$ converge almost everywhere to f (see also Zygmund [85, Vol. I, p.94]).

In this paper we consider the *Fejér* and *Cesàro* (or (C,α)) means defined by

$$\sigma_n f(x) := \frac{1}{n}\sum_{k=0}^{n-1} s_k f(x) = \sum_{|j|\le n}\left(1-\frac{|j|}{n}\right)\widehat{f}(j)\phi_j(x) = \int_{\mathbb{X}} f(x-u)K_n(u)\,du$$

and

$$\begin{aligned}\sigma_n^\alpha f(x) &:= \frac{1}{A_{n-1}^\alpha}\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} s_k f(x)\\ &= \frac{1}{A_{n-1}^\alpha}\sum_{|j|\le n} A_{n-1-|j|}^{\alpha}\widehat{f}(j)\phi_j(x) = \int_{\mathbb{X}} f(x-u)K_n^\alpha(u)\,du,\end{aligned}$$

where

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+k)}{k!}$$

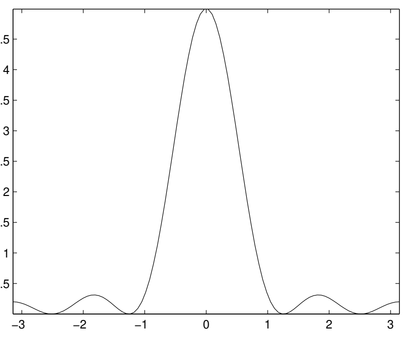
and the *Fejér* and *Cesàro kernels* are given by

$$K_n(u) := \sum_{|j|\le n}\left(1-\frac{|j|}{n}\right)\phi_j(u) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(u)$$

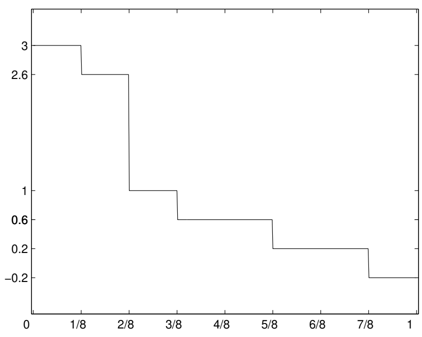
and

$$K_n^\alpha(u) := \frac{1}{A_{n-1}^\alpha}\sum_{|j|\le n} A_{n-1-|j|}^{\alpha}\phi_j(u) = \frac{1}{A_{n-1}^\alpha}\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} D_k(u)$$

(see Figure 3). It is known (Zygmund [85]) that $A_k^\alpha \sim k^\alpha$ $(k\in\mathbb{N})$. The Cesàro means are generalizations of the Fejér means, if $\alpha=1$, then we get back the Fejér means. We will suppose always that $0<\alpha\le 1$. The case $\alpha>1$ can



(a) The trigonometric Fejér kernel



(b) The Walsh-Fejér kernel

FIGURE 3. The Fejér kernels K_n with $n = 5$.

be led back to $\alpha = 1$. The next result extends Theorem 1 to the summability means (see Zygmund [85] and Paley [43]).

Theorem 5. *If $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$, then*

$$\|\sigma_n^\alpha f\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbb{X}), n \in \mathbb{N}).$$

Moreover, for all $f \in L_p(\mathbb{X})$ $(1 \leq p < \infty)$,

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \quad \textit{in the } L_p\textit{-norm.}$$

The *maximal operator* of the Cesàro means are defined by

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f| \, .$$

Applying Theorem 4, we have extended the previous result to the $L_p(\mathbb{X})$ spaces $(0 < p < 1)$ and to the maximal operator in [67, 76, 77]. The first inequality of Theorem 6 was proved by Fujii [18] in the Walsh case for $p = 1$ (see also Schipp and Simon [49]).

Theorem 6. *If $0 < \alpha \leq 1$ and $1/(\alpha+1) < p \leq \infty$, then*

$$\|\sigma_*^\alpha f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p(\mathbb{X}))$$

and for $f \in H_{1/(\alpha+1)}(\mathbb{X})$,

$$\|\sigma_*^\alpha f\|_{1/(\alpha+1),\infty} = \sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho)^{\alpha+1} \leq C\|f\|_{H_{1/(\alpha+1)}}.$$

The critical index is $p = 1/(\alpha+1)$, if p is smaller than or equal to this critical index, then σ_*^α is not bounded anymore (see Stein, Taibleson and Weiss [60], Simon and Weisz [57], Simon [55] and Gát and Goginava [23]).

Theorem 7. *The operator σ_*^α $(0 < \alpha \leq 1)$ is not bounded from $H_p(\mathbb{X})$ to $L_p(\mathbb{X})$ if $0 < p \leq 1/(\alpha+1)$.*

We get the next weak type $(1,1)$ inequality from Theorem 6 by interpolation (Weisz [67, 76, 77], Zygmund [85] for the trigonometric system, for $\alpha = p = 1$ Móricz [39], for $\alpha = 1$ and for the Walsh system Schipp [46] and Simon [54]).

