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# THE INDEX OF COMPOSITION OF THE ITERATES OF THE EULER FUNCTION

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*Dedicated to Professor Ferenc Schipp on the occasion of his 75th birthday, to Professor William Wade on the occasion of his 70th birthday and to Professor Péter Simon on the occasion of his 65th birthday.*

Abstract. The index of composition of an integer $n \geq 2$ is defined as $\lambda(n) = (\log n)/(\log \gamma(n))$, where $\gamma(n)$ stands for the largest square-free divisor of $n$. Let $\varphi$ stand for the Euler totient function. We show that the index of composition of the $k$-fold iterate of $\varphi(n)$ is 1 on a set of density 1 and that an analogous result holds if $n$ runs over the set of shifted primes.

## 1. Introduction and notation

The index of composition of an integer $n \geq 2$ is defined as $\lambda(n) = (\log n)/(\log \gamma(n))$, where $\gamma(n)$ stands the largest square-free divisor of $n$. For convenience, we set $\lambda(1) = \gamma(1) = 1$. The index of composition was introduced by Browkin in 2000. Later, De Koninck and Doyon [3] obtained various results concerning its global and local behaviour. In particular, they proved that the average value of $\lambda(n)$ is 1. This function was also the subject of various papers, namely De Koninck and Kátai [4], De Koninck, Kátai and Subbarao [5], Zhai [8], Zhang, Lü and Zhai [9], Zhang and W. Zhai [9] as well as Robert and Tenenbaum [7]. Recently, De Koninck and Luca [6] proved that the average value of $\lambda(\varphi(n))$, where $\varphi$ is the Euler totient function, is also 1.

For each integer $k \geq 1$, let $\varphi_k = \varphi \circ \varphi_{k-1}$, with $\varphi_0(n) = n$ for all $n \in \mathbb{N}$, stand for the $k$-fold iterate of the Euler $\varphi$ function. Here, we show that the index of composition of the $k$-fold iterate of $\varphi(n)$ is 1 on a set of density 1 and that an analogous result holds if $n$ runs over the set of shifted primes.

Let $\omega(n)$ stand for the number of distinct prime divisors of the integer $n \geq 2$, setting $\omega(1) = 0$. Bassily, Kátai and Wijsmuller [1] obtained the distribution

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of the function $\omega(\varphi_k(n))$ which counts the number of distinct prime factors of the $k$-fold iterate of the Euler function.

We let $\pi(x)$ stand for the number primes not exceeding $x$ and $\Omega(n)$ stand for the number of prime divisors of $n$ counting their multiplicity, setting $\Omega(1)=0$. As usual, we let $\mathrm{li}(x) := \displaystyle\int_2^x \frac{dt}{\log t}$ and let $\pi(x;k,\ell)$ be the number of primes $p \le x$ such that $p \equiv \ell \pmod k$. We let $P(n)$ stand for the largest prime factor of $n \ge 2$ and set $P(1)=1$. For convenience, we shall write $\log_2 x$ for $\max(1, \log\log x)$, $\log_3 x$ for $\max(1, \log\log_2 x)$, and so on. From here on, the letter $c$, with or without subscript, stands for an absolute positive constant, but not necessarily the same at each occurrence, while the letters $p, q, Q, \pi$, with or without subscript, will always denote primes.

## 2. Preliminary results

Writing $\Phi(z) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_z^\infty e^{-w^2/2}\, dw$ $(z \in \mathbb{R})$ for the normal distribution function, and setting

$$a_k = \frac{1}{(k+1)!}, \quad b_k = \frac{1}{k!\sqrt{2k+1}} \quad (k = 1, 2, \ldots),$$

Bassily, Kátai and Wijsmuller [1] proved that

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \left| \frac{\omega(\varphi_k(n)) - a_k(\log_2 x)^{k+1}}{b_k(\log_2 x)^{k+1/2}} \right| < z \right\} = \Phi(z), \tag{1}$$

