Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
33 (2017), 39–44
www.emis.de/journals
ISSN 1786-0091

ON THE SOLVABILITY OF NON-HOMOGENEOUS STURM-LIOUVILLE PROBLEM

ANTON I. POPOV

Abstract. Non-homogeneous Sturm-Liouville problems can arise when trying to solve non-homogeneous partial differential equations or when constructing the asymptotic series for partial differential equation solution. The present paper gives a condition of solvability for the non-homogeneous Sturm-Liouville problem in general case for formal power series.

1. Introduction

Sturm-Liouville theory is a powerful instrument of the spectral theory. It is well described in many books (see, e.g. [5, 9] and references therein). Numerous physical problems (both quantum and classical) reduce to the Sturm-Liouville problem. One meet this problem when dealing with quantum wells, quantum graphs, wave guides, etc. [6, 7, 8, 11]. We mention also asymptotical approach in waves theory. It is applied when one has a small parameter (coupling constant, perturbation parameter, etc.). Formally, an asymptotic approach reduces to construction of the asymptotic expansion in powers of this small parameter [1, 4, 10]. The series is constructed consequently, term by term. To find a term, it is necessary to solve the non-homogeneous Sturm-Liouville problem for formal power series with the right hand side depending on the previous terms. Correspondingly, the question appears about the solution existence for this problem. One observe this situation, e.g. in asymptotic expansions related with space-time ray method [3, 12]. The present paper gives necessary and sufficient condition of solvability for the non-homogeneous Sturm-Liouville problem in general case.

2010 *Mathematics Subject Classification.* 34B24, 34E05.

Key words and phrases. Sturm-Liouville problem, asymptotic expansion, power series.

This work was partially financially supported by the Government of the Russian Federation (grant 074-U01), MK-5161.2016.1 of the President of the Russian Federation, by grant 16-11-10330 of Russian Science Foundation.

2. The main theorem

Theorem. *Let us consider homogeneous Sturm-Liouville problem*

$$\begin{cases} \boldsymbol{L}y = (p(x)y')' - q(x)y = 0, \\ \ell_0 y = (\alpha_0 y + \alpha_1 y')\Big|_{x=x_0} = 0, \\ \ell_1 y = (\beta_0 y + \beta_1 y')\Big|_{x=x_1} = 0. \end{cases} \tag{1}$$

$$p(x) > 0, \quad Im\ q = 0. \tag{2}$$

$\alpha_j,\ \beta_j,\ j = 0, 1$ *are real.* $p(x),\ q(x)$ *are formal power series. Let there exist a solution* $y_0 \neq 0$ *in the form of a formal power series. Then the necessary and sufficient condition for the existence of the solution in the form of a formal power series of non-homogeneous Sturm-Liouville problem*

$$\begin{cases} \boldsymbol{L}y = -F, \\ \ell_0 y = A, \\ \ell_1 y = B. \end{cases} \tag{3}$$

is as follows:

$$p(x_1)\frac{B}{\beta_1}y_0\Big|_{x=x_1} - p(x_0)\frac{A}{\alpha_1}y_0\Big|_{x=x_0} = -\int_{x_0}^{x_1} F(x)y_0(x)dx. \tag{4}$$

Here $F(x),\ A,\ B$ *is formal power series.*

Proof. *Necessary condition.*

$$(p(x)y')' - q(x)y = -F. \tag{5}$$

We multiply (5) by y_0 and integrate from x_0 to x_1:

$$\begin{aligned} \int_{x_0}^{x_1}((py')' - qy)y_0 dx &= \int_{x_0}^{x_1}(py')'y_0 dx - \int_{x_0}^{x_1} qyy_0 dx \\ &= py'y_0\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} py'y_0 dx - \int_{x_0}^{x_1} qyy_0 dx \\ &= py'y_0\Big|_{x_0}^{x_1} - py_0'y\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} y(py_0')'dx - \int_{x_0}^{x_1} qyy_0 dx. \end{aligned} \tag{6}$$

