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SEMI-SYMMETRY TYPE LP-SASAKIAN MANIFOLDS

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Abstract. Recently the present authors have introduced the notion of *generalized quasi-conformal curvature tensor* $\mathcal{W}$, which bridges Conformal curvature tensor, Concircular curvature tensor, Projective curvature tensor and Conharmonic curvature tensor. The present paper attempts to investigate the curvature conditions (like flatness, semi-symmetry type and Ricci semi-symmetry type) of *LP-Sasakian manifold*. Further, the paper seeks to table the nature of the Ricci tensor for respective semi-symmetry type curvature conditions.

1. Introduction

In tune with Yano and Sawaki [30], recently the present authors [1] have defined and studied *generalized quasi-conformal curvature tensor* $\mathcal{W}$, in the context of $N(k,\mu)$-contact metric manifold. The beauty of *generalized quasi-conformal curvature tensor* $\mathcal{W}$ lies in the fact that it has the flavour of Riemann curvature tensor R, conformal curvature tensor C [8], conharmonic curvature tensor $\hat{C}$ [9], concircular curvature tensor E [29, p. 84] projective curvature tensor P [29, p. 84] and m-projective curvature tensor H [17], as special cases.

In 1989 K. Matsumoto [11] introduced the notion of Lorentzian para-Sasakian (LP-Sasakian for short) manifold. In 1992, Mihai and Rosca [12] defined the same notion independently. This type of manifold is also discussed in [22], [23], [24] and the references therein.

An LP-Sasakian manifold is said to be semi-symmetry type (respectively Ricci semi-symmetry type) if the *generalized quasi-conformal curvature tensor* $\mathcal{W}$(respectively Ricci tensor S) admits the condition

$$\omega(X,Y).\mathcal{W}=0, \text{ (respectively } \mathcal{W}(X,Y).S=0), \text{ for any } X,Y \text{ on } M, \tag{1.1}$$

where the dot means that $\omega(X,Y)$ acts on $\mathcal{W}$ (respectively on S) as derivation. Here ω and $\mathcal{W}$ stand for *generalized quasi-conformal curvature tensor* with the associated scalar triples $(\bar{a},\bar{b},\bar{c})$ and (a,b,c) respectively. In particular,

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manifold satisfying the condition $R(X,Y)\cdot R=0$ (obtained from 1.1 by setting $\bar{a}=\bar{b}=\bar{c}$=0 $=a=b=c$) is said to be semi-symmetric in the sense of Cartan ([2, p. 265], and named by N. S. Sinjukov [25]). A full classification of such space is given by Z. I. Szabó ([27], [26], [28]). This type of the manifolds have been studied by several authors such as Sekigawa and Tanno [21], Sekigawa and Takagi [20], Papantoniou [15], Perrone [16], Kowalski [10], Sekigawa [19] and the references therein.

Our work is structured as follows. The section 2 is a very brief account of LP-Sasakian manifolds. Definition and some basic results of the *generalized quasi-conformal curvature tensor* $\mathcal{W}$ is discussed in section 3. LP-Sasakian manifold with vanishing *generalized quasi-conformal curvature tensor* is studied in section 4 and it is found that such a manifold M is either an Einstein space or an η-Einstein space or isometric to the Lorentz sphere $S^{2n+1}(1)$. Furthermore, the nature of the Ricci tensors for the flatness of different curvature tensors are characterized. In section 5, we investigate LP-Sasakian manifold satisfying the condition $\omega(\xi,X)\cdot\mathcal{W}=0$. Based on this conditions and by taking into account the permutation of different curvature tensors, we obtained and tabled the expression of Ricci tensors for different semi-symmetry type conditions. The last section is devoted to study LP-Sasakian manifold admitting the condition $\mathcal{W}\cdot S=0$. Among others an equivalent conditions that (a) M is an Einstein space, (b) M is Ricci symmetric i.e., $\nabla S=0$, (c) $P(\xi,X)\cdot S=0$ (or $E(\xi,X)\cdot S=0$) for all $X\in\chi(M)$ is brought out.