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \left| \frac{\omega(\varphi_k(p-1)) - (\log_2 x)^{k+1}}{b_k(\log_2 x)^{k+1/2}} \right| < z \right\} = \Phi(z). \tag{2}$$

Letting $\Delta(n) := \Omega(n) - \omega(n)$, Bassily, Kátai and Wijsmuller [2] proved that, given any positive integer $k$, as $x \to \infty$,

$$\Delta(\varphi_k(n)) = (1+o(1)) a_{k-1} (\log_2 x)^k \, (\log_4 x) \quad \text{for almost all } n \le x. \tag{3}$$

**Lemma 1.** *Given a positive integer $D$ and any fixed integer $\ell$, let*

$$s(x; D, \ell) := \sum_{\substack{p \le x \\ p \equiv \ell \pmod D}} \frac{1}{p}.$$

*Then, uniformly for $D \in [1, x]$, $x \ge 3$, if $\ell = 1$ or $-1$,*

$$s(x; D, \ell) \le \frac{c \log_2 x}{\varphi(D)}.$$

*Proof.* This is Lemma 2.5 in Bassily, Kátai and Wijsmuller [1]. $\square$

**Lemma 2.** *Given integers $k \ge 0$ and $D \ge 1$, let $U_k(x, D) := \#\{n \le x : D \mid \varphi_k(n)\}$. Then, for every integer $k \ge 0$, there exist constants $C(k,0), C(k,1)$,*

$C(k,2),\ldots$ *satisfying* $C(k,r)\le C(k,r+1)$ *for* $r=0,1,2,\ldots$, *such that*

$$U_k(x,D)\le C(k,\Omega(D))\frac{x(\log_2 x)^{k\Omega(D)}}{D}\qquad(1\le D\le x,\ x>e^e).$$

*Proof.* The proof of this result can be found in Bassily, Kátai and Wijsmuller [2]. □

## 3. Main results

**Theorem 1.** *Given any positive integer* $k$,

$$\lambda(\varphi_k(n))\to 1\qquad \text{as } n\to\infty \qquad \text{on a set of density 1.}$$

**Theorem 2.** *Let* $k$ *be a positive integer. Given an arbitrarily small number* $\varepsilon>0$,

$$\frac{1}{\pi(x)}\#\{p\le x : |\lambda(\varphi_k(p-1))-1|>\varepsilon\}\to 0\qquad \text{as } x\to\infty.$$

## 4. Proof of Theorem 1

One can write any integer $n\ge 2$ as $n=A(n)\cdot B(n)$, where $A(n)$ is the square-full part of $n$ and $B(n)$ its square-free part. In light of Lemma 2, we have

$$U_k^*(x,p^2):=\#\left\{\frac{x}{2}<n\le x : p^2\mid\varphi_k(n)\right\}\le C(k,2)\frac{x(\log_2 x)^{2k}}{p^2}.$$

As a consequence of this inequality, we have that

$$\begin{aligned}\sum_{p>(\log_2 x)^k} U_k^*(x,p^2)&\le 2C(k,2)x(\log_2 x)^{2k}\int_{(\log_2 x)^k}^{\infty}\frac{du}{u^2\log u}\\ &\le \frac{4C(k,2)x}{k\log_3 x}=o(x)\quad (x\to\infty).\end{aligned}$$

It follows from this that

$$P(A(\varphi_k(n)))\le(\log_2 x)^k\qquad\text{for almost all } n\le x. \tag{4}$$

