Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
33 (2017), 61–66
www.emis.de/journals
ISSN 1786-0091

STUDY OF KÄHLER MANIFOLDS ENDOWED WITH LIFT OF SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, a condition on the manifold M for being a Kähler manifold with respect to lift of symmetric non-metric connection is obtained. Further, contravariant almost analytic vector field is discussed in such a manifold.

1. Introduction

The lift function plays an important role in the study of Differentiable manifolds. In last few decades, the theory of lifts and tangent bundles is studied by several authors. The study of tangent bundles is generalized by L. S. Das and M. N. I. Khan [1] (2005). They [1] consider the manifold with almost r-contact structure and obtained almost complex structure on the tangent bundle. Recently, M. Tekkoyun and S. Civelek [5] (2008) studied and extended the concept of lifts by considering the structures on complex manifolds. In 2014, the lifts are studied with quarter-symmetric semi-metric connection on tangent bundles by M. N. I. Khan [3]. The same author [4] also studied the lift of semi-symmetric metric connection on a Kähler manifold in 2015.

1.1. **Kähler manifold.** Let M be an n-(even) dimensional differentiable manifold. If, for a tensor field F of type (1,1) and a Riemannian metric g, the conditions

$$F^2(X) + X = 0,\ g(FX, FY) = g(X, Y),\ (\nabla_X F)Y = 0$$

hold, then M is called Kähler manifold, X, Y being arbitrary vector fields.

2010 *Mathematics Subject Classification.* 53C07, 53C35, 53C55.
Key words and phrases. Tangent bundle, complete Lift, vertical Lift, symmetric non-metric connection, Kähler manifold, contravariant almost analytic vector field.
The second author expresses his thanks to UGC, New Delhi, India for the financial support.

1.2. **Symmetric non-metric connection.** A linear connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X Y = \nabla_X Y + g(X,Y)\rho \tag{1.1}$$

is called symmetric non-metric connection, ∇ being Riemannian connection, ρ is associated vector field defined by $g(X,\rho) = \omega(X)$ [2].

1.3. **Tangent Bundle.** let M be a differentiable manifold and T_pM denotes the tangent space of M at any point $p \in M$ then the collection of all tangent spaces at point $p \in M$ is called tangent bundle of M and denote by $T(M) = \cup_{p\in M} T_pM$. Let $\tilde{p} \in T(M)$ then the projection $\pi\colon T(M) \to M$ defined by $\pi(\tilde{p}) = p$ is called bundle projection of $T(M)$ over M and the set $\pi^{-1}(p)$ is called fiber over $p \in M$ and M the base space [1, 4].

Vertical lift: For a smooth function f in M, a function $f^V = f o\pi$ defined on the composition of $\pi\colon T(M) \to M$ and $f\colon M \to \mathbb{R}$ is called the vertical lift. For $\tilde{p} \in \pi^{-1}(U)$ with induced coordinates (x^h, y^h), the value of $f^V(\tilde{p})$ is constant along each fiber $T_P(M)$ and equal to $f(p)$ *i.e.* $f^V(\tilde{p}) = f^V(x,y) = f o\pi(\tilde{p}) = f(p) = f(x)$.

Complete lift: For a smooth function f in M, a function f^C defined by $f^C = i(df)$ on $T(M)$ is called complete lift of the function f. If ∂f is denoted locally by $y^i\partial_i f$ then complete lift of f is locally denoted by $f^C = i(df) = \partial f$.

Let X be a vector field then for a smooth function f on M, a vector field $X^C \in T(M)$ defined by $X^C f^C = (Xf)^C$ is called complete lift of X in $T(M)$. If X has component x^h in M then component of complete lift X^C in $T(M)$ is given by $X^C : (x^h, \partial x^h)$ with respect to induced coordinates in $T(M)$.

For a 1-form ω in M and an arbitrary vector field X, the complete lift of ω is denoted by ω^C and defined by $\omega^C(X^C) = (\omega(X))^C$.