2. LP-Sasakian manifolds

An $(2n+1)$-dimensional differentiable manifold M is said to be an LP-Sasakian manifold ([4], [11]) if it admits a $(1,1)$ tensor field ϕ, a unit timelike contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi)=-1,\quad g(X,\xi)=\eta(X),\quad \phi^2X=X+\eta(X)\xi, \tag{2.1}$$

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y),\quad \nabla_X\xi=\phi X, \tag{2.2}$$

$$(\nabla_X\phi)(Y)=g(X,Y)\xi+\eta(Y)X+2\eta(X)\eta(Y)\xi, \tag{2.3}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold, the following relations hold :

$$\phi\xi=0,\quad \eta(\phi X)=0,\quad \text{Rank }\phi=2n. \tag{2.4}$$

Again, if we put

$$\Omega(X,Y)=g(X,\phi Y)$$

for any vector fields X,Y then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$ tensor field [12]. Also, since the vector field η is closed in an LP-Sasakian manifold, we have ([11], [12])

$$(\nabla_X\eta)(Y)=\Omega(X,Y),\quad \Omega(X,\xi)=0, \tag{2.5}$$

for any vector fields X and Y.

Let M be an $(2n+1)$-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g). Then the following relations hold ([11], [12]):

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.6}$$

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.7}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.8}$$

$$S(X,\xi) = 2n\eta(X), \tag{2.9}$$

for any vector fields X, Y, Z where R is the Riemannian curvature tensor of the manifold.

3. The generalized quasi conformal curvature tensor

The *generalized quasi-conformal curvature tensor* is defined as

$$\begin{aligned}\mathcal{W}(X,Y)Z = \frac{2n-1}{2n+1}\left[(1+2na-b) - \{1+2n(a+b)\}c\right] C(X,Y)Z \\ +\Big[1-b+2na\Big] E(X,Y)Z + 2n(b-a)P(X,Y)Z \\ + \frac{2n-1}{2n+1}(c-1)\{1+2n(a+b)\}\hat{C}(X,Y)Z\end{aligned} \tag{3.1}$$

for all X, Y and $Z \in \chi(M)$, the set of all vector field of the manifold M, where a, b and c are real constants. And C, E, P and $\hat{C}$ stand for Conformal, Concircular, Projective and Conharmonic curvature tensor respectively. These curvature tensor are defined as follows

$$\begin{aligned}C(X,Y)Z = R(X,Y)Z \\ - \frac{1}{2n-1}\Big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\Big] \\ + \frac{r}{2n(2n-1)}\Big[g(Y,Z)X - g(X,Z)Y\Big],\end{aligned} \tag{3.2}$$

$$E(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\Big[g(Y,Z)X - g(X,Z)Y\Big], \tag{3.3}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\Big[S(Y,Z)X - S(X,Z)Y\Big], \tag{3.4}$$

$$\begin{aligned}\hat{C}(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\Big[S(Y,Z)X - S(X,Z)Y \\ + g(Y,Z)QX - g(X,Z)QY\Big],\end{aligned} \tag{3.5}$$

for all X, Y & $Z \in \chi(M)$, where R, S, Q and r being Christoffel Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively.

In particular, the *generalized quasi-conformal curvature tensor* $\mathcal{W}$ reduced to

(1) Riemann curvature tensor R, if $a = b = c = 0$,
(2) conformal curvature tensor C, if $a = b = -\frac{1}{2n-1}$, $c = 1$,
(3) conharmonic curvature tensor $\hat{C}$, if $a = b = -\frac{1}{2n-1}$, $c = 0$,
(4) concircular curvature tensor E, if $a = b = 0$ and $c = 1$,
(5) projective curvature tensor P, if $a = -\frac{1}{2n}$, $b = 0$, $c = 0$ and
(6) m-projective curvature tensor H, if $a = b = -\frac{1}{4n}$, $c = 0$.