Hence, assuming (4), we have, in light of (3),

$$\begin{aligned}\frac{A(\varphi_k(n))}{\gamma(A(\varphi_k(n)))}&=\prod_{p^\alpha\|\varphi_k(n)}p^{\alpha-1}\le((\log_2 x)^k)^{\Delta(\varphi_k(n))}\\ &\le\exp\{2ka_{k-1}(\log_2 x)^k\log_3 x\log_4 x\}\le\exp\{\varepsilon_x\cdot\log x\},\end{aligned}$$

say. It follows from this that, for almost all $n\le x$, we have

$$\begin{aligned}\gamma(\varphi_k(n))&=\gamma(A(\varphi_k(n)))\cdot\gamma(B(\varphi_k(n)))\\ &\ge\gamma(B(\varphi_k(n)))A(\varphi_k(n))\exp\{-\varepsilon_x\cdot\log x\}\\ &=B(\varphi_k(n))A(\varphi_k(n))\exp\{-\varepsilon_x\cdot\log x\}\\ &=\varphi_k(n)\exp\{-\varepsilon_x\cdot\log x\}.\end{aligned}$$

Hence, for all but $o(x)$ of integers $n \in [x/2, x]$, we have

$$\lambda(\varphi_k(n)) = \frac{\log \varphi_k(n)}{\log \gamma(\varphi_k(n))} \leq \frac{\log \varphi_k(n)}{\log \varphi_k(n) - \varepsilon_x \cdot \log x}$$

$$\leq 1 + \frac{\varepsilon_x \log x}{\log \varphi_k(n)}. \tag{5}$$

On the other hand,

$$\frac{\varphi_k(n)}{n} = \prod_{j=0}^{k-1} \frac{\varphi_{j+1}(n)}{\varphi_j(n)} \geq \left( \prod_{p \leq x} \left(1 - \frac{1}{p}\right)\right)^k \geq c(\log x)^{-k},$$

thereby implying that

$$\log \varphi_k(n) \geq c \log n - ck \log_2 x \geq c \log(x/2) - ck \log_2 x \quad (n \in [x/2, x]),$$

which substituted in (5) yields

$$\lambda(\varphi_k(n)) \leq 1 + \frac{\varepsilon_x \log x}{c(\log(x/2) - k \log_2 x)} \leq 1 + o(1),$$

thus completing the proof of Theorem 1.

## 5. Proof of Theorem 2

As in the paper of Bassily, Kátai and Wijsmuller [1], we say that a $k+1$-tuple of primes $(q_0, q_1, \ldots, q_k)$ is a *$k$-chain* if $q_{i-1} \mid q_i - 1$ for $i = 1, 2, \ldots, k$.

Before we start the proof of Theorem 2, we shall prove the following lemma.

**Lemma 3.** *If $p_0 \mid \varphi_k(n)$, then there exists an $\ell$-chain $(p_0, p_1, \ldots, p_\ell)$ with $\ell \leq k$ and $p_\ell \mid n$.*

*Proof.* We use an induction argument. First of all, in the case $k = 1$, if $p_0 \mid \varphi(n)$, then either $p_0 \mid n$, in which case we are done, or $p_0 \mid p_1 - 1$ with $p_1 \mid n$, in which case the result holds also. So let us assume that the result is true up to $k - 1$. If $p_0 \mid \varphi_k(n)$, then either $p_0 \mid \varphi_{k-1}(n)$, in which case we are done, or $p_1 \mid \varphi_{k-1}(n)$ with $p_1 \equiv 1 \pmod{p_0}$. By applying the induction argument, the result is then also true for $k$. □

We are now ready to prove Theorem 2.

*Proof of Theorem 2.* Let $\delta > 0$ be small number and let $x$ be a fixed large number.

We first drop those primes $p \leq x$ for which any of the following three conditions holds:

(i) $|\omega(\varphi_j(p-1)) - a_j(\log_2 x)^{j+1}| > \delta(\log_2 x)^{j+1}$ for at least one $j \in \{1, \ldots, k\}$;

(ii) $\displaystyle\sum_{\substack{q \mid p-1 \\ x^\delta < q < x^{1/3}}} 1 \leq 2$;

(iii) $P(p-1) > x^{1-\delta}$.