1.4. **Induced metric and connection.** Let $\tau\colon S \to M$ be an immersion of $(n-1)$-dimensional manifold S in M. If we denote the differentiable map $d\tau\colon T(S) \to T(M)$ of τ by B called tangent map of τ, $T(S)$ and $T(M)$ being tangent bundles of S and M respectively, then the tangent map of B is denoted by $\tilde{B}\colon T(T(S)) \to T(T(M))$ [4].

Let g be Riemannian metric in M and the complete lift of g is g^C in $T(M)$. If $\tilde{g}$ denotes the induced metric of g^C on $T(S)$ then we have $\tilde{g}(\tilde{B}X^C, \tilde{B}Y^C) = g^C(\tilde{B}X, \tilde{B}Y)$, for X, Y being vector fields in S. If ∇ denotes the Riemannian connection on M then ∇^C, the complete lift of ∇, is also Riemannian connection satisfying $\nabla^C_{X^C} Y^C = (\nabla_X Y)^C$ and $\nabla^C_{X^C} Y^V = (\nabla_X Y)^V$, for the vector fields X, Y in M.

The lift has the following properties [4],

$$\begin{aligned}
\omega^V(\tilde{B}X^C) &= \omega^V(\tilde{B}X)^{\overline{C}} = \#(\omega^V(X^C)) = \#((\omega(X))^V) = (\omega(BX))^{\overline{V}},\\
\omega^C(\tilde{B}X^C) &= \omega^C(\tilde{B}X)^{\overline{C}} = \#(\omega^C(X^C)) = \#((\omega(X))^C) = (\omega(BX))^{\overline{C}},\\
[X^C, Y^C] &= [X,Y]^C,\ F^C(X^C) = (F(X))^C,\ \omega^V(X^C) = (\omega(X))^V,\\
\omega^C(X^C) &= (\omega(X))^C,\ g^C(X^V, Y^C) = g^C(X^C, Y^V) = (g(X,Y))^V,
\end{aligned} \tag{1.2}$$

where X^C, ω^C, F^C, g^C and X^V, ω^V, F^V, g^V are the complete and vertical lifts of X, ω, F, g. $\#$, $\overline{V}$ and $\overline{C}$ denote the operation of restriction, vertical lift and complete lift on $\pi_M^{-1}(\tau(S))$ respectively, X, Y are arbitrary vector fields in S.

2. Kähler manifold equipped with lift of symmetric non-metric connection

Taking complete lift of equation (1.1), we get

$$(\overline{\nabla}_{BX}BY)^{\overline{C}} = (\nabla_{BX}BY)^{\overline{C}} + (g(BX, BY)B\rho)^{\overline{C}}. \tag{2.1}$$

Simplifying (2.1), we have

$$\begin{aligned}\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C = \nabla^C_{\tilde{B}X^C}\tilde{B}Y^C \\ + g^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}\rho^V + g^V(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}\rho^C.\end{aligned} \tag{2.2}$$

Replacing Y by FY, equation (2.2) gives

$$\begin{aligned}\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}(FY)^C = \nabla^C_{\tilde{B}X^C}\tilde{B}(FY)^C \\ + g^C(\tilde{B}X^C, \tilde{B}(FY)^C)\tilde{B}\rho^V + g^V(\tilde{B}X^C, \tilde{B}(FY)^C)\tilde{B}\rho^C.\end{aligned} \tag{2.3}$$

Also, operating F^C on equation (2.2), we get

$$\begin{aligned}F^C(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C) = F^C(\nabla^C_{\tilde{B}X^C}\tilde{B}Y^C) \\ + g^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}(F\rho)^V + g^V(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}(F\rho)^C.\end{aligned} \tag{2.4}$$

Subtracting (3.4) from (2.3), we have

$$\begin{aligned}(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}F^C)(\tilde{B}Y^C) = \omega^C(\tilde{B}(FY)^C)\tilde{B}(FX)^V \\ + g^C(\tilde{B}X^C, \tilde{B}(FY)^C)\tilde{B}\rho^V + g^V(\tilde{B}X^C, \tilde{B}(FY)^C)\tilde{B}\rho^C \\ - g^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}(F\rho)^V - g^V(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}(F\rho)^C.\end{aligned} \tag{2.5}$$