The m-projective curvature tensor is introduced by G. P. Pokhariyal and R. S. Mishra [17]. Which is defined as follows

$$H(X,Y)Z = R(X,Y) - \frac{1}{4n}\Big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\Big]. \tag{3.6}$$

Using (3.2), (3.3), (3.4) and (3.5) in (3.1), the *generalized quasi-conformal curvature tensor* $\mathcal{W}$ becomes

$$\begin{aligned}\mathcal{W}(X,Y)Z = R(X,Y)Z &+ a\Big[S(Y,Z)X - S(X,Z)Y\Big] \\ &+ b\Big[g(Y,Z)QX - g(X,Z)QY\Big] \\ &- \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\Big]g(Y,Z)X - g(X,Z)Y\Big].\end{aligned} \tag{3.7}$$

Note that our *generalized quasi-conformal curvature tensor* $\mathcal{W}$ is not a generalized curvature tensor (see, [13], [6], [5], [7]), as it does not satisfy the condition $\mathcal{W}(X_1, X_2, X_3, X_4) = \mathcal{W}(X_3, X_4, X_1, X_2)$, where

$$\mathcal{W}(X_1, X_2, X_3, X_4) = g(\mathcal{W}(X_1, X_2)X_3, X_4),$$

for all X_1, X_2, X_3, X_4. Moreover our $\mathcal{W}$ is not a proper generalized curvature tensor[13], as it does not satisfy the second Bianchi identity

$$(\nabla_{X_1}\mathcal{W})(X_2, X_3)X_4 + (\nabla_{X_2}\mathcal{W})(X_3, X_1)X_4 + (\nabla_{X_3}\mathcal{W})(X_1, X_2)X_4 = 0. \tag{3.8}$$

4. LP-Sasakian manifolds with flat generalized quasi-conformal curvature tensor

Definition 4.1. An LP-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be η-Einstein, if its Ricci tensor$S$ of the metric g satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

for some real constant α and β.

Such notion was first introduced and studied by Okumura [14] and named by Sasaki [18] in his lecture notes 1965. In particular, if $\beta = 0$, we say that the manifold is Einstein.

Let the manifold be generalized quasi-conformally flat. Therefore, using (2.7) and (2.9) in (3.7) we can easily bring out after a straightforward calculation that

$$S(X,Z) = \kappa g(X,Z) + (\kappa - 2n)\eta(X)\eta(Z), \tag{4.1}$$

where

$$\kappa = -\frac{1}{a}\left\{1 + 2nb - \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right\}. \tag{4.2}$$

Remark 4.2. The Ricci tensor S in the equation (4.1) can also be written as

$$S(X,Z) = \kappa g(X,Z) + (\kappa - 2n)\eta(X)\eta(Z), \tag{4.3}$$

where

$$\kappa = -\frac{1}{b}\left\{1 + 2na - \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right\}. \tag{4.4}$$

Thus, we see that LP-Sasakian manifold with $\mathcal{W}(X,Y)Z = 0$ is an η-Einstein space provided $a \neq 0$ (or $b \neq 0$).

Again, an LP-Sasakian manifold with $R = 0$ or $E = 0$ (i.e., for the case $a = 0$ and $b = 0$), one can easily determine that such manifold is an Einstein. This leads to the followings:

Theorem 4.3. *Let $M^{2n+1}(\phi, \xi, \eta, g)$, $n > 1$; be an LP-Sasakian manifold with vanishing generalized quasi-conformal curvature tensor. Then M is either an Einstein space or an η-Einstein space or isometric to the Lorentz sphere $S^{2n+1}(1)$.*

Again, from (4.1), one can easily determine the following theorem-

Theorem 4.4. *Every LP-Sasakian manifold (M^{2n+1}, g), $n > 1$ with vanishing generalized quasi-conformal curvature tensor is necessarily η-parallel.*

Further, if we choose $a = b$, then from (4.1) we can state the following.

Theorem 4.5. *A generalized quasi-conformally flat LP-Sasakian manifold (M^{2n+1}, g), $(n > 1)$ is a manifold of quasi-constant curvature with associated scalars*

$$A = -\left(1 + \frac{b}{a}\right)\kappa a + \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right), \ B = a\left(\kappa + 2n\right),$$

provided that $a \neq 0$.