Indeed, the number of primes $p \le x$ thereby dropped is at most $c\delta x/\log x$. To see this, observe that, in light of (2), the number of those primes $p \le x$ dropped by condition (i) does not exceed $c_1\delta\pi(x)$, while the number of those primes $p \le x$ dropped through condition (iii) clearly does not exceed $c_2\delta x/\log x$. Finally, to account for the number of primes $p \le x$ dropped through condition (ii), observe that, setting

$$\omega_1(p-1) := \sum_{\substack{q \mid p-1 \\ x^\delta < q < x^{1/3}}} 1,$$

we have

$$\sum_{p \le x} \left( \omega_1(p-1) - \sum_{x^\delta < q < x^{1/3}} \frac{1}{q-1} \right)^2 \le c_3 \mathrm{li}(x) \sum_{x^\delta < q < x^{1/3}} \frac{1}{q}$$

and therefore, since $\displaystyle\sum_{x^\delta < q < x^{1/3}} \frac{1}{q-1} = \log(1/3) + \log(1/\delta) + o(1)$ (as $x$ becomes large), we may conclude that the number of those primes $p \le x$ dropped through condition (ii) is at most $c_3\delta\ \mathrm{li}(x)$. It follows from these observations that, indeed, the number of primes $p \le x$ thereby dropped by conditions (i), (ii) and (iii) is at most $c\delta x/\log x$.

We shall now denote by $\overline{\wp}_x$ the set of those primes $p \le x$ not dropped by any of the above three conditions and further introduce the quantities

$$z = z(x) = \frac{\log x}{2^{k+1} \cdot (\log_2 x)^{5k+1}}, \quad T = T(x) = \lfloor (\log_2 x)^{5k+1} \rfloor. \tag{6}$$

Given a prime $Q \le z$, let us count the number of those primes $p \in \overline{\wp}_x$ for which $Q^{2^k T} \mid \varphi_k(p-1)$ and write $\varphi_{k-1}(p-1) = \pi_1^{\alpha_1} \cdots \pi_m^{\alpha_m}$. For each $k \in \mathbb{N}$, two separate cases can occur:

- Case $I(k)$: $Q^{2^{k-1}T} \mid \varphi_{k-1}(p-1)$;
- Case $II(k)$: there exists a prime $\pi_j$ for which $\pi_j - 1 \equiv 0 \pmod{Q^{\lfloor 2^{k-1}T/m \rfloor}}$.

We start with Case $II(k)$. Set $q_0 := \pi_j$. For any given such $q_0$, there exists a prime $p$ and a $\ell$-chain $(q_0, q_1, \ldots, q_\ell)$ with $\ell \le k$ for which $q_\ell \mid p-1$. We then have two possibilities: either $\ell = k$ or $\ell < k$. Assume first that $\ell = k$. In this case, let $S$ denote the scenario

$$q_{j+1} - 1 \equiv 0 \pmod{q_j} \text{ for } j = 1, \ldots, k-1 \quad \text{and } q_0 - 1 \equiv 0 \pmod{Q^{\lfloor 2^{k-1}T/m \rfloor}}.$$

Summing over all such possible scenarios $S$, we get

$$\sum_S \pi(x; q_{k-1}, 1) \le C(\delta)\, \mathrm{li}(x) \sum_S \frac{1}{q_{k-1}}. \tag{7}$$

Observe that

$$\sum_S \frac{1}{q_{k-1}} = \sum_{q_0} \frac{1}{q_0} \cdots \sum_{q_{k-1}} \frac{1}{q_{k-1}}$$

$$\begin{aligned}
&\leq c\log_2 x \sum_{q_0} \frac{1}{q_0} \dots \sum_{q_{k-2}} \frac{1}{q_{k-2}} \\
&\vdots \\
&\leq c(\log_2 x)^{k-1} \sum_{q_0-1\equiv 0 \pmod{Q^{\lfloor 2^{k-1}T/m\rfloor}}} \frac{1}{q_0}. \\
&\leq \frac{c(\log_2 x)^k}{Q^{\lfloor 2^{k-1}T/m\rfloor}}, \qquad (8)
\end{aligned}$$