Corollary 1. *If $0 < \alpha \leq 1$ and $f \in L_1(\mathbb{X})$ then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \leq C\|f\|_1.$$

This weak type $(1,1)$ inequality and the density argument of Marcinkiewicz and Zygmund [37] imply the well known theorem of Fejér [14] and Lebesgue [35] with $\alpha = 1$. Riesz [44] proved it for other α's and Fine [15], Schipp [46] and Weisz [76] for the Walsh system.

Corollary 2. *If $0 < \alpha \leq 1$ and $f \in L_1(\mathbb{X})$ then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \quad a.e.$$

With the help of the conjugate functions we ([77]) proved also

Theorem 8. *If $0 < \alpha \leq 1$ and $1/(\alpha+1) < p \leq \infty$ then*

$$\|\sigma_n^\alpha f\|_{H_p} \leq C_p \|f\|_{H_p} \quad (f \in H_p(\mathbb{X})).$$

Corollary 3. *If $0 < \alpha \leq 1$, $1/(\alpha+1) < p < \infty$ and $f \in H_p(\mathbb{X})$ then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \quad \text{in the } H_p\text{-norm.}$$

6. Partial sums of multi-dimensional Fourier series

Let us fix $d \geq 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself d-times. The $L_p(\mathbb{X}^d)$ spaces are defined in the usual way. The *d-dimensional trigonometric* and *Walsh system* is introduced as a Kronecker product by

$$\phi_k(x) := \phi_{k_1}(x_1) \cdots \phi_{k_d}(x_d),$$

where $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{X}^d$. The multi-dimensional *Fourier coefficients* of an integrable function f are defined by

$$\widehat{f}(k) := \int_{\mathbb{X}^d} f\phi_k \, d\lambda \quad (k \in \mathbb{N}^d).$$

The definition of the Fourier coefficients can be again extended to distributions resp. to martingales (see Weisz [77, 82]).

For $f \in L_1(\mathbb{X}^d)$ the nth *rectangular partial sum* $s_n f$ of the Fourier series of f is introduced by

$$s_n f(x) := \sum_{|k_1|\leq n_1} \cdots \sum_{|k_d|\leq n_d} \widehat{f}(k)\phi_k(x) = \int_{\mathbb{X}^d} f(x-u) D_n(u) \, du \quad (n \in \mathbb{N}^d),$$

where

$$D_n(u) := \prod_{j=1}^d D_{n_j}(u_j)$$

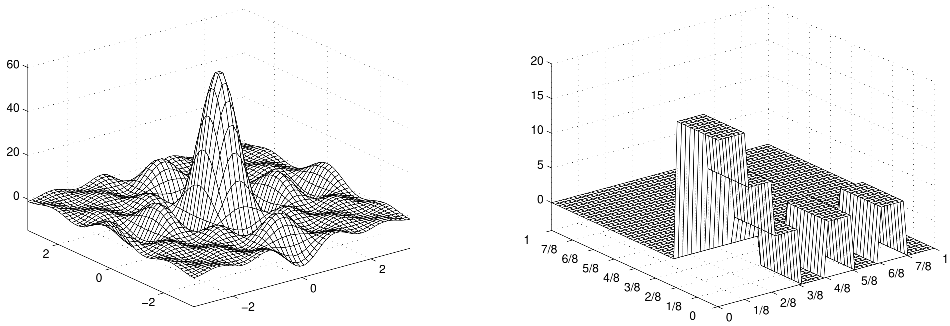
is the nth *multi-dimensional trigonometric* or *Walsh-Dirichlet kernel* (see Figure 4). By iterating the one-dimensional result, we get easily the next theorem.

Theorem 9. *If $f \in L_p(\mathbb{X}^d)$ for some $1 < p < \infty$, then*

$$\|s_n f\|_p \le C_p \|f\|_p \quad (n \in \mathbb{N}^d)$$

and

$$\lim_{n\to\infty} s_n f = f \quad \textit{in the } L_p\textit{-norm.}$$



(a) The trigonometric Dirichlet kernel

(b) The Walsh-Dirichlet kernel

FIGURE 4. The Dirichlet kernels D_n with $n_1 = 5$, $n_2 = 4$.

Other types of partial sums are considered e.g. in Weisz [77, 82]. The analogue of the Carleson's theorem is not true, i.e., $s_n f$ is not convergent (Fefferman [11, 12]). However, investigating the partial sums over the diagonal, only, Carleson's theorem holds also for higher dimensions and for the trigonometric system (see Fefferman [11] and Grafakos [28]), and it holds for the Walsh system if $p = 2$ (see Móricz [38] or Schipp, Wade, Simon and Pál [50]).

Theorem 10. *If $f \in L_p(\mathbb{X}^d)$ for some $1 < p < \infty$, then for the trigonometric Fourier series*

$$\left\| \sup_{n\in\mathbb{N}} |s_{n,\dots,n} f| \right\|_p \le C_p \|f\|_p$$

and

$$\lim_{n\to\infty} s_{n,\dots,n} f = f \quad a.e.$$

The same result holds for the Walsh-Fourier series if $p = 2$.