The notion of a *manifold of quasi-constant curvature* was first introduced by Chen and Yano [3] for a Riemannian geometry.

5. LP-Sasakian manifolds with semi-symmetry type curvature condition

Definition 5.1. A $(2n+1)$-dimensional $(n>1)$ LP-Sasakian manifold is said to be semi-symmetry type if the condition $\omega(X,Y)\cdot\mathcal{W}=0$ holds, for any vector fields X, Y on the manifold and $\omega(X,Y)$ acts on $\mathcal{W}$ as derivation, where ω and $\mathcal{W}$ stand for *generalized quasi-conformal curvature tensor* with the associated scalar triples $(\bar{a},\bar{b},\bar{c})$ and (a,b,c) respectively.

Now, let us consider a $(2n+1)$-dimensional LP-Sasakian manifold M, satisfying the condition

$$(\omega(\xi,X)\cdot\mathcal{W})(Y,Z)U=0, \tag{5.1}$$

which is equivalent to

$$\begin{aligned} g(\omega(\xi,X)\mathcal{W}(Y,Z)U,\xi)-g(\mathcal{W}(\omega(\xi,X)Y,Z)U,\xi)& \\ -g(\mathcal{W}(Y,\omega(\xi,X)Z)U,\xi)-g(\mathcal{W}(Y,Z)\omega(\xi,X)U,\xi)&=0. \end{aligned} \tag{5.2}$$

Putting $X=Y=\{e_i\}$ in (5.2) where $\{e_1,\ e_2,\ e_3,\dots,e_{2n},\ e_{2n+1}=\xi\}$ is an orthonormal basis of the tangent space at each point of the manifold M and then taking the summation over i, $1\le i\le 2n+1$, we get

$$\begin{aligned} \sum_{i=1}^{2n+1}\Big[g(\omega(\xi,e_i)\mathcal{W}(e_i,Z)U,\xi)-g(\mathcal{W}(\omega(\xi,e_i)e_i,Z)U,\xi)& \\ -g(\mathcal{W}(e_i,\omega(\xi,e_i)Z)U,\xi)-g(\mathcal{W}(e_i,Z)\omega(\xi,e_i)U,\xi)\Big]&=0. \end{aligned} \tag{5.3}$$

From the equation (3.7), we can easily estimate the followings

$$\begin{aligned} \eta(\mathcal{W}(\xi,U)Z)=&\left[\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)-2na-2nb-1\right]\eta(Z)\eta(U) \\ &+\left[\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)-2nb-1\right]g(Z,U)-aS(Z,U), \end{aligned} \tag{5.4}$$

$$\begin{aligned} \sum_{i=1}^{2n+1}\tilde{\mathcal{W}}(e_i,Z,U,e_i)=&(1-b+2na)S(Z,U) \\ &+\left\{br-\frac{2ncr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right\}g(Z,U), \end{aligned} \tag{5.5}$$

$$\begin{aligned} \sum_{i=1}^{2n+1}\eta(\mathcal{W}(e_i,Z)e_i)=&-2n(1-a+2nb)\eta(Z) \\ &-\left\{ar-\frac{2ncr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right\}\eta(Z), \end{aligned} \tag{5.6}$$

$$(5.7)\quad \sum_{i=1}^{2n+1} S(\mathcal{W}(e_i,Z)U,e_i) = \left\{ar + \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right\} S(Z,U)$$
$$-(a+b-1)S^2(Z,U) - \frac{cr^2}{2n+1}\left(\frac{1}{2n}+a+b\right) g(Z,U) + bS(Qe_i,e_i)g(Z,U),$$

$$(5.8)\quad \sum_{i=1}^{2n+1} \eta(e_i)\eta(\mathcal{W}(Qe_i,Z)U)$$
$$= -2n\left[1+2nb - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right] g(Z,U) - 2naS(Z,U)$$
$$- 2n\left[1+2n(a+b) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right]\eta(Z)\eta(U)$$

and

$$(5.9)\quad \sum_{i=1}^{2n+1} S(e_i,Z)\eta(\mathcal{W}(e_i,\xi)U)$$
$$= 2n\left[1+2n(a+b) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right]\eta(Z)\eta(U)$$
$$+\left\{1+2nb - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right\} S(Z,U) + aS^2(Z,U).$$