Thus, we can state

Theorem 2.1. *Let M be a Kähler manifold equipped with lift of symmetric non-metric connection $\overline{\nabla}^C$ then M is a Kähler manifold with respect to $\overline{\nabla}^C$ if and only if*

$$\begin{aligned}g^C(\tilde{B}X^C, \tilde{B}(FY)^C)\tilde{B}\rho^V + g^V(\tilde{B}X^C, \tilde{B}(FY)^C)\tilde{B}\rho^C \\ - g^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}(F\rho)^V - g^V(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}(F\rho)^C = 0.\end{aligned} \tag{2.6}$$

Let $'F$ denotes the 2-form of Riemannian metric g defined by $'F(Y, Z) = g(FY, Z)$ then the complete lift of $'F$ is denoted and defined by

$$'F^C(\tilde{B}Y^C, \tilde{B}Z^C) = g^C(\tilde{B}(FY)^C, \tilde{B}Z^C). \tag{2.7}$$

Taking covariant differentiation of (2.7), we get

$$(\overline{\nabla}^C_{\tilde{B}X^C}{'F^C})(\tilde{B}Y^C, \tilde{B}Z^C) = (\nabla^C_{\tilde{B}X^C}{'F^C})(\tilde{B}Y^C, \tilde{B}Z^C) \tag{2.8}$$

$$- g^C(\tilde{B}(F\rho)^V, \tilde{B}Z^C)g^C(\tilde{B}X^C, \tilde{B}Y^C) - g^V(\tilde{B}X^C, \tilde{B}Y^C)g^C(\tilde{B}(F\rho)^C, \tilde{B}Z^C)$$
$$- g^C(\tilde{B}X^C, \tilde{B}Z^C)g^C(\tilde{B}\rho^V, \tilde{B}(FY)^C) - g^V(\tilde{B}X^C, \tilde{B}Z^C)g^C(\tilde{B}\rho^C, \tilde{B}(FY)^C).$$

By taking cyclic sum over X, Y, Z of equation (2.8), we obtained

(2.9)
$$(\overline{\nabla}^C_{\tilde{B}X^C}{}'F^C)(\tilde{B}Y^C, \tilde{B}Z^C) + (\overline{\nabla}^C_{\tilde{B}Y^C}{}'F^C)(\tilde{B}Z^C, \tilde{B}X^C)$$
$$+ (\overline{\nabla}^C_{\tilde{B}Z^C}{}'F^C)(\tilde{B}X^C, \tilde{B}Y^C)$$
$$= -g^C(\tilde{B}(F\rho)^V, \tilde{B}Z^C)g^C(\tilde{B}X^C, \tilde{B}Y^C) - g^C(\tilde{B}(F\rho)^V, \tilde{B}X^C)g^C(\tilde{B}Y^C, \tilde{B}Z^C)$$
$$- g^C(\tilde{B}(F\rho)^V, \tilde{B}Y^C)g^C(\tilde{B}Z^C, \tilde{B}X^C) - g^V(\tilde{B}X^C, \tilde{B}Y^C)g^C(\tilde{B}(F\rho)^C, \tilde{B}Z^C)$$
$$- g^V(\tilde{B}Y^C, \tilde{B}Z^C)g^C(\tilde{B}(F\rho)^C, \tilde{B}X^C) - g^V(\tilde{B}Z^C, \tilde{B}X^C)g^C(\tilde{B}(F\rho)^C, \tilde{B}Y^C)$$
$$- g^C(\tilde{B}X^C, \tilde{B}Z^C)g^C(\tilde{B}\rho^V, \tilde{B}(FY)^C) - g^C(\tilde{B}Y^C, \tilde{B}X^C)g^C(\tilde{B}\rho^V, \tilde{B}(FZ)^C)$$
$$- g^C(\tilde{B}Z^C, \tilde{B}Y^C)g^C(\tilde{B}\rho^V, \tilde{B}(FX)^C) - g^V(\tilde{B}X^C, \tilde{B}Z^C)g^C(\tilde{B}\rho^C, \tilde{B}(FY)^C)$$
$$- g^V(\tilde{B}Y^C, \tilde{B}X^C)g^C(\tilde{B}\rho^C, \tilde{B}(FZ)^C) - g^V(\tilde{B}Z^C, \tilde{B}Y^C)g^C(\tilde{B}\rho^C, \tilde{B}(FX)^C).$$