It is an open question, whether this theorem holds for the Walsh system and for $p \neq 2$ (cf. Schipp, Wade, Simon and Pál [50]).

7. Multi-dimensional Hardy spaces

In this section we introduce three types of multi-dimensional classical Hardy spaces for the trigonometric system and three types of multi-dimensional dyadic Hardy spaces for the Walsh system.

7.1. **Multi-dimensional classical Hardy spaces.** A distribution f is in the *classical Hardy space* $H_p^\square(\mathbb{T}^d)$, in the *product Hardy space* $H_p(\mathbb{T}^d)$ and in the *hybrid Hardy space* $H_p^i(\mathbb{T}^d)$ $(0 < p \le \infty)$ if

$$\|f\|_{H_p^\square} := \left\| \sup_{t>0} |f * (P_t \otimes \cdots \otimes P_t)| \right\|_p < \infty,$$

$$\|f\|_{H_p} := \left\| \sup_{t_k>0, k=1,\dots,d} |f * (P_{t_1} \otimes \cdots \otimes P_{t_d})| \right\|_p < \infty$$

and

$$\|f\|_{H_p^i} := \left\| \sup_{t_k>0, k=1,\dots,d; k\neq i} \left| f * (P_{t_1} \otimes \cdots \otimes P_{t_{i-1}} \otimes P_{t_{i+1}} \otimes \cdots \otimes P_{t_d}) \right| \right\|_p < \infty,$$

respectively, where P_t the one-dimensional Poisson kernel and $i = 1, \dots, d$.

7.2. **Multi-dimensional dyadic Hardy spaces.** By a *dyadic rectangle* we mean a Cartesian product of d dyadic intervals. For $n \in \mathbb{N}^d$ and $x = (x_1, \dots, x_d) \in [0,1)^d$ let $I_n(x) := I_{n_1}(x_1) \times \cdots \times I_{n_d}(x_d)$ be a dyadic rectangle. The σ-algebra generated by the dyadic rectangles $\{I_n(x) : x \in [0,1)^d\}$ will be denoted again by $\mathcal{F}_n$ $(n \in \mathbb{N}^d)$.

For $0 < p \le \infty$ the *martingale Hardy space* $H_p^\square[0,1)^d$, the *product Hardy space* $H_p[0,1)^d$ and the *hybrid Hardy space* $H_p^i[0,1)^d$ consist of all d-parameter dyadic martingales $f = (f_n, n \in \mathbb{N}^d)$ with respect to $(\mathcal{F}_n, n \in \mathbb{N}^d)$, for which

$$\|f\|_{H_p^\square} := \left\| \sup_{n\in\mathbb{N}} |f_{n,\dots,n}| \right\|_p < \infty,$$

$$\|f\|_{H_p} := \left\| \sup_{n\in\mathbb{N}^d} |f_{n_1,\dots,n_d}| \right\|_p < \infty,$$

and

$$\|f\|_{H_p^i} := \left\| \sup_{n_k\in\mathbb{N}, k\neq i} \left| E_{n_1} \cdots E_{n_{i-1}} E_{n_{i+1}} \cdots E_{n_d} f \right| \right\|_p < \infty,$$

respectively, where E_{n_i} denotes the conditional expectation operator relative to $\mathcal{F}_{n_i}$ $(i = 1, \dots, d)$. We can show again that for a martingale $f = (f_n, n \in \mathbb{N}^d)$ we have $s_{2^{n_1},\dots,2^{n_d}} f = f_n$.

7.3. **Atomic decomposition of $H_p^\square(\mathbb{X}^d)$.** It is known again (see e.g. Stein [59] or Weisz [77, 82]) that

$$H_p^\square(\mathbb{X}^d) \sim H_p(\mathbb{X}^d) \sim H_p^i(\mathbb{X}^d) \sim L_p(\mathbb{X}^d) \quad (1 < p \le \infty).$$

and

$$H_1^i(\mathbb{X}^d) \supset L(\log L)^{d-1}(\mathbb{X}^d) \quad (i = 1, \dots, d),$$

i.e.,

$$\|f\|_{H_1^i} \le C + C \left\| |f| (\log^+ |f|)^{d-1} \right\|_1 \quad (f \in L(\log L)^{d-1}(\mathbb{X}^d))$$

where $\log^+ u = 1_{\{u>1\}} \log u$.

To obtain some convergence results of the summability means over the diagonal or over a cone we consider the Hardy space $H_p^\square(\mathbb{X}^d)$. Now the situation is similar to the one-dimensional case. A function $a \in L_\infty(\mathbb{T}^d)$ is a *multi-dimensional classical p-atom* if there exists a cube $I \subset \mathbb{T}^d$ such that

(1) $\operatorname{supp} a \subset I$,
(2) $\|a\|_\infty \leq |I|^{-1/p}$,
(3) $\int_I a(x)x^k \, dx = 0$ for all multi-indices $k = (k_1, \ldots, k_d)$ for which $\|k\|_2 \leq \lfloor d(1/p - 1) \rfloor$.