Then, we substitute the boundary conditions into (6):

$$y_0'(x_0) = -\frac{\alpha}{\alpha_1}y_0(x_0), \tag{7}$$

$$y'(x_0) = \frac{A}{\alpha_1} - \frac{\alpha}{\alpha_1}y(x_0), \tag{8}$$

$$y_0'(x_1) = -\frac{\beta_0}{\beta_1} y_0(x_1), \tag{9}$$

$$y'(x_1) = \frac{B}{\beta_1} - \frac{\beta_0}{\beta_1} y(x_1). \tag{10}$$

Then

$$\begin{aligned}
\int_{x_0}^{x_1} ((py')' - qy) y_0 dx &= p(x_1)\frac{B}{\beta_1} y_0(x_1) - p(x_1)\frac{\beta_0}{\beta_1} y y_0(x_1) \\
&\quad -p(x_0)\frac{A}{\alpha_1} y_0(x_0) + p(x_0)\frac{\alpha_0}{\alpha_1} y y_0(x_0) + p(x_1)\frac{\beta_0}{\beta_1} y_0 y(x_1) \\
&\quad -p(x_0)\frac{\alpha_0}{\alpha_1} y_0 y(x_0) + \int_{x_0}^{x_1} y((py_0')' - qy_0) dx \\
&= p(x_1)\frac{B}{\beta_1} y_0\Big|_{x=x_1} - p(x_0)\frac{A}{\alpha_1} y_0\Big|_{x=x_0}.
\end{aligned} \tag{11}$$

From the other side,

$$\int_{x_0}^{x_1} ((py')' - qy) y_0 dx = -\int_{x_0}^{x_1} F y_0 dx. \tag{12}$$

Equations (11) and (12) lead to (4), so we get necessary condition.

Sufficient condition.

Let us assume that ψ is a solution of the Cauchy problem:

$$\begin{cases}
(p\psi')' - q\psi = -F, \\
\psi\Big|_{x=x_0} = A, \\
\psi'\Big|_{x=x_0} = B.
\end{cases} \tag{13}$$

The Cauchy problem always has a solution. Consequently, ψ exists. Let us consider $y = \psi - y_0$.

$$(py')' - qy = (p\psi')' - q\psi - (py_0')' + qy_0 = -F, \tag{14}$$

i.e. y satisfies the proper equation. Check the boundary conditions. We multiply the first equation in (13) by y_0 and integrate from x_0 to x_1:

$$\begin{aligned}
\int_{x_0}^{x_1} ((p\psi')' - q\psi) y_0 dx &= \int_{x_0}^{x_1} (p\psi')' y_0 dx - \int_{x_0}^{x_1} q\psi y_0 dx \\
&= p\psi' y_0\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} p\psi' y_0' dx - \int_{x_0}^{x_1} q\psi y_0 dx \\
&= p(\psi' y_0 - y_0'\psi)\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} \psi((py_0')' - qy_0) dx
\end{aligned}$$

$$= p((y' + y_0')y_0 - y_0'(y + y_0))\Big|_{x_0}^{x_1} = p(y'y_0 - y_0'y)\Big|_{x_0}^{x_1}, \tag{15}$$

so,

$$\int_{x_0}^{x_1} ((p\psi')' - q\psi)y_0 dx = p(x_1)y'(x_1)y_0(x_1) - p(x_1)y_0'(x_1)y(x_1) - \\ - p(x_0)y'(x_0)y_0(x_0) + p(x_0)y_0'(x_0)y(x_0). \tag{16}$$

We substitute (9)-(10) into (16) and come to the equation:

$$\int_{x_0}^{x_1} ((p\psi')' - q\psi)y_0 dx = p(x_1)y'(x_1)y_0(x_1) + p(x_1)\frac{\beta_0}{\beta_1}y_0(x_1)y(x_1) - \\ - p(x_0)y'(x_0)y_0(x_0) - p(x_0)\frac{\alpha_0}{\alpha_1}y_0(x_0)y(x_0). \tag{17}$$

On the other side,

$$\int_{x_0}^{x_1} ((py')' - qy)y_0 dx = -\int_{x_0}^{x_1} Fy_0 dx. \tag{18}$$

Let

$$p(x_1)\frac{B}{\beta_1}y_0\Big|_{x=x_1} - p(x_0)\frac{A}{\alpha_1}y_0\Big|_{x=x_0} = -\int_{x_0}^{x_1} F(x)y_0(x)dx. \tag{19}$$

Then, relations (17)-(19) gives us:

$$p(x_1)\frac{B}{\beta_1}y_0\Big|_{x=x_1} - p(x_0)\frac{A}{\alpha_1}y_0\Big|_{x=x_0} = p(x_1)y'(x_1)y_0(x_1) \\ + p(x_1)\frac{\beta_0}{\beta_1}y_0(x_1)y(x_1) - p(x_0)y'(x_0)y_0(x_0) - p(x_0)\frac{\alpha_0}{\alpha_1}y_0(x_0)y(x_0). \tag{20}$$

