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ON THE CONCIRCULAR VECTOR FIELDS OF SPACES WITH AFFINE CONNECTION

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Abstract. In this paper we study concircular vector fields of spaces with affine connection. We found the fundamental equation of these fields for the minimal requirements on the differentiability of the connection. The maximal numbers of linearly independent fields (with constant coefficients) is equal to $n+1$ and is realized only on projective flat spaces. Further we found a criterion on the Weyl tensor of the projective curvature of spaces, in which exist exactly $n-1$ independent concircular vector fields.

1. Introduction

Under a geodesic circle we understand a curve for which the first curvature is constant and the second curvature is zero. In 1944 K. Yano [26] introduced a conformal mapping of (pseudo-) Riemannian spaces which preserves geodesic circles and is called *concircular*.

The existence of these mappings is connected with the existence of concircular vector fields φ, which satisfy the equations $\nabla\varphi = \rho\,\mathrm{Id}$. Special types of these fields were studied earlier in the case when they are covariantly constant ($\rho = 0$) by T. Levi-Civita [11], for convergent vector fields ($\rho = \mathrm{const}$) by P. A. Shirokov [23], and for concircular ones by H. W. Brinkmann [2] and H. L. Vries [4]. Later they were independently studied as geodesic fields by Ya.L. Shapiro [22, 21], and as equidistant fields by N. S. Sinyukov [24, p. 92–98].

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Concircular vector fields and their generalizations play great role in many problems in differential geometry, for example in the theory of geodesic, almost geodesic mappings, see [1, 3, 6, 9, 12, 13, 15, 14, 16, 17, 19, 20, 24, 25].

In our work we study concircular vector fields on spaces with affine connection. In our paper we going to show some results connected to basic notations under the conditions of minimal differentiability of affine connection and other geometric objects which define concircular vector fields.

2. Fundamental equations of concircular vector fields

Let $A_n = (M, \nabla)$ be n-dimensional manifolds with affine connection ∇, $n \geq 2$.

Definition 1. A *concircular vector field* is a vector field φ in A_n such that for all points $x \in M$ the following relation is satisfied

$$\nabla \varphi = \varrho \,\mathrm{Id}, \tag{1}$$

where ϱ is a function on M.

If ϱ is constant then the vector field is *convergent*, and, moreover, if $\varrho = 0$ then the vector field is *covariantly constant.* this definition is analogous for such vector fields on (pseudo-) Riemannian manifolds [5, 16, 12, 24, 17, 26].

In a local coordinate neighbourhood (U, x), $U \subset M$, the equation (1) has the form $\nabla_j \varphi^h = \varrho\, \delta^h_j$, where δ^h_j is the Kronecker symbol. We can write it in the following form

$$\nabla_j \varphi^h \equiv \partial_j \varphi^h + \Gamma^h_{\alpha j} \varphi^\alpha = \varrho \cdot \delta^h_j, \tag{2}$$

where $\partial_j = \partial/\partial x^j$.

It is easily seen that formula (2), and also (1), is true when

$$A_n \in C^0 \text{ (i.e. } \Gamma^h_{ij}(x) \in C^0), \ \varphi^i(x) \in C^1 \text{ and } \varrho(x) \in C^0.$$

The following lemma holds.

Lemma 1 (Hinterleitner, Mikeš [7])**.** *Let $\lambda^h(x) \in C^1$ be a vector field and $\rho(x)$ a function. If $\dfrac{\partial \lambda^h}{\partial x^i} - \rho\, \delta^h_i \in C^1$ then $\lambda^h \in C^2$ and $\rho \in C^1$.*

Validity of Lemma 1 follows from the following more general lemma.

Lemma 2. *Let $\lambda^h(x) \in C^1$ be a vector field, $\varrho(x)$ a function and*

$$D^h_i = \begin{pmatrix} \delta^a_b & 0 \\ 0 & 0 \end{pmatrix}, \quad a, b = 1, \ldots, r, \quad 1 < r \leq n.$$

If $\partial_i \lambda^h - \varrho\, D^h_i \in C^1$ then $\lambda^h \in C^2$ and $\varrho \in C^1$.

Proof. The condition $\partial_i \lambda^h - \varrho\, D^h_i \in C^1$ can be written in the following form

$$\partial_i \lambda^h - \varrho\, D^h_i = f^h_i(x), \tag{3}$$

where $f_i^h(x)$ are functions of class C^1. Evidently, from (3) follows that ϱ is smooth function. For fixed but arbitrary indices $i \neq a$, $1 \leq i \leq n$, $1 \leq a \leq r$, we integrate (3) with respect to dx^i:

$$\lambda^a = \Lambda^a + \int_{x_0^i}^{x^i} f_i^a(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n)\, dt,$$

where Λ^a is a function, which does not depend on x^i.