A function $a \in L_\infty[0,1)^d$ is called a *multi-dimensional dyadic p-atom* if there exists a dyadic cube $I \subset [0,1)^d$ such that (i), (ii) and (iii) with $k = 0$ hold.

The atomic decomposition holds for the multi-dimensional Hardy spaces, too (see Latter [34], Lu [36], Wilson [83, 84], Stein [59] and Weisz [66, 77]).

Theorem 11. *A distribution (resp. martingale) f is in $H_p^\square(\mathbb{X}^d)$ $(0 < p \leq 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of multi-dimensional classical (resp. dyadic) p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^\infty |\mu_k|^p < \infty \quad \textit{and} \quad \sum_{k=0}^\infty \mu_k a^k = f$$

in the sense of distributions (resp. martingales). Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^\infty |\mu_k|^p \right)^{1/p}.$$

For a cube

$$I = I_1 \times \cdots \times I_d \subset \mathbb{X}^d \quad \text{let} \quad I^r = I_1^r \times \cdots \times I_d^r.$$

For the proof of the next theorem see Weisz [77, 81].

Theorem 12. *For each $n \in \mathbb{N}^d$, let $V_n \colon L_1(\mathbb{X}^d) \to L_1(\mathbb{X}^d)$ be a bounded linear operator and let*

$$V_* f := \sup_{n \in \mathbb{N}^d} |V_n f|.$$

Suppose that

$$\int_{\mathbb{X}^d \setminus I^r} |V_* a|^{p_0} \, d\lambda \leq C_{p_0}$$

for all classical (resp. dyadic) p_0-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \leq 1$, where the cube I is the support of the atom. If V_* is bounded from $L_{p_1}(\mathbb{X}^d)$ to $L_{p_1}(\mathbb{X}^d)$ for some $1 < p_1 \leq \infty$, then

$$\|V_* f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p(\mathbb{X}^d))$$

for all $p_0 \leq p \leq p_1$. Moreover, if $p_0 < 1$, then

$$\sup_{\rho > 0} \rho \, \lambda(|V_* f| > \rho) \leq C \|f\|_1 \quad (f \in L_1(\mathbb{X}^d)).$$

7.4. **Atomic decomposition of $H_p(\mathbb{X}^d)$.** In the investigation of the convergence in the Prighheim's sense (i.e., over all n) we use the Hardy spaces $H_p(\mathbb{X}^d)$. The atomic decomposition for $H_p(\mathbb{X}^d)$ is much more complicated. One reason of this is that the support of an atom is not a rectangle but an open set. Moreover, here we have to choose the atoms from $L_2(\mathbb{X}^d)$ instead of $L_\infty(\mathbb{X}^d)$. This atomic decomposition was proved by Chang and Fefferman [9, 13] and Weisz [72, 77]. For an open set $F \subset (\mathbb{X}^d)$ denote by $\mathcal{M}(F)$ the maximal dyadic subrectangles of F.

A function $a \in L_2(\mathbb{X}^d)$ is a *classical H_p-atom* if

(a) supp $a \subset F$ for some open set $F \subset \mathbb{X}^d$,
(b) $\|a\|_2 \le |F|^{1/2-1/p}$,
(c) a can be further decomposed into the sum of "elementary particles" $a_R \in L_2(\mathbb{X}^d)$, $a = \sum_{R\in\mathcal{M}(F)} a_R$ in $L_2(\mathbb{X}^d)$, satisfying
  (a) supp $a_R \subset R \subset F$,
  (b) for $i = 1,\dots,d$, $k \le \lfloor 2/p - 3/2 \rfloor$ and $R \in \mathcal{M}(F)$, we have
$$\int_{\mathbb{X}} a_R(x) x_i^k \, dx_i = 0,$$
  (c) for every disjoint partition $\mathcal{P}_l$ $(l = 1, 2, \dots)$ of $\mathcal{M}(F)$,
$$\left(\sum_l \left\| \sum_{R\in\mathcal{P}_l} a_R \right\|_2^2 \right)^{1/2} \le |F|^{1/2-1/p}.$$

We get the definition of *dyadic H_p-atoms* if $k = 0$ in (b).

The analogue of Theorem 11 holds in this case, too, however, the proof is much more complicated (see Chang and Fefferman [9, 13] and Weisz [72, 77]).

Theorem 13. *A distribution (resp. martingale) f is in $H_p(\mathbb{X}^d)$ $(0 < p \le 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of classical (resp. dyadic) H_p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*
$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} \mu_k a^k = f$$
in the sense of distributions (resp. martingales). Moreover,
$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.$$

The corresponding result to Theorem 12 for the $H_p(\mathbb{X}^d)$ space are much more complicated again. Since the definition of the H_p-atom is very complex, to obtain a usable condition about the boundedness of the operators, we have to introduce simpler atoms.

