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ON SOME LOCAL PROPERTIES OF THE CONJUGATE FUNCTION AND THE MODULUS OF CONTINUITY OF k-TH ORDER

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Abstract. In the present paper we study a local smoothness of the conjugate functions of several variables in the space $\mathbb{C}(T^n)$. The direct estimates are obtained and exactness of these estimates are established by proper examples.

1. Introduction

Let $\mathbb{R}^n$ ($n = 1, 2, \ldots$; $\mathbb{R}^1 \equiv \mathbb{R}$) be the n-dimensional Euclidean space of points $\overline{x} = (x_1, \ldots, x_n)$ with real coordinates. Let B be an arbitrary non-empty subset of the set $M = \{1, \ldots, n\}$. Denote by $|B|$ the cardinality of B. Let x_B be such a point in $\mathbb{R}^n$ whose coordinates with indices in $M \backslash B$ are zero.

As usual $\mathbb{C}(T^n)$ ($\mathbb{C}(T^1) \equiv \mathbb{C}(T)$), where $T = [-\pi, \pi]$, denotes the space of all continuous functions $f\colon \mathbb{R}^n \to \mathbb{R}$ that are 2π-periodic in each variable, endowed with the norm

$$\|f\| = \max_{\overline{x} \in T^n} |f(\overline{x})|.$$

If $f \in L(T^n)$, then following Zhizhiashvili [10], we call the expression

$$\widetilde{f}_B(\overline{x}) = \Big(-\frac{1}{2\pi}\Big)^{|B|} \int_{T^{|B|}} f(\overline{x} + s_B) \prod_{i \in B} \cot \frac{s_i}{2} \, ds_B$$

the conjugate function of n variables with respect to those variables whose indices form the set B (with $\widetilde{f}_B \equiv \widetilde{f}$ for $n = 1$).

Suppose that $f \in \mathbb{C}(T^n)$, $1 \le i \le n$, and $h \in T$. Then for each $\overline{x} \in T^n$ let us consider the difference of k-th order

$$\Delta_i^k(h)\, f(\overline{x}) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x_1, \ldots, x_{i-1}, x_i + j\,h, x_{i+1}, \ldots, x_n)$$

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and define the partial modulus of continuity of k-th order of the function f with respect to the variable x_i by the equality

$$\omega_{k,i}\,(f;\delta)=\sup_{|h|\le\delta}\left\|\Delta_i^k(h)\,f\right\|.$$

$(\Delta_i^k(h)\,f(x)\equiv\Delta^k(h)\,f(x)$ and $\omega_{k,i}\,(f;\delta)\equiv\omega_k\,(f;\delta)$ for $n=1$).

Definition 1. We say that a function φ is almost decreasing in $[a,b]$ if there exists a positive constant A such that $\varphi(t_1)\ge A\,\varphi(t_2)$ for $a\le t_1\le t_2\le b$.

Definition 2. A function $\omega_k\colon[0,\pi]\to\mathbb{R}$ which satisfies the following four conditions:

(1) $\omega_k(0)=0$,
(2) ω_k is nondecreasing,
(3) ω_k is continuous,
(4) $\frac{\omega_k(t)}{t^k}$ is almost decreasing in $[0,\pi]$,

we call the modulus of continuity of k-th order.

Definition 3. We say that the modulus of continuity of k-th order ω_k satisfies Zygmund's condition if

$$\int_0^\delta\frac{\omega_k(t)}{t}dt+\delta^k\int_\delta^\pi\frac{\omega_k(t)}{t^{k+1}}dt=O(\omega_k(\delta)),\quad\delta\to0+\,.$$

Let ω_k be a modulus of continuity of k-th order. Then we denote by $H_i(\omega_k;\mathbb{C}(T^n))$ $(i=1,\dots,n)$ the set of all functions $f\in\mathbb{C}(T^n)$ such that

$$\omega_{k,i}(f;\delta)=O\big(\omega_k(\delta)\big),\quad\delta\to0+,\quad i=1,\dots,n.$$

We set

$$H\big(\omega_k;\mathbb{C}(T^n)\big)=\bigcap_{i=1}^n H_i\big(\omega_k;\mathbb{C}(T^n)\big).$$

By I we denote the following subset of the set $\mathbb{R}^n$:

$$\{\overline{x}:\overline{x}=(\underbrace{x,\dots,x}_{n});x\in T\}.$$

Moduli of smoothness play a basic role in approximation theory, Fourier analysis and their applications. For a given function f, they essentially measure the structure or smoothness of the function via the k-th difference $\Delta_i^k(h)\,f(\overline{x})$. In fact, for the functions f belonging to the Lebesgue space $L^p(1\le p<+\infty)$ or the space of continuous functions $\mathbb{C}$, the classical k-th modulus of continuity has turned out to be a rather good measure for determining the rate of convergence of best approximation. On this direction one could see books by V. K. Dzyadyk, I. A. Shevchuk [4] and by R. Trigub, E. Belinsky [9].