Now,

(5.10)

$$\sum_{i=1}^{2n+1} g(\omega(\xi,e_i)\mathcal{W}(e_i,Z)U,\xi)$$
$$= \left[\frac{\bar{c}r}{2n+1}\left(\frac{1}{2n}+\bar{a}+\bar{b}\right) - 2n\bar{b} - 1\right]\left\{\bar{\mathcal{W}}(e_i,Z,U,e_i) + \eta(\mathcal{W}(e_i,Z)U)\eta(e_i)\right\}$$
$$- \bar{a}\left[S(\mathcal{W}(e_i,Z)U,e_i) + 2n\eta(\mathcal{W}(e_i,Z)U)\eta(e_i)\right]$$
$$= \left[\frac{\bar{c}r}{2n+1}\left(\frac{1}{2n}+\bar{a}+\bar{b}\right) - 2n\bar{b} - 1\right]\bar{\mathcal{W}}(e_i,Z,U,e_i) - \bar{a}S(\mathcal{W}(e_i,Z)U,e_i)$$
$$+ \left[\frac{\bar{c}r}{2n+1}\left(\frac{1}{2n}+\bar{a}+\bar{b}\right) - 2n\bar{b} - 1 - 2n\bar{a}\right]\eta(\mathcal{W}(\xi,U)Z).$$

In consequence of (5.5) and (5.7), the equation (5.10) becomes

(5.11)

$$\sum_{i=1}^{2n+1} g(\omega(\xi,e_i)\mathcal{W}(e_i,Z)U,\xi) =$$
$$\Big[\left\{\frac{\bar{c}r}{2n+1}\left(\frac{1}{2n}+\bar{a}+\bar{b}\right) - 2n\bar{b} - 1\right\}(1+2na-b)$$

$$- \bar{a}\left\{ar + \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right\}\Big]S(Z,U) + \bar{a}(a+b-1)S^{2}(Z,U)$$
$$+ \left\{\frac{\bar{c}r}{2n+1}\left(\frac{1}{2n} + \bar{a} + \bar{b}\right) - 2n(\bar{a}+\bar{b}) - 1\right\}\eta(\mathcal{W}(\xi,U)Z)$$
$$+ \Big[\left\{\frac{\bar{c}r}{2n+1}\left(\frac{1}{2n} + \bar{a} + \bar{b}\right) - 2n\bar{b} - 1\right\}\left\{br - \frac{2ncr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right\}$$
$$- \bar{a}\left\{b \parallel Q \parallel^2 - \frac{cr^2}{2n+1}\left(\frac{1}{2n} + a + b\right)\right\}\Big]g(Z,U).$$

Again,

(5.12)
$$\sum_{i=1}^{2n+1} g(\mathcal{W}(\omega(\xi,e_i)e_i, Z)U,\xi)$$
$$= \left[2n\left\{1 + 2n\bar{b} - \frac{\bar{c}r}{2n+1}\left(\frac{1}{2n} + \bar{a} + \bar{b}\right)\right\} + \bar{a}r + 2n\bar{b} - 2n\bar{a}\right]\eta(\mathcal{W}(\xi,U)Z)$$
$$+ 2n\bar{b}\left[1 + 2nb - \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right]g(Z,U) + 2na\bar{b}S(Z,U)$$
$$+ 2n\bar{b}\left[1 + 2n(a+b) - \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right]\eta(Z)\eta(U).$$

By virtue of (5.9), we have

$$\sum_{i=1}^{2n+1} g(\mathcal{W}(e_i, \omega(\xi,e_i)Z)U,\xi) \tag{5.13}$$
$$= \left\{\frac{\bar{c}r}{2n+1}\left(\frac{1}{2n} + \bar{a} + \bar{b}\right) - 2n\bar{b} - 1\right\}\eta(\mathcal{W}(\xi,U)Z)$$
$$+ 2n\bar{a}\left[1 + 2n(a+b) - \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right]\eta(Z)\eta(U)$$
$$+ \bar{a}\left\{1 + 2nb - \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right\}S(Z,U) + a\bar{a}S^2(Z,U).$$