First suppose that $d = 2$. A function a is a *simple H_p-atom* if

(1) supp $a \subset R$ for some rectangle $R \subset \mathbb{T}^2$,

(2) $\|a\|_2 \le |R|^{1/2-1/p}$,

(3) $\int_{\mathbb{T}} a(x) x_i^k \, dx_i = 0$ for all $i = 1, 2$ and $k \le \lfloor 2/p - 3/2 \rfloor$.

A function a is called a *simple dyadic H_p-atom* if there exists a dyadic rectangle $R \subset [0,1)^2$ such that (i), (ii) and (iii) with $k = 0$ hold.

Note that there are not enough simple H_p-atoms, more exactly, $H_p(\mathbb{X}^2)$ cannot be decomposed into simple H_p-atoms, a counterexample can be found in Weisz [66]. However, the following result says that for an operator V to be bounded from $H_p(\mathbb{X}^d)$ to $L_p(\mathbb{X}^d)$ ($0 < p \le 1$) it is enough to check V_* on simple H_p-atoms and the boundedness of V_* on $L_2(\mathbb{X}^d)$.

Theorem 14. *For each $n \in \mathbb{N}^2$, let $V_n \colon L_1(\mathbb{X}^2) \to L_1(\mathbb{X}^2)$ be a bounded linear operator and*

$$V_* f := \sup_{n \in \mathbb{N}^2} |V_n f|.$$

Let $d = 2$ and $0 < p_0 \le 1$. Suppose that there exists $\eta > 0$ such that for every simple H_{p_0}-atom a and for every $r \ge 1$

$$\int_{\mathbb{X}^2 \setminus R^r} |V_* a|^{p_0} \, d\lambda \le C_{p_0} 2^{-\eta r},$$

where R is the support of a. If V_* is bounded from $L_2(\mathbb{X}^2)$ to $L_2(\mathbb{X}^2)$, then

$$\|V_* f\|_p \le C_p \|f\|_{H_p} \quad (f \in H_p(\mathbb{X}^2))$$

for all $p_0 \le p \le 2$. Moreover, if $p_0 < 1$, then

$$\sup_{\rho > 0} \rho \, \lambda(|V_* f| > \rho) \le C \|f\|_{H_1^i} \quad (f \in H_1^i(\mathbb{X}^2), i = 1, 2).$$

Theorem 14 for two-dimensional classical Hardy spaces is due to Fefferman [13] and for martingale Hardy spaces to Weisz [70]. Journé [31] verified that the preceding result do not hold for dimensions greater than 2. So there are fundamental differences between the theory in the two-parameter and three- or more-parameter cases. Now we present the analogous theorem for higher dimensions.

If $d \ge 3$, a function $a \in L_2(\mathbb{T}^d)$ is called a *simple H_p-atom* if there exist intervals $I_i \subset \mathbb{T}$, $i = 1, \ldots, j$ for some $1 \le j \le d-1$, such that

(1) $\operatorname{supp} a \subset I_1 \times \cdots \times I_j \times A$ for some measurable set $A \subset \mathbb{T}^{d-j}$,

(2) $\|a\|_2 \le (|I_1| \cdots |I_j| |A|)^{1/2-1/p}$,

(3) $\int_{\mathbb{T}} a(x) x_i^k \, dx_i = \int_A a \, d\lambda = 0$ for all $i = 1, \ldots, j$ and $k \le \lfloor 2/p - 3/2 \rfloor$.

If $j = d-1$, we may suppose that $A = I_d$ is also an interval. Of course if $a \in L_2(\mathbb{T}^d)$ satisfies these conditions for another subset of $\{1, \ldots, d\}$ than $\{1, \ldots, j\}$, then it is also called a simple H_p-atom. If the intervals are dyadic and $k = 0$, then we get the definition of *simple dyadic H_p-atoms.*

Note that $H_p(\mathbb{X}^d)$ cannot be decomposed into simple p-atoms. The following result is due to the author [72, 77]. Let H^c denote the complement of the set H.

Theorem 15. *For each $n \in \mathbb{N}^d$, let $V_n \colon L_1(\mathbb{X}^d) \to L_1(\mathbb{X}^d)$ be a bounded linear operator and*

$$V_* f := \sup_{n \in \mathbb{N}^d} |V_n f|.$$

Let $d \geq 3$ and $0 < p_0 \leq 1$. Suppose that there exist $\eta_1, \ldots, \eta_d > 0$ such that for every simple H_{p_0}-atom a and for every $r_1 \ldots, r_d \geq 1$

$$\int_{(I_1^{r_1})^c \times \cdots \times (I_j^{r_j})^c} \int_A |V_* a|^{p_0} \, d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \cdots 2^{-\eta_j r_j},$$

where $I_1 \times \cdots \times I_j \times A$ is the support of a. If $j = d-1$ and $A = I_d$ is an interval, then we also assume that

$$\int_{(I_1^{r_1})^c \times \cdots \times (I_{d-1}^{r_{d-1}})^c} \int_{(I_d)^c} |V_* a|^{p_0} \, d\lambda \leq C_{p_0} 2^{-\eta_1 r_1} \cdots 2^{-\eta_{d-1} r_{d-1}}.$$

If V_* is bounded from $L_2(\mathbb{X}^d)$ to $L_2(\mathbb{X}^d)$, then

$$\|V_* f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p(\mathbb{X}^d))$$

for all $p_0 \leq p \leq 2$. Moreover, if $p_0 < 1$, then

$$\sup_{\rho > 0} \rho \, \lambda(|V_* f| > \rho) \leq C \|f\|_{H_1^i} \quad (f \in H_1^i(\mathbb{X}^d), i = 1, \ldots, d).$$

In some sense the space $H_1^i(\mathbb{X}^d)$ plays the role of the one-dimensional $L_1(\mathbb{X})$ space.