where we used Lemma 1. It follows from the definition of $T$ given in (6) that

$$\lfloor 2^{k-1}T/m\rfloor \geq (1-2\delta)2^{k-1}(\log_2 x)^{3k} \geq 2^{k-2}(\log_2 x)^{4k},$$

say. Using this last estimate in (8), we obtain that

$$\sum_S \frac{1}{q_{k-1}} \leq \frac{c(\log_2 x)^k}{Q^{2^{k-2}(\log_2 x)^{4k}}}. \qquad (9)$$

In the case where $\ell < k$, one can easily show that the same bound (as in (9)) still holds. Hence, in both cases, by substituting (9) in (7), it follows that the number of primes $p \in \overline{\wp}_x$ satisfying Case II($k$) is no more than

$$\frac{c(\log_2 x)^k}{Q^{2^{k-2}(\log_2 x)^{4k}}} C(\delta)\,\mathrm{li}(x).$$

The Case $I(k)$ can be reduced to the Case $I(k-1)$, for which the estimate is similar. In Case $I(0)$, we have $Q^T \mid p-1$, which occurs less than $c\,\mathrm{li}(x)/Q^T$ times. Hence, from the above reasoning, it follows that

$$\#\{p \in \overline{\wp}_x : Q^{2^kT} \mid \varphi_k(p-1)\} \leq cC(\delta)\mathrm{li}(x)\frac{(\log_2 x)^k}{Q^{2^{k-2}(\log_2 x)^{4k}}}.$$

Thus, the number of primes $p \in \overline{\wp}_x$ for which there is at least one prime $Q \leq z$ for which $Q^{2^kT} \mid \varphi_k(p-1)$ is at most $o(\mathrm{li}(x))$ as $x \to \infty$.

Let us now introduce the function

$$E_k(p;z) := \prod_{\substack{q^{\alpha_q}\|\varphi_k(p-1)\\ q\leq z}} q^{\alpha_q}.$$

Observe that we have just established that for every $q \leq z$ with $q^{\alpha_q}\|\varphi_k(p-1)$, we can assume that $\alpha_q \leq 2^kT$ and that this holds for all $p \in \overline{\wp}_x$ with at most $o(\mathrm{li}(x))$ exceptions.

Hence, using (6), it follows that

$$E_k(p;z) \leq \left(\prod_{q\leq z} q\right)^{2^kT} \leq \exp\{2^{k+1}Tz\} \leq \exp\left\{\frac{\log x}{\log\log x}\right\}. \qquad (10)$$

Let $P_2(n) := \max_{Q^2|n} Q^2$ stand for the largest prime squared divisor of $n$, setting $P_2(n) = 1$ if $n$ is square-free. We then have

$$\#\{p \le x : Q^2 \mid \varphi_k(p-1)\} \le \#\{n \le x : Q^2 \mid \varphi_k(n)\} \le \frac{C(k,2)x \cdot (\log_2 x)^{2k}}{Q^2},$$

from which it follows that

$$\sum_{Q>Y} \#\{p \le x : Q^2 \mid \varphi_k(p-1)\} \le c\frac{x \cdot (\log_2 x)^{2k}}{Y \log Y},$$

which itself implies that

$$\frac{1}{\pi(x)}\#\{p \le x : P_2(\varphi_k(p-1)) > (\log x \cdot (\log_2 x)^{2k})^2\} \to 0 \text{ as } x \to \infty.$$

It follows from this estimate that we only need to consider those primes $p \le x$ such that $P_2(\varphi_k(p-1)) \le (\log x \cdot (\log_2 x)^{2k})^2$.