Finally,

$$\sum_{i=1}^{2n+1} g(\mathcal{W}(e_i, Z)\omega(\xi,e_i)U,\xi) \tag{5.14}$$
$$= \left\{1 + 2n\bar{b} - \frac{\bar{c}r}{2n+1}\left(\frac{1}{2n} + \bar{a} + \bar{b}\right)\right\}g(e_i,U)\eta(\mathcal{W}(e_i,Z)\xi)$$
$$+ \left\{\frac{\bar{c}r}{2n+1}\left(\frac{1}{2n} + \bar{a} + \bar{b}\right) - 2n\bar{a} - 1\right\}\eta(\mathcal{W}(e_i,Z)e_i)\eta(U)$$
$$+ \bar{a}S(e_i,U)\eta(\mathcal{W}(e_i,Z)\xi)$$

$$= \left\{ \frac{\bar{c}r}{2n+1} \left(\frac{1}{2n} + \bar{a} + \bar{b} \right) - 2n\bar{a} - 1 \right\} \eta(\mathcal{W}(e_i, Z)e_i)\eta(U).$$

In view of (5.6), the equation (5.14) turns into

$$(5.15) \quad \sum_{i=1}^{2n+1} g(\mathcal{W}(e_i, Z)\omega(\xi, e_i)U, \xi)$$
$$= - \left\{ \frac{\bar{c}r}{2n+1} \left(\frac{1}{2n} + \bar{a} + \bar{b} \right) - 2n\bar{a} - 1 \right\} \Big[2n(1 - a + 2nb) + \left\{ ar - \frac{2ncr}{2n+1} \left(\frac{1}{2n} + a + b \right) \right\} \Big] \eta(U)\eta(Z).$$

By virtue of (5.4), (5.11), (5.12), (5.13)and (5.15), we obtain from (5.3) that

$$(5.16) \quad \Big[\left\{ \frac{\bar{c}r}{2n+1} \left(\frac{1}{2n} + \bar{a} + \bar{b} \right) - 2n\bar{b} - 1 \right\} (1 + 2na - b)$$
$$-\bar{a} \left\{ ar + \frac{cr}{2n+1} \left(\frac{1}{2n} + a + b \right) \right\} + a\bar{a}r$$
$$+ 2na \left\{ 1 + 2n\bar{b} - \frac{\bar{c}r}{2n+1} \left(\frac{1}{2n} + \bar{a} + \bar{b} \right) \right\}$$
$$-\bar{a} \left\{ 1 + 2nb - \frac{cr}{2n+1} \left(\frac{1}{2n} + a + b \right) \right\} \Big] S(Z, U)$$
$$+ \Big[\left\{ 2n(1 - a + 2nb) + ar - \frac{2ncr}{2n+1} \left(\frac{1}{2n} + a + b \right) \right\} \times$$
$$\times \left\{ \frac{\bar{c}r}{2n+1} \left(\frac{1}{2n} + \bar{a} + \bar{b} \right) - 2n\bar{a} - 1 \right\}$$
$$+ \left\{ \frac{cr}{2n+1} \left(\frac{1}{2n} + a + b \right) - 2n(a + b) - 1 \right\} \times$$
$$\times \left\{ 2n \Big\{ - \bar{b} + \frac{\bar{c}r}{2n+1} \left(\frac{1}{2n} + \bar{a} + \bar{b} \right) - 1 - 2n\bar{b} \Big\} - \bar{a}r \right\}$$
$$-2n(\bar{a} + \bar{b}) \left\{ 1 + 2n(a + b) - \frac{cr}{2n+1} \left(\frac{1}{2n} + a + b \right) \right\} \Big] \eta(U)\eta(Z)$$
$$+ \Big[\left\{ \frac{\bar{c}r}{2n+1} \left(\frac{1}{2n} + \bar{a} + \bar{b} \right) - 2n\bar{b} - 1 \right\} \left\{ br - \frac{2ncr}{2n+1} \left(\frac{1}{2n} + a + b \right) \right\}$$
$$+ \left\{ \frac{cr}{2n+1} \left(\frac{1}{2n} + a + b \right) - 2nb - 1 \right\} \times$$
$$\times \Big\{ - 2n \left\{ 1 + 2n\bar{b} - \frac{\bar{c}r}{2n+1} \left(\frac{1}{2n} + \bar{a} + \bar{b} \right) \right\} - \bar{a}r - 2n\bar{b} \Big\}$$
$$- \bar{a} \left\{ b \parallel Q \parallel^2 - \frac{cr^2}{2n+1} \left(\frac{1}{2n} + a + b \right) \right\}$$