8. Summability of multi-dimensional Fourier series

The multi-dimensional *Fejér* and *Cesàro means* of a distribution resp. martingale f are defined by

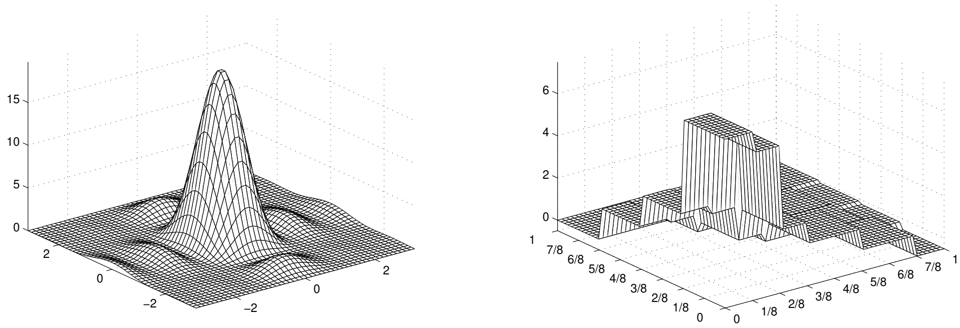
$$\sigma_n f := \frac{1}{\prod_{i=1}^d n_i} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} s_k f = \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \prod_{i=1}^d \left(1 - \frac{|k_i|}{n_i}\right) \widehat{f}(k) \phi_k$$
$$= \int_{\mathbb{X}^d} f(x-u) K_n(u) \, du$$

and

$$\sigma_n^\alpha f := \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} A_{n_j-1-k_j}^{\alpha_j-1} s_k f$$
$$= \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \left(\prod_{i=1}^d A_{n_i-1-|k_i|}^{\alpha_i} \right) \widehat{f}(k) \phi_k$$
$$= \int_{\mathbb{X}^d} f(x-u) K_n^\alpha(u) \, du,$$

where the *Fejér* and *Cesàro kernels* (see Figure 5) are given by

$$K_n(u) := \prod_{j=1}^{d} K_{n_j}(u_j) \quad \text{and} \quad K_n^{a}(u) := \prod_{j=1}^{d} K_{n_j}^{\alpha_j}(u_j).$$



(a) The trigonometric Fejér kernel (b) The Walsh-Fejér kernel

FIGURE 5. The Fejér kernels K_n with $n_1 = 5$ and $n_2 = 4$.

Theorem 16. *If* $0 < \alpha_j \le 1$ $(j = 1, \dots, d)$ *and* $1 \le p \le \infty$, *then*

$$\|\sigma_n^{\alpha} f\|_p \le C_p \|f\|_p \quad (f \in L_p(\mathbb{X}^d), n \in \mathbb{N}^d).$$

Moreover, for all $f \in L_p(\mathbb{X}^d)$ $(1 \le p < \infty)$,

$$\lim_{n\to\infty} \sigma_n^{\alpha} f = f \quad \textit{in the } L_p\textit{-norm.}$$

This theorem can be found e.g. in Zygmund [85] and Weisz [77]. Here the convergence is understood in Pringsheim's sense, i.e., $n \to \infty$ means that $\min(n_1, \dots, n_d) \to \infty$.

9. Restricted convergence of summability means

For a given $\tau \ge 1$ we define a cone (see Figure 6) by

$$\mathbb{N}_\tau^d := \{n \in \mathbb{N}^d : \tau^{-1} \le n_i/n_j \le \tau, i, j = 1, \dots, d\}.$$

In this section we investigate the convergence of the summability means over this cone, the multi-dimensional Hardy space $H_p^{\square}(\mathbb{X}^d)$ and the *restricted maximal operator* defined by

$$\sigma_{\square}^{\alpha} f := \sup_{n \in \mathbb{N}_\tau^d} |\sigma_n^{\alpha} f|.$$

For the Walsh system let

$$p_0 := \max\left\{\frac{1}{\alpha_1 + 1}, \dots, \frac{1}{\alpha_d + 1}\right\}$$

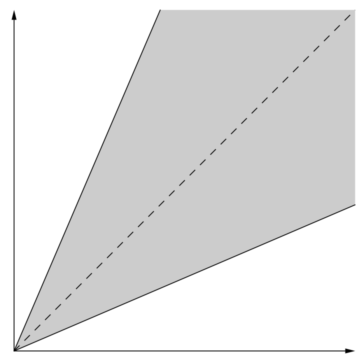


FIGURE 6. The cone for $d = 2$.