In the theory of functions of real variables there is a well-known theorem of Privalov on the invariance of the functional class $\mathrm{Lip}(\alpha,C(T))(0<\alpha<1)$ under the conjugate function $\widetilde{f}$. If $\alpha=1$ the invariance of the functional class fails. Later Zygmund [10] established that the analogous theorem is valid in the case $\alpha = 1$ for the modulus of continuity of the second order.Afterwards Bari and Stechkin obtained results connected with behaviour of the moduli of continuity of k-th order of the function f and its conjugate function.They obtained the necessary and sufficient condition on the modulus of continuity of k-th order ω_k for the invariance of $H(\omega_k; \mathbb{C}(T))$class under the conjugate function $\widetilde{f}$. As to the functions of many variables, the first result in this direction belongs to Cesari and Zhak. They showed that the class $\mathrm{Lip}(\alpha, C(T^2))(0 < \alpha < 1)$ is not invariant under the conjugate operators of two variables. Later, there were obtained the sharp estimates for partial moduli of continuity of different orders in the space of continuous functions [2, 3, 7]. The cases when moduli of continuity of different orders satisfy Zygmund's condition were considered in works [1, 5, 6]. In the present work, we study the behaviour of the smoothness of the conjugate functions $\widetilde{f}_B$ on the set I. If we restrict the function $\widetilde{f}_B$ on the set I, we can consider it as a function of one variable.the following question arises: what we can say about the smoothness of this 'new function' if the function f belongs to $H\big(\omega_k; \mathbb{C}(T^n)\big)$ and the modulus of continuity ω_k satisfies Zygmund's condition.

We now state the facts on which the proof of the main results is based.

Lemma 1 (see [3, p. 283])**.** *Let ω_k be a modulus of continuity of k-th order, I_l be a system of pairwise disjoint intervals, $I_l \subset T$ for each $l\,(l = 1, 2, \ldots)$. Let $(f_l)_{l\geq 1}$ be a sequence of functions such that for each l, $f_l \in \mathbb{C}(T)$ and $f_l(x) = 0$ when $x \in T \backslash I_l$. If*

$$\omega_k(f_l; \delta) \leq \omega_k(\delta), \quad 0 \leq \delta \leq \pi, \quad l = 1, 2, \ldots$$

and the function f is defined by the equality $f(x) = \sum\limits_{l=1}^{\infty} f_l(x)$, then

$$\omega_k(f; \delta) \leq (k + 1)\, \omega_k(\delta), \quad 0 \leq \delta \leq \pi.$$

Note that the case $k = 1$ is considered in [8, Lemma 1].

Remark 1. [3, p. 285] By the definition of the partial modulus of continuity of the multivariable function $f \in \mathbb{C}(T^n)$, it is easy to obtain the multivariable versions of Lemma 1 for partial moduli of continuity.

2. Main results

We can state and prove the following Theorem.

Theorem 1. a) *Let $f \in H(\omega_k, \mathbb{C}(T^n))$ and modulus of continuity of k-th order satisfies Zygmund's condition. Then*

$$\sup_{\overline{h}\in I, |h|\leq\delta} \sup_{\overline{x}\in I} \mid \Delta_j^k(h)\, \widetilde{f}_B(\overline{x}) \mid = O(\omega_k(\delta) \mid \ln\delta \mid^{|B|-1}), \quad j \in B, \quad \delta \to 0+, \tag{1}$$

$$\sup_{\overline{h}\in I, |h|\leq\delta} \sup_{\overline{x}\in I} \mid \Delta_j^k(h)\, \widetilde{f}_B(\overline{x}) \mid = O(\omega_k(\delta) \mid \ln\delta \mid^{|B|}), \quad j \in M \setminus B, \quad \delta \to 0+. \tag{2}$$

b) *For each* $B \subseteq M$ *there exist functions* F *and* G *such that* $F, G \in H(\omega_k; \mathbb{C}(T^n))$ *and*