Because of the existence of the partial derivatives of the functions λ^a and the above integrals (see [10, p. 300]), also the derivatives $\partial_a \Lambda^a$ exist; in this proof we don't use Einstein's summation convention. Then we can write (3) for $h = i = a$:

$$\varrho = -f_a^a + \partial_a \Lambda^a + \int_{x_0^i}^{x^i} \partial_a f_i^a(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n)\, dt. \tag{4}$$

Because the derivative with respect to x^i of the right-hand side of (4) exists, the derivative of the function ϱ exists, too. Obviously $\partial_i \varrho = \partial_h f_i^h - \partial_i f_h^h$, therefore $\varrho \in C^1$ and from (3) follows $\lambda^h \in C^2$. □

In a similar way we can prove that for $r = 1$ Lemma 2 is not valid.

If $A_n \in C^1$ (i.e. $\Gamma_{ij}^h(x) \in C^1$) holds, then from formula (2) follows $\dfrac{\partial \varphi^i}{\partial x^j} - \varrho \cdot \delta_j^i \in C^1$, and from Lemma 2 we get:

$$\varphi^i(x) \in C^2 \quad \text{and} \quad \varrho(x) \in C^1.$$

From this viewpoint we specify and generalize the results involving concircular vector fields below.

After differentiation of (1) we have $\varphi^h_{,jk} = \nabla_k \varrho\, \delta_j^h$ and alternation with respect to indices j and k we obtain the formula $\varphi^h_{,jk} - \varphi^h_{,kj} = \nabla_k \varrho\, \delta_j^h - \nabla_j \varrho\, \delta_k^h$. From the Ricci identity follow the integrability conditions of equation (1):

$$\varphi^\alpha R^h_{\alpha jk} = \nabla_j \varrho\, \delta_k^h - \nabla_k \varrho\, \delta_j^h, \tag{5}$$

where R^h_{ijk} are components of the curvature tensor.

We contract the indices h and k in (5) and get

$$\nabla_j \varrho = -\frac{1}{n-1}\, \varphi^\alpha R_{\alpha j},$$

where $R_{ij} = R^k_{ikj}$ are components of the Ricci tensor and (5) has the form

$$\varphi^\alpha \tilde{W}^h_{\alpha jk} = 0, \tag{6}$$

where

$$\tilde{W}^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\, (R_{ij}\, \delta_k^h - R_{ik}\, \delta_j^h). \tag{7}$$

The tensor $\tilde{W}$ is similar to the Weyl tensor of projective curvature W, see [16, p. 133]. In an equiaffine space A_n (where the Ricci tensor is symmetric, i.e. $R_{ij} = R_{ji}$) the tensor $\tilde{W}$ is identical to W.

Moreover, after contraction of (6) with respect to the indices h and k for $n > 2$ we get

$$\varphi^\alpha(R_{\alpha i} - R_{i\alpha}) = 0. \tag{8}$$

The system of equations

$$\begin{aligned} \nabla_j \varphi^h &= \rho \cdot \delta^h_j, \\ \nabla_j \varrho \ &= -\frac{1}{n-1}\ \varphi^\alpha R_{\alpha j}. \end{aligned} \tag{9}$$

is closed.

It is a system of linear differential equations of Cauchy type, in first order covariant derivatives of the vector $\varphi^h(x)$ and the function $\varrho(x)$, with coefficients uniquely determined by the connection ∇ of the manifold $A_n \in C^1$.

For any family of initial values $\varphi^h(x_0) = \varphi^h_0$ and $\rho(x_0) = \rho_0$ of the functions under consideration in the given point x_0, it admits at most one solution. Consequently, the number of free parameters in the general solution of the system is at most $n + 1$.

Similar research of (pseudo-) Riemannian spaces V_n was published in [3]. For higher differentiability of the vector field ϕ, see [4, 9], and for manifolds with affine connections, see [6, 18, 19].

Since the system is linear, it admits at most $n + 1$ linearly independent solutions corresponding to constant coefficients. This is obvious the cardinality of the system of independent concircular vector fields of the space $A_n \in C^1$.

It follows from the analysis of the system of equations (9) that if $A_n \in C^r$, $r \geq 1$, then $\varphi^h \in C^{r+1}$ and $\rho \in C^r$. From this we obtain the following theorem.