and for the trigonometric system

$$p_0 := \max\left\{\frac{d}{d+1}, \frac{1}{\alpha_1+1}, \dots, \frac{1}{\alpha_d+1}\right\}.$$

Theorem 17. *If $0 < \alpha_j \leq 1$ $(j = 1, \dots, d)$ and $p_0 < p < \infty$, then*

$$\|\sigma_\square^\alpha f\|_p \leq C_p \|f\|_{H_p^\square} \quad (f \in H_p^\square(\mathbb{X}^d)).$$

This theorem is due to the author [71, 75, 77, 78, 82] (for Walsh-Kaczmarz system see Simon [55]). For the Fejér means (i.e., $\alpha_j = 1$, $j = 1, \dots, d$) there are counterexamples for the boundedness of $\sigma_\square^\alpha$ if $p \leq p_0 = 1/2$ (Goginava and Nagy [25, 27]).

Theorem 18. *For the Walsh system the operator $\sigma_\square^1$ $(\alpha_j = 1, j = 1, \dots, d)$ is not bounded from $H_p^\square(\mathbb{X}^d)$ to $L_p(\mathbb{X}^d)$ if $0 < p \leq 1/2$.*

By interpolation we obtain ([71, 75])

Corollary 4. *If $0 < \alpha_j \leq 1$ $(j = 1, \dots, d)$ and $f \in L_1(\mathbb{X}^d)$ then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_\square^\alpha f > \rho) \leq C \|f\|_1.$$

The usual density argument and Corollary 4 imply the generalization of the Marcinkiewicz-Zygmund result.

Corollary 5. *If $0 < \alpha_j \leq 1$ $(j = 1, \dots, d)$ and $f \in L_1(\mathbb{X}^d)$, then*

$$\lim_{n \to \infty, n \in \mathbb{N}_\tau^d} \sigma_n^\alpha f = f \quad a.e.$$

Note that this corollary is due to the author [71, 68, 75, 78]. For Fejér means of two-dimensional Walsh-Fourier series it can also be found in Gát [19] (see also Móricz, Schipp and Wade [40], Simon [56] for Walsh-Kaczmarz system and Blahota and Gát [5] for a general orthonormal system).

The following results are known ([71, 75]) for the norm convergence of $\sigma_n f$.

Theorem 19. *If $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$, $p_0 < p < \infty$ and $n \in \mathbb{N}_\tau^d$, then*

$$\|\sigma_n^\alpha f\|_{H_p^\square} \leq C_p \|f\|_{H_p^\square} \quad (f \in H_p^\square(\mathbb{X}^d)).$$

Corollary 6. *If $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$, $p_0 < p < \infty$ and $f \in H_p^\square(\mathbb{X}^d)$, then*

$$\lim_{n\to\infty, n\in\mathbb{N}_\tau^d} \sigma_n^\alpha f = f \quad \textit{in the } H_p^\square\textit{-norm.}$$

10. Convergence of summability means over a cone-like set

Here we extend the results of the preceding section. First we introduce the cone-like sets, which are generalizations of the cones investigated before. Suppose that for all $j = 2, \ldots, d$, $\gamma_j \colon \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing continuous functions such that $\lim_{j\to\infty} \gamma_j = \infty$ and $\lim_{j\to+0} \gamma_j = 0$. Moreover, suppose that there exist $c_{j,1}, c_{j,2}, \xi > 1$ such that

$$c_{j,1}\gamma_j(x) \leq \gamma_j(\xi x) \leq c_{j,2}\gamma_j(x) \quad (x > 0).$$

For a fixed $\tau \geq 1$ we define the cone-like set (see Figure 7) by

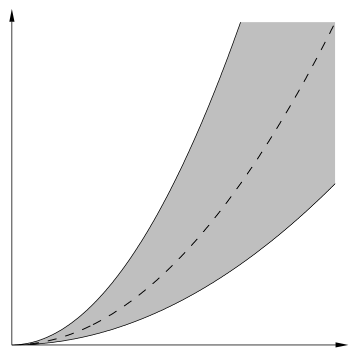


Figure 7. Cone-like set for $d = 2$.

$$\mathbb{N}_{\tau,\gamma}^d := \{n \in \mathbb{N}^d : \tau^{-1}\gamma_j(n_1) \leq n_j \leq \tau\gamma_j(n_1), j = 2, \ldots, d\}.$$

To investigate the convergence of the summability means over these cone-like sets, we have to introduce another maximal operator and other Hardy spaces. Now we introduce the *maximal operator*

$$\sigma_\gamma^\alpha f := \sup_{n\in\mathbb{N}_{\tau,\gamma}^d} |\sigma_n^\alpha f|.$$