Recalling the definition of $z = z(x)$ provided in (6), we now introduce the interval

$$\mathcal{L} = \mathcal{L}(x) = [z(x), (\log x)(\log_2 x)^{2k}].$$

For each $Q \in \mathcal{L}$, we will now estimate

$$H(Q) := \#\{p \in \overline{\wp}_x : Q^2 \mid \varphi_k(p-1)\}.$$

One can easily see that if $Q^2 \mid \varphi_k(p-1)$, then one of the following situation occurs:

(a) $Q^3 \mid \varphi_{k-1}(p-1)$;
(b) $Q^2 \| \varphi_{k-1}(p-1)$ and $\pi \mid \varphi_{k-1}(p-1)$ with $\pi \equiv 1 \pmod Q$;
(c) there exist $\pi_1, K_1 \in \wp$ such that $Q \mid \pi_1 - 1$, $Q \mid K_1 - 1$, $\pi_1 \ne K_1$, $\pi_1 K_1 \mid \varphi_{k-1}(p-1)$.

In the worst case scenario, that is in case (c), we have the following situation:

$$(R): \quad \begin{array}{l} Q \to \pi_1 \to \pi_2 \to \cdots \to \pi_k \\ Q \to K_1 \to K_2 \to \cdots \to K_k \end{array} \quad \pi_k K_k \mid p-1.$$

(Here, $\pi_j \to \pi_{j+1}$ means that $\pi_{j+1} - 1 \equiv 0 \pmod{\pi_j}$.)

Now, using Lemma 1, the number of such primes $p \le x$ does not exceed

$$\begin{aligned} \sum_{(R)} \pi(x; \pi_k K_k, 1) &\le C(\delta)\mathrm{li}(x) \sum_{(R)} \frac{1}{\pi_k K_k} \\ &\le C(\delta)\mathrm{li}(x)(\log_2 x)^2 \sum_{(R)} \frac{1}{\pi_{k-1} K_{k-1}} \\ &\vdots \\ &\le C(\delta)\mathrm{li}(x)(\log_2 x)^{2(k-1)} \sum_{\substack{\pi_1 - 1 \equiv 0 \pmod Q \\ K_1 - 1 \equiv 0 \pmod Q}} \frac{1}{\pi_1 K_1} \end{aligned}$$

$$\ll C(\delta)\mathrm{li}(x)\frac{(\log_2 x)^{2k}}{Q^2}.$$

Since it is clear that

$$\sum_{Q\in\mathcal{L}}\frac{1}{Q^2}\le\frac{c}{z(x)\log z(x)}$$

and since the contribution of the other cases, namely cases (a) and (b), is no larger than the worst case, we obtain that

$$P_2(\varphi_k(p-1))\le z^2(x)\quad\text{ for all but } o(\pi(x)) \text{ of the primes } p\in\overline{\wp}_x.$$

Let us now set

$$V_k(p;z):=\prod_{\substack{q^{\alpha_q}\|\varphi_k(p-1)\\ q>z}} q^{\alpha_q}.$$

Since $V_k(p;z)=\prod\limits_{\substack{q|\varphi_k(p-1)\\ q>z}} q$ for all $p\in\overline{\wp}_x$, with the possible exception of $o(\pi(x))$ primes, it follows, using (10), that

$$\gamma(\varphi_k(p-1))\ge\frac{\varphi_k(p-1)}{E_k(p;z)}\ge\varphi_k(p-1)\cdot x^{-\varepsilon_x},$$

where $\varepsilon_x\to 0$ as $x\to\infty$, so that

$$\lambda(\varphi_k(p-1))=\frac{\log\varphi_k(p-1)}{\log\gamma(\varphi_k(p-1))}\le\frac{\log\varphi_k(p-1)}{\log\varphi_k(p-1)-\varepsilon_x\log x}=1+\frac{\varepsilon_x\log x}{\log\varphi_k(p-1)},$$

thereby implying

$$\#\{p\le x:|\lambda(\varphi_k(p-1))-1|\ge\varepsilon\}\le C(\delta)\pi(x)+o(\pi(x)),$$

thus completing the proof of Theorem 2. □

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