Thus, we can state the following

Theorem 2.2. *Let M be a Kähler manifold equipped with lift of symmetric non-metric connection $\overline{\nabla}^C$ then the relation (2.9) holds.*

3. Contravariant almost analytic vector field on a Kähler manifold

We know that in an Almost Hermitian manifold, a necessary and sufficient condition for a vector field W to be contravariant almost analytic vector field is that

$$\nabla_{FX}W = (\nabla_W F)X + F(\nabla_X W). \tag{3.1}$$

For a Kähler manifold (4.1) reduces to

$$\nabla_{FX}W - F(\nabla_X W) = 0. \tag{3.2}$$

Replacing X by FX and Y by W in (2.2), we have

$$\overline{\nabla}^C_{\tilde{B}(FX)^C}\tilde{B}W^C = \nabla^C_{\tilde{B}(FX)^C}\tilde{B}W^C + g^C(\tilde{B}(FX)^C, \tilde{B}W^C)\tilde{B}\rho^V + g^V(\tilde{B}(FX)^C, \tilde{B}W^C)\tilde{B}\rho^C. \tag{3.3}$$

Again, replacing Y by W and then taking F^C in (3.2), we get

$$F^C(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}W^C) = F^C(\nabla^C_{\tilde{B}X^C}\tilde{B}W^C) + g^C(\tilde{B}X^C, \tilde{B}W^C)\tilde{B}(F\rho)^V + g^V(\tilde{B}X^C, \tilde{B}W^C)\tilde{B}(F\rho)^C. \tag{3.4}$$

Subtracting (3.4) from (3.3), we obtain

$$\overline{\nabla}^C_{\tilde{B}(FX)^C}\tilde{B}W^C - F^C(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}W^C) = \nabla^C_{\tilde{B}(FX)^C}\tilde{B}W^C - F^C(\nabla^C_{\tilde{B}X^C}\tilde{B}W^C) + g^C(\tilde{B}(FX)^C, \tilde{B}W^C)\tilde{B}\rho^V \tag{3.5}$$

$$+ g^V(\tilde{B}(FX)^C, \tilde{B}W^C)\tilde{B} - g^C(\tilde{B}X^C, \tilde{B}W^C)\tilde{B}(F\rho)^V - g^V(\tilde{B}X^C, \tilde{B}W^C)\tilde{B}(F\rho)^C.$$

Thus, we have the following

Theorem 3.1. *Let M be a Kähler manifold equipped with lift of symmetric non-metric connection $\overline{\nabla}^C$ and W be a contravariant almost analytic vector field with respect to connection ∇ then W is contravariant almost analytic vector field with respect to connection $\overline{\nabla}^C$ ii and only if*

$$g^C(\tilde{B}(FX)^C, \tilde{B}W^C)\tilde{B}\rho^V + g^V(\tilde{B}(FX)^C, \tilde{B}W^C)\tilde{B} - g^C(\tilde{B}X^C, \tilde{B}W^C)\tilde{B}(F\rho)^V - g^V(\tilde{B}X^C, \tilde{B}W^C)\tilde{B}(F\rho)^C. \tag{3.6}$$