The Hardy spaces $H_p^\gamma(\mathbb{T}^d)$ and $H_p^\gamma[0,1)^d$ are given with the norms

$$\|f\|_{H_p^\gamma} := \left\| \sup_{t>0} \left| f * (P_t \otimes P_{\gamma_2(t)} \otimes \cdots \otimes P_{\gamma_d(t)}) \right| \right\|_p$$

and

$$\|f\|_{H_p^\gamma} := \left\| \sup_{n_1 \in \mathbb{N}} |s_{2^{n_1},\dots,2^{n_d}} f| \right\|_p < \infty,$$

respectively, where

$$2^{n_j} \le \gamma_j(2^{n_1}) < 2^{n_j+1} \quad (j = 2, \dots, d).$$

For the Walsh system let $p_1 = 0$. For the trigonometric system we can define a number $p_1 < 1$ depending only on the functions γ_j (see Weisz [79]). The results of the preceding section can be generalized as follows (see Weisz [79, 80, 82], for two-dimensional Walsh-Kaczmarz-Fejér means Nagy [41]).

Theorem 20. *If $0 < \alpha_j \le 1$ $(j = 1, \dots, d)$ and $p_0 \vee p_1 < p < \infty$, then*

$$\left\|\sigma_\gamma^\alpha f\right\|_p \le C_p \|f\|_{H_p^\gamma} \quad (f \in H_p^\gamma(\mathbb{X}^d)).$$

Corollary 7. *If $0 < \alpha_j \le 1$ $(j = 1, \dots, d)$ and $f \in L_1(\mathbb{X}^d)$, then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_\gamma^\alpha f > \rho) \le C \|f\|_1.$$

Corollary 8. *If $0 < \alpha_j \le 1$ $(j = 1, \dots, d)$ and $f \in L_1(\mathbb{X}^d)$, then*

$$\lim_{n \to \infty, n \in \mathbb{N}^d_{\tau,\gamma}} \sigma_n^\alpha f = f \quad a.e.$$

In the two-dimensional case, Corollaries 7 and 8 were proved by Gát and Nagy [22, 24] for Fejér summability, for a general orthonormal system by Nagy [42]. For two-dimensional Fejér means the border point p_0 is essential [41].

11. Unrestricted convergence of summability means

In this section we deal with the Hardy spaces $H_p(\mathbb{X}^d)$ and the *non-restricted maximal operator* introduced by

$$\sigma_*^\alpha f := \sup_{n \in \mathbb{N}^d} |\sigma_n^\alpha f|.$$

Now we investigate the convergence of $\sigma_n^\alpha f$ in Pringsheim's sense, that is, $\min(n_1, \dots, n_d) \to \infty$. The next result is due to the author ([69, 74, 70, 73, 72]) (for Walsh-Kaczmarz systems see Simon [55]). Let

$$p_2 := \max\left\{\frac{1}{\alpha_1 + 1}, \dots, \frac{1}{\alpha_d + 1}\right\}.$$

Theorem 21. *If $0 < \alpha_j \le 1$ $(j = 1, \dots, d)$ and $p_2 < p < \infty$, then*

$$\|\sigma_*^\alpha f\|_p \le C_p \|f\|_{H_p} \quad (f \in H_p(\mathbb{X}^d)).$$

The following unboundedness result was proved by Goginava [25, 26].

Theorem 22. *For the Walsh system the operator σ_*^1 $(\alpha_j = 1, j = 1, \dots, d)$ is not bounded from $H_p(\mathbb{X}^d)$ to $L_p(\mathbb{X}^d)$ if $0 < p \le p_2$.*

By interpolation we get here almost everywhere convergence for functions from the spaces $H_1^i(\mathbb{X}^d)$ instead of $L_1(\mathbb{X}^d)$.

Corollary 9. *If* $0 < \alpha_j \leq 1$ *and* $f \in H_1^i(\mathbb{X}^d)$ $(i, j = 1, \dots, d)$ *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \leq C\|f\|_{H_1^i}.$$

Recall that $H_1^i(\mathbb{X}^d) \supset L(\log L)^{d-1}(\mathbb{X}^d)$ for all $i = 1, \dots, d$.

Corollary 10. *If* $0 < \alpha_j \leq 1$ *and* $f \in H_1^i(\mathbb{X}^d)$ $(i, j = 1, \dots, d)$ *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \quad a.e.$$

For the $L(\log L)[0, 1)^2$ space and Walsh system see also Móricz, Schipp and Wade [40]. Gát [20, 21] proved for the Fejér means that this corollary does not hold for all integrable functions.

Theorem 23. *The almost everywhere convergence is not true for all* $f \in L_1(\mathbb{X}^d)$.

Theorem 24. *If* $0 < \alpha_j \leq 1$ $(j = 1, \dots, d)$ *and* $p_2 < p < \infty$, *then*

$$\|\sigma_n^\alpha f\|_{H_p} \leq C_p\|f\|_{H_p} \quad (f \in H_p(\mathbb{X}^d), n \in \mathbb{N}^d).$$

Corollary 11. *If* $0 < \alpha_j \leq 1$ $(j = 1, \dots, d)$, $p_2 < p < \infty$ *and* $f \in H_p(\mathbb{X}^d)$ *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \quad \text{in the } H_p\text{-norm.}$$

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