Theorem 1. *If the manifold A_n with affine connection ($A_n \in C^r$, $r \geq 1$) admits a concircular vector field $\varphi^h \in C^1$, then φ^h belongs to C^{r+1}.*

We suppose that the differentiability class r is equal to $2, 3, \ldots, \infty, \omega$, where ∞ and ω denote infinitely differentiable, and real analytic functions, respectively.

It is known that only projective flat manifolds admit the maximal number of $n+1$ linearly independent concircular vector fields. This holds locally. This fact follows from the study of the integrability conditions of (9) and their differential prolongations (explicitly $\tilde{W}^h_{ijk} = 0$, $R_{ij} = R_{ji}$ and $R_{ij,k} = R_{ik,j}$).

In Riemannian spaces $V_n = (M, g)$ the equations (9) were modified as follows [3]:

$$\begin{aligned} \nabla_j \varphi_i &= \rho \cdot g_{ij}, \\ \nabla_j \varrho \ &= B\, \varphi_j, \end{aligned}$$

where B is a function on M, $\varphi_i = \varphi^\alpha g_{\alpha i}$, and g_{ij} are components of metric g. If equations (9) have more than one solution, then the function B is constant.

If a Riemannian space $V_n \in C^2$ ($n > 2$) admits at least two linearly independent concircular vector fields $\varphi_i(x) \in C^1$ with constant coefficients, then B is a constant, uniquely determined by the metric of the space V_n, see [3].

Remark 1. In [9] and [8, p. 88] a similar theorem was published, but the proof was done only for $V_n \in C^3$, $\varphi_i(x) \in C^3$ and $\varrho(x) \in C^2$, and, moreover, it has local validity.

3. A space with affine connection which admits at least two linearly independent concircular vector fields

The initial conditions $\varphi^h(x_0) = 0$ and $\rho(x_0) = 0$ have only the trivial solution $\varphi^h(x) = 0$ and $\rho(x) = 0$ on $A_n = (M, \nabla) \in C^1$ for the system of equations (9). For this reason $\varphi^h(x)$ and $\rho(x)$ are vanishing on A_n, if $\varphi^h(x) = 0$ in the neighborhood U_{x_0} of the point x_0. Then the following lemma holds.

Lemma 3. *The non-vanishing concircular vector field $\varphi^h(x)$ can be equal to zero only on point sets of zero measure.*

By mathematical induction we have the following lemma.

Lemma 4. *A set of r $(r < n)$ linear independent concircular vector fields $\{\varphi^h_{1|}, \varphi^h_{2|}, \dots, \varphi^h_{r|}\}$ on A_n can be linearly dependent only on point sets of zero measure.*

Proof. Successively we can substitute $r = 1, 2, \dots, n-1$. Let $\{\varphi^h_{1|}, \varphi^h_{2|}, \dots, \varphi^h_{r|}\}$ be linearly independent (except at point sets of zero measure) concircular vector fields on A_n, which satisfy the following equations $\varphi^h_{s|,j} = \rho_{s|} \cdot \delta^h_j$, where $\rho_{s|}$ are functions on A_n.

Let these vectors be linearly independent at the point $x_0 \in M$, then they are linearly independent at a point x in a certain neighborhood $U_{x_0} \subset M$. Finally, let φ^h be a concircular vector field on M and

$$\varphi^h(x) = \sum_{s=1}^{r} \alpha_{s|}(x) \cdot \varphi^h_{s|}(x) \text{ for } x \in U_{x_0}\,, \tag{10}$$

where $\alpha_{s|}(x)$ are functions on U_{x_0}. Because $\varphi^h(x), \varphi^h_{s|}(x) \in C^1$, the functions $a_{s|}(x)$ are differentiable. Covariantly differentiating (10) with respect to x^j we find $(\rho \ - \sum\limits_{s=1}^{r} \alpha_{s|} \cdot \varrho_{s|})\, \delta^h_j = \sum\limits_{s=1}^{r} \nabla_j \alpha_{s|} \cdot \varphi^h_{s|}$. From this follows that $\rho = \sum\limits_{s=1}^{r} \alpha_{s|} \cdot \rho_{s|}$ and $\nabla_j \alpha_{s|} = 0$ (i.e. $\alpha_{s|} = \text{const}$) on U_{x_0}.