If we denote

$$\overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C) = g^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}\rho^V + g^V(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}\rho^C. \tag{3.7}$$

and define a tensor $'\overline{H}^C$ of type (0,3) by

$$'\overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C, \tilde{B}Z^C) = g^C(\overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C), \tilde{B}Z^C), \tag{3.8}$$

then, equations (3.7) and (3.8) together give

$$\begin{aligned} '\overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C, \tilde{B}Z^C) &= g^C(\tilde{B}X^C, \tilde{B}Y^C)g^C(\tilde{B}Z^C, \tilde{B}\rho^V) \\ &+ g^V(\tilde{B}X^C, \tilde{B}Y^C)g^C(\tilde{B}Z^C, \tilde{B}\rho^C). \end{aligned} \tag{3.9}$$

Replacing Y and Z by FY and FZ in (3.8) respectively, we get

$$\begin{aligned} &'\overline{H}^C(\tilde{B}X^C, \tilde{B}(FY)^C, \tilde{B}(FZ)^C) \\ &= g^C(\tilde{B}X^C, \tilde{B}(FY)^C)g^C(\tilde{B}(FZ)^C, \tilde{B}\rho^V) \\ &\quad + g^V(\tilde{B}X^C, \tilde{B}(FY)^C)g^C(\tilde{B}(FZ)^C, \tilde{B}\rho^C). \end{aligned} \tag{3.10}$$

Subtracting (3.10) from (3.9), we find

$$\begin{aligned} &'\overline{H}^C(\tilde{B}X^C, \tilde{B}(FY)^C, \tilde{B}(FZ)^C) -' \overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C, \tilde{B}Z^C) \\ &= g^C(\tilde{B}X^C, \tilde{B}(FY)^C)g^C(\tilde{B}(FZ)^C, \tilde{B}\rho^V) \\ &+ g^V(\tilde{B}X^C, \tilde{B}(FY)^C)g^C(\tilde{B}(FZ)^C, \tilde{B}\rho^C) \\ &- g^C(\tilde{B}X^C, \tilde{B}Y^C)g^C(\tilde{B}Z^C, \tilde{B}\rho^V) - g^V(\tilde{B}X^C, \tilde{B}Y^C)g^C(\tilde{B}Z^C, \tilde{B}\rho^C). \end{aligned} \tag{3.11}$$

Which shows that $'H$ is hybrid in last two slots if and only if right hand side of (3.11) vanishes.

We also know that a necessary and sufficient condition to be a Kähler manifold with respect to connection D defined by $D_XY = \nabla_XY + H(X, Y)$ is that $'H$ defined by $'H(X, Y, Z) = g(H(X, Y), Z)$ is hybrid in last two slots. Hence, from above discussion we have the following

Theorem 3.2. *Let M be a Kähler manifold equipped with lift of symmetric non-metric connection $\overline{\overline{\nabla}}^C$ then a necessary and sufficient condition for M to be a Kähler manifold with respect to connection $\overline{\overline{\nabla}}^C$ is that*

(3.12)
$$g^C(\tilde{B}X^C, \tilde{B}(FY)^C)g^C(\tilde{B}(FZ)^C, \tilde{B}\rho^V) - g^C(\tilde{B}X^C, \tilde{B}Y^C)g^C(\tilde{B}Z^C, \tilde{B}\rho^V)$$
$$+g^V(\tilde{B}X^C, \tilde{B}(FY)^C)g^C(\tilde{B}(FZ)^C, \tilde{B}\rho^C) - g^V(\tilde{B}X^C, \tilde{B}Y^C)g^C(\tilde{B}Z^C, \tilde{B}\rho^C) = 0.$$

Corollary 3.1. *Also, replacing X by FX in (3.11), we have*

$$\begin{aligned} g^C(\tilde{B}X^C, \tilde{B}(FY)^C)\tilde{B}\rho^V + g^V(\tilde{B}X^C, \tilde{B}(FY)^C)\tilde{B}\rho^C \\ - g^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}(F\rho)^V - g^V(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}(F\rho)^C = 0. \end{aligned} \tag{3.13}$$

which verifies the condition of Kähler manifold obtained in (3.6).

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Received April 4, 2016.

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