$$-2n\bar{b}\left\{1+2nb-\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right\}\Big]g(Z,U)$$
$$+\bar{a}(b-1)S^2(Z,U)=0.$$

From (5.16), one can easily bring out the following.

Theorem 5.2. *Let (M^{2n+1},g), $n>1$ be an LP-Sasakian manifold. Then the nature of Ricci tensor for respective semi-symmetry type curvature conditions are given in Table 1.*

6. LP-Sasakian manifold satisfying the condition $\mathcal{W}\cdot S=0$

Let $M^{2n+1}(\phi,\ \xi,\ \eta,\ g)(n>1)$, be an LP-Sasakian manifold, satisfying the condition

$$\mathcal{W}(\xi,Y)\cdot S=0, \tag{6.1}$$

i.e.

$$\mathcal{W}(\xi,Y)S(Z,U)-S(\mathcal{W}(\xi,Y)Z,U)-S(Z,\mathcal{W}(\xi,Y)U)=0,$$

i.e.

$$S(\mathcal{W}(\xi,Y)Z,U)+S(Z,\mathcal{W}(\xi,Y)U)=0. \tag{6.2}$$

Taking $U=\xi$ in (6.2) and using (2.9), we get

$$2n\eta(\mathcal{W}(\xi,Y)Z)+S(Z,\mathcal{W}(\xi,Y)\xi)=0. \tag{6.3}$$

In view of (2.7), (2.9) and (3.7), we have

$$\eta(\mathcal{W}(\xi,Y)Z)=\left[\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)-2na-2nb-1\right]\eta(Z)\eta(Y) \tag{6.4}$$
$$+\left[\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)-2nb-1\right]g(Z,Y)-aS(Z,Y).$$

$$S(Z,\mathcal{W}(\xi,Y)\xi)=\left[1+2na-\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right]S(Y,Z) \tag{6.5}$$
$$+2n\left[1+2n(a+b)-\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right]\eta(Y)\eta(Z)+bS^2(Y,Z).$$

By virtue of(6.4) and (6.5), the equation (6.3) yields

$$S^2(Y,Z)=\frac{2n}{b}\left[1+2nb-\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right]g(Y,Z) \tag{6.6}$$
$$+\frac{1}{b}\left[\frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)-1\right]S(Y,Z)$$

for all Y,Z where $b\neq 0$, provided $S^2(X,Y)=S(QX,Y)$.

This leads to Theorem 6.1.