For the initial conditions $\varphi^h(x_0) = \sum\limits_{s=1}^{r} \alpha_{s|} \cdot \varphi^h_{s|}(x_0)$ and $\rho(x_0) = \sum\limits_{s=1}^{r} \alpha_{s|} \cdot \rho_{s|}(x_0)$ the equations (9) have only one solution: $\varphi^h(x) = \sum\limits_{s=1}^{r} \alpha_{s|} \ \cdot \varphi^h_{s|}(x)$ on A_n. □

Theorem 2. *There are no manifolds with affine connection $A_n \in C^1$, $(n > 2)$, except equiaffine projective flat spaces, which admit more than $(n-1)$ linearly independent concircular vector fields $\varphi_i(x) \in C^1$ corresponding to constant coefficients (of linearly dependence).*

Proof. Let us suppose the opposite. Let A_n be a space which is not equiaffine projective flat and yet admits more than $(n-1)$ linearly independent concircular vector fields with constant coefficients. The conditions (6) read

$$\varphi^\alpha \tilde{W}^h_{\alpha jk} = 0. \tag{11}$$

We can write the tensor $\tilde{W}^h_{ijk}$ as $\tilde{W}^h_{ijk} = \sum\limits_{s=1}^{m} a_{s|i}\Omega^h_{s|ijk}$ where $a_{s|i}$ are some linearly independent covectors, and $\Omega^h_{s|jk}$ are linearly independent tensors. Since A_n is not equiaffine projectively flat, $m \geq 1$ holds.

From the conditions (11) we obtain

$$\varphi^\alpha a_{1|\alpha} = 0, \quad \varphi^\alpha a_{2|\alpha} = 0, \quad \dots\ , \quad \varphi^\alpha a_{m|\alpha} = 0. \tag{12}$$

Since $m \geq 1$, among the equations of the system (12) there is at least one substantial equation. From the previous facts it follows that there exist less or equal to $n-1$ linearly independent vector fields φ^h, a contradiction. This proves the Theorem 2. □

From the Theorem 2 the following two Theorems follow

Theorem 3. *Let $A_n \in C^1$, $(n > 2)$, be a space with affine connection ∇ in which there are $(n-1)$ linearly independent concircular vector fields $\varphi^h(x) \in C^1$. Then the tensor $\tilde{W}$ has the following expression*

$$\tilde{W}^h_{ijk} = a_i \Omega^h_{jk}, \tag{13}$$

where a_i and Ω^h_{jk} are a non vanishing covector and tensor of type $(1,2)$, respectively.

Theorem 4. *$A_n \in C^3$ $(n > 2)$ admits $(n-1)$ linearly independent concircular vector fields $\varphi_i(x) \in C^1$, if and only if in A_n the following relations are satisfied*

$$\begin{aligned}
\tilde{W}^h_{ijk} &= a_i \Omega^h_{jk}, \\
a_{i,j} &= \mu_j a_i + b_i a_j; \\
b_{i,j} &= \nu_j a_i + b_i b_j - \frac{1}{n-1}\, R_{ij},
\end{aligned}$$

where a_i is a non-vanishing covector; c_i, μ_i, ν_i are some covectors.

4. Examples

Assume that A_n has the components of affine connection ∇ defined in the following way $\Gamma^1_{11} \in C^r$, $r \geq 0$, $\exists i \neq 1$ $\partial_i \Gamma^1_{11} \neq 0$ and the other components of Γ^h_{ij} are vanishing. If $\Gamma^1_{11} \in C^1$ then $R^h_{ijk} = a_i \Omega^h_{jk}$, $R_{ij} = a_i \Omega^h_{jh}$ and (13).

We can easily convince ourselves that in A_n exist exactly $n-1$ non-linear covariantly constant vector fields ξ^h, i.e. for which is $\nabla \xi^h = 0$. These fields have $\xi^h_{s|} = \delta^h_s$, $s = 2, 3, \cdots, n$. This is a general solution of equations (1). The condition $\varrho = 0$ is necessary.

The above solution is valid even though $\Gamma^1_{11} \in C^0$ and $\Gamma^1_{11} \notin C^1$. In this case there may be formula (13).

Let $\tilde{A}_n$ be a space with affine connection $\tilde{\nabla}$ for which $\tilde{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i$, where $\psi_i = \partial_i \Psi$. The vector fields $\varphi^h_{s|} = \exp(-\Psi) \cdot \xi^h_{s|}$ satisfy the equations (1). When $\tilde{\Gamma}^h_{ij} \in C^1$, then formula (13) is valid in $\tilde{A}_n$. Notice that A_n and $\tilde{A}_n$ have common geodesics.

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