$$\sup_{\overline{h}\in I, |h|\le\delta} \sup_{\overline{x}\in I} \mid \Delta_j^k(h)\, \widetilde{F}_B(\overline{x}) \mid \ge C\omega_k(\delta) \mid \ln\delta \mid^{|B|-1}, \quad j \in B, \quad 0 \le \delta \le \delta_0, \tag{3}$$

$$\sup_{\overline{h}\in I, |h|\le\delta} \sup_{\overline{x}\in I} \mid \Delta_j^k(h)\, \widetilde{G}_B(\overline{x}) \mid \ge C\omega_k(\delta) \mid \ln\delta \mid^{|B|}, \quad j \in M \setminus B, \quad 0 \le \delta \le \delta_0, \tag{4}$$

where C *and* δ_0 *are positive constants.*

Proof. a) Part a) is the particular case of the first part of the theorem given in [3].

b) Without loss of generality, we shall carry out the proof of part (b) for the case $B = \{1, \dots, m\}$ $(1 \le m \le n)$.

Let first $B = \{1, \dots, m\}$ $(1 \le m < n)$. Let us consider a strictly decreasing sequence of positive numbers $(b_l)_{l\ge1}$ such that

$$\sum_{l=0}^{\infty} b_l \le 1 \quad (b_0 = 0).$$

We set

$$\begin{aligned}
\tau_p &= 2\sum_{j=0}^{p-1} \omega_k^{-1}(b_j),\\
\tau_p^* &= \tau_p + \frac{2}{k}\omega_k^{-1}(b_p),\\
\tau_{p,q} &= \tau_p + q(\tau_p^* - \tau_p), \quad q = 2, \dots, k-1; p = 1, 2, \dots,
\end{aligned}$$

where $\omega_k^{-1}(b_p)$, $(p = 1, 2, \dots)$ is a certain element of the set $\{t : \omega_k(t) = b_p\}$.

Let $\tau_p \equiv \tau_{p,1}$ and $\tau_{p+1} \equiv \tau_{p,k}$. We define the functions $g_{p,q}$ and h_p $(p = 1, 2, \dots; q = 1, \dots, k-1)$ in T as follows:

$$g_{p,q}(x) = \begin{cases} \dfrac{(x - \tau_{p,q})^k (\tau_{p,q+1} - x)^k}{(\tau_p^* - \tau_p)^{2k}}, & x \in [\tau_{p,q}; \tau_{p,q+1}],\\ 0, & \text{otherwise.} \end{cases}$$

$$h_p(x) = \begin{cases} 0, & x \in [-\pi; 2\,\tau_p - \tau_p^*],\\ \dfrac{(x + \tau_p^* - 2\,\tau_p)^k}{(\tau_p^* - \tau_p)^k}, & x \in (2\,\tau_p - \tau_p^*; \tau_p],\\ 1, & x \in (\tau_p; \pi - \tau_p^* + \tau_p]\\ \dfrac{(\pi - x)^k}{(\tau_p^* - \tau_p)^k}, & x \in (\pi - \tau_p^* + \tau_p; \pi]. \end{cases}$$

We define the function $G_{p,q}$ ($p = 1, 2, \ldots; q = 1, \ldots, k-1$) in T^n as follows:

$$G_{p,q}(x_1, \ldots, x_n) = b_p \prod_{i=1}^{m} h_p(x_i) g_{p,q}(x_{m+1})$$

Consider the function G defined by the series

$$G(x_1, \ldots, x_n) = \sum_{p=1}^{\infty} \sum_{q=1}^{k-1} G_{p,q}.$$

We extend this function G 2π-periodically in each variable to the whole space $\mathbb{R}^n$.

We claim that

$$G \in H\big(\omega_k; \mathbb{C}(T^n)\big).$$

It is known [4, p. 195] that if the function of one variable f has kth derivative on $(x, x+kh)$ then

$$\Delta_k(h)\, f(x) = h^k f^{(k)}(x + k\theta h), \quad 0 < \theta < 1.$$

In our situation using the definition of the function $G_{p,q}$ and this fact we can conclude that

$$\mid \Delta_i^k(G_{p,q}; h) \mid \leq D_1 \mid h^k \mid \frac{b_p}{(\tau_p^* - \tau_p)^k}, \quad D_1 = \text{const}, \quad i = 1, \ldots, m+1.$$

Using this fact and the fact that $\frac{\omega_k(t)}{t^k}$ is almost decreasing we get

$$\omega_{k,i}(G_{p,q}; \delta) \leq D_2 \omega_k(\delta), \ \delta \to 0+, \quad D_2 = \text{const}\,.$$

By Remark for Lemma 1 we conclude

$$\omega_{k,i}(G; \delta) = O\big(\omega_k(\delta)\big), \quad \delta \to 0+, i = 1, \ldots, m+1.$$

If $i \in m+2, \ldots, n$ then it is easy to conclude that

$$\omega_{k,i}(G; \delta) = O\big(\omega_k(\delta)\big), \quad \delta \to 0+\,.$$

Hence

$$G \in H\big(\omega_k; \mathbb{C}(T^n)\big).$$

Let $h = \tau_p^* - \tau_p$ and $x_i = \tau_p\,, i = 1, \ldots, n$.