Condition (20) must be fulfilled for any x_0 and x_1. We fix x_0, and will change x_1. Since the ratio of (20) must always be performed, then parts of the equation, corresponding to x_0 and x_1 should be independent of each other:

$$-p(x_0)\frac{A}{\alpha_1}y_0(x_0) = -p(x_0)y'(x_0)y_0(x_0) - p(x_0)\frac{\alpha_0}{\alpha_1}y_0(x_0)y(x_0). \tag{21}$$

$$p(x_1)\frac{B}{\beta_1}y_0(x_1) = p(x_1)y'(x_1)y_0(x_1) + p(x_1)\frac{\beta_0}{\beta_1}y_0(x_1)y(x_1). \tag{22}$$

$y_0(x_1) \neq 0$. Proof by contradiction. If $y_0(x_1) = 0$ then from the boundary condition $\ell_1 y = (\beta_0 y + \beta_1 y')\Big|_{x=x_1} = 0$ we get: $y_0'\Big|_{x=x_1} = 0$. Consequently, y_0 is a solution of the Cauchy problem:

$$\begin{cases} (p(x)y_0')' - q(x)y_0 = 0, \\ y_0\Big|_{x=x_1} = 0, \\ y_0'\Big|_{x=x_1} = 0. \end{cases}$$

Therefore, $y_0 \equiv 0$, which contradicts to the hypothesis of the theorem. Taking into account that $p > 0$, $\beta_1 \neq 0$, we can divide the both sides of (22) by $p(x_1)y_0(x_1)$ and multiply by β_1. As a result, we obtain:

$$B = \beta_1 y'(x_1) + \beta_0 y(x_1), \tag{23}$$

i.e. we come to the necessary condition for $x = x_1$.

Let's go back to relation (21):

$$-p(x_0)\frac{A}{\alpha_1}y_0(x_0) = -p(x_0)y'(x_0)y_0(x_0) - p(x_0)\frac{\alpha_0}{\alpha_1}y_0(x_0)y(x_0). \tag{24}$$

Taking into account that $p > 0$, $y_0(x_0) \neq 0$ ((proved in a similar way as for $y_0(x_1)$), $\alpha_1 \neq 0$, we come to the proper condition at $x = x_0$:

$$\alpha_1 y'(x_0) + \alpha_0 y(x_0) = A. \tag{25}$$

This completes the proof. □

References

[1] V. M. Babich. Quasiphotons and the space-time ray method. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 342(Mat. Vopr. Teor. Rasprostr. Voln. 36):5–13, 257, 2007.

[2] V. M. Babich. Formal power series and their applications in the mathematical theory of diffraction. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 409(Matematicheskie Voprosy Teorii Rasprostraneniya Voln. 42):5–16, 240, 2012.

[3] V. M. Babich and A. I. Popov. Quasiphotons of waves on the surface of a heavy fluid. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 379(Matematicheskie Voprosy Teorii Rasprostraneniya Voln. 39):5–23, 179, 2010.

[4] V. G. Bagrov, V. V. Belov, and A. Y. Trifonov. Semiclassical trajectory-coherent approximation in quantum mechanics. I. High-order corrections to multidimensional time-dependent equations of Schrödinger type. *Ann. Physics*, 246(2):231–290, 1996.

[5] J. R. Brannan and W. E. Boyce. *Differential equations with boundary value problems. An introduction to modern methods and applications.* Hoboken, NJ: John Wiley & Sons, 2010.

[6] C. Cacciapuoti, A. Mantile, and A. Posilicano. Time dependent delta-prime interactions in dimension one. *Nanosystems: Phys. Chem. Math.*, 7(2):303–314, 2016.

[7] L. Jódar. Explicit solutions for nonhomogeneous Sturm-Liouville operator problems. *Publ. Mat.*, 33(1):47–57, 1989.

[8] L. Kong and Q. Kong. Second-order boundary value problems with nonhomogeneous boundary conditions. II. *J. Math. Anal. Appl.*, 330(2):1393–1411, 2007.

[9] B. M. Levitan and I. S. Sargsyan. *Operatory Shturma-Liuvillya i Diraka.* Moskva: Nauka, 1988.

[10] T. F. Pankratova. Tunneling in multidimensional wells. *Nanosystems: Phys. Chem. Math*, 6(1):113–121, 2015.

[11] Y. V. Pokornyĭ and V. L. Pryadiev. Some problems in the qualitative Sturm-Liouville theory on a spatial network. *Uspekhi Mat. Nauk*, 59(3(357)):115–150, 2004.

[12] A. I. Popov. Wave walls for waves on the surface of a heavy liquid. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 409(Matematicheskie Voprosy Teorii Rasprostraneniya Voln. 42):151–175, 243–244, 2012.

Received June 20, 2015.

ITMO University,
Kronverkskiy 49, 197101, St. Petersburg,
Russia
E-mail address: popov239@gmail.com