According to the definition of the function G we obtain

$$\begin{aligned}\big|\Delta_n^k(h)\, \widetilde{G}_{\{1,\ldots,m\}}(\tau_p, \ldots, \tau_p)\big| &\geq D_3 b_p \int_{[\tau_p^* - \tau_p, 1]^m} \prod_{i=1}^{m} s_i^{-1}\, ds_i \\ &= D_3 \omega_k(\tau_p^* - \tau_p) \mid \ln(\tau_p^* - \tau_p) \mid^m, \ D_3 = \text{const}\,.\end{aligned}$$

Therefore, the inequality (4) is proved.

To prove the inequality (3) we use the function F considered in [3, p. 289]

$$F(x) = F(x_1, x_2, \ldots, x_n) =$$

$$\begin{cases} \prod_{i=1}^{m} x_i^{k-1}(\pi - x_i)^{k-1} \int\limits_{x_1(\pi-x_1)}^{2x_1(\pi-x_1)} \cdots \int\limits_{x_m(\pi-x_m)}^{2x_m(\pi-x_m)} \frac{\min\limits_{1\le i\le m} \omega_k(t_i)}{\prod\limits_{i=1}^{m} t_i^k} \times & x_i \in [0,\pi],\ i=1,\dots,m, \\ \times \prod_{i=1}^{m} \frac{(t_i - x_i(\pi-x_i))^k}{t_i^k} \frac{(2x_i(\pi-x_i)-t_i)^k}{t_i^k}\, dt, & x_j \in [-\pi,\pi], \\ & j = m+1,\dots,n, \\ 0, & \text{if at least one } x_i \in [-\pi, 0] \\ & (i=1,\dots,m). \end{cases}$$

We extend the function F 2π-periodically in each variable to the whole space $\mathbb{R}^n$.

In [3] we have proved that $F \in H\big(\omega_k; \mathbb{C}(T^n)\big)$ and

$$\begin{aligned} &\big|\Delta_m^k(-h)\widetilde{F}_{\{1,\dots,m\}}(0,\dots,0,\dots,0)\big| \\ &\qquad \ge D_4 \int_{[0,\pi]^m} \min(h^k, s_m^k) s_m^{-k} \min_{1\le i\le m} \omega_k(s_i) \prod_{i=1}^{m} s_i^{-1}\, ds_i, \end{aligned}$$

where D_4 is a positive constant.

Using the fact that ω_k satisfies Zygmund's condition we get

$$\begin{aligned} &\big|\Delta_m^k(-h)\widetilde{F}_{\{1,\dots,m\}}(0,\dots,0,\dots,0)\big| \\ &\qquad \ge D_5 \omega_k(\tau_p^* - \tau_p) \mid \ln(\tau_p^* - \tau_p) \mid^{m-1}, \quad D_5 = \text{const}. \end{aligned}$$

The inequality (3) is proved. □

Corollary 1. *Let the modulus of continuity* ω *satisfies Zygmund's condition. Then for each* $B \subset \{1,\dots,n\}$ *there exist a function* $f \in H(\omega, \mathbb{C}(T^n))$ *and* C, δ_0 *positive constants for which we have*

$$\sup_{\overline{h}\in I, |h|\le\delta} \sup_{\overline{x}\in I} \mid \widetilde{f}_B(\overline{x}+\overline{h}) - \widetilde{f}_B(\overline{x}) \mid \ge \omega(\delta) \mid \ln\delta \mid^{|B|},$$

where $0 \le \delta \le \delta_0,\ B \ne \{1,2,\dots,n\}$.

$$\sup_{\overline{h}\in I, |h|\le\delta} \sup_{\overline{x}\in I} \mid \widetilde{f}_B(\overline{x}+\overline{h}) - \widetilde{f}_B(\overline{x}) \mid \ge \omega(\delta) \mid \ln\delta \mid^{n-1},$$

where $0 \le \delta \le \delta_0, B = \{1,2,\dots,n\}$.

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