Curvature condition	*Expression for Ricci tensor*
$R(\xi,X)\cdot R=0$ *(Obtain by* $\bar{a}=\bar{b}=\bar{c}=0$ & $a=b=c=0$ *)*	$S=2ng$, *an Einstein space.*
$R(\xi,X)\cdot C=0$ *(Obtain by* $\bar{a}=\bar{b}=\bar{c}=0$, $a=b=-\frac{1}{2n-1}$ & $c=1$ *)*	$S=\left(\frac{r}{2n}-1\right)g+(\frac{r}{2n}-2n-1)\eta\otimes\eta$, *an* $\eta-$ *Einstein space.*
$R(\xi,X)\cdot\hat{C}=0$ *(Obtain by* $\bar{a}=\bar{b}=\bar{c}=0$, $a=b=-\frac{1}{2n-1}$ & $c=0$ *)*	$S=\left(\frac{r}{2n}-1\right)g+(\frac{r}{2n}-2n-1)\eta\otimes\eta$, *an* $\eta-$ *Einstein space.*
$R(\xi,X)\cdot E=0$ *(Obtain by* $\bar{a}=\bar{b}=\bar{c}=0$, $a=b=0$ & $c=1$ *)*	$S=2ng$, *an Einstein space.*
$R(\xi,X)\cdot P=0$ *(Obtain by* $\bar{a}=\bar{b}=\bar{c}=0$, $a=-\frac{1}{2n}$ & $b=\ c=0$ *)*	$S=2ng+(\frac{r}{2n}-2n-1)\eta\otimes\eta$, *an* $\eta-$ *Einstein space.*
$R(\xi,X)\cdot H=0$ *(Obtain by* $\bar{a}=\bar{b}=\bar{c}=0$, $a=b=-\frac{1}{4n}$ & $c=0$ *)*	$S=(\frac{r+4n^2}{4n+1})g+\{\frac{r-2n(2n+1)}{4n+1}\}\eta\otimes\eta$, *an* $\eta-$ *Einstein space.*
$E(\xi,X)\cdot R=0$ *(Obtain by* $\bar{a}=\bar{b}=0$, $\bar{c}=1$, & $a=b=\ c=0$ *)*	$S=2ng$, *an Einstein space.*
$E(\xi,X)\cdot C=0$ *(Obtain by*$\bar{a}=0=\bar{b},\bar{c}=1$ $a=b=-\frac{1}{2n-1},\quad c=1)$	$S=\left(\frac{r}{2n}-1\right)g+(\frac{r}{2n}-2n-1)\eta\otimes\eta$, *an* $\eta-$ *Einstein space.*
$E(\xi,X)\cdot\hat{C}=0$ *(Obtain by* $\bar{a}=0=\bar{b},\bar{c}=1$ $a=b=-\frac{1}{2n-1}$ & $c=1)$	$S=\left(\frac{r}{2n}-1\right)g+(\frac{r}{2n}-2n-1)\eta\otimes\eta$, *an Einstein space*
$E(\xi,X)\cdot E=0$ *(Obtain by* $\bar{a}=0=\bar{b},\bar{c}=1$ $a=b=0$ & $c=1)$	$S=2ng$, *an Einstein space.*
$E(\xi,X)\cdot P=0$ *(Obtain by* $\bar{a}=0=\bar{b},\bar{c}=1$ $a=-\frac{1}{2n},b=0$ & $c=1)$	$S=2ng+(\frac{r}{2n}-2n-1)\eta\otimes\eta$, *an* $\eta-$ *Einstein space.*

TABLE 1

Curvature condition	*Expression for Ricci tensor*
$E(\xi,X)\cdot H=0$ *(Obtain by* $\bar{a}=0=\bar{b},\bar{c}=1$ $a=b=-\frac{1}{4n}$ & $c=1)$	$S=\left(\frac{r+4n^2}{4n+1}\right)g+\left\{\frac{r-2n(2n+1)}{4n+1}\right\}\eta\otimes\eta,$ *an* $\eta-$ *Einstein space.*
$\hat{C}(\xi,X)\cdot R=0$ *(Obtain by* $\bar{a}=\bar{b}=-\frac{1}{2n-1},\quad \bar{c}=0$ & $a=b=c=0$)	$2S=(r+2n)g+(r-2n)\eta\otimes\eta$ $-S^2.$
$\hat{C}(\xi,X)\cdot\hat{C}=0$ *(Obtain by* $\bar{a}=\bar{b}=-\frac{1}{2n-1},\quad \bar{c}=0,$ $a=b=-\frac{1}{2n-1}$ & $c=0)$	$(2n-1)S=(\Vert Q\Vert^2-2n)g$ $-2nr\eta\otimes\eta+2nS^2.$
$\hat{C}(\xi,X)\cdot C=0$ *(Obtain by* $\bar{a}=\bar{b}=-\frac{1}{2n-1},\quad \bar{c}=0,$ $a=b=-\frac{1}{2n-1}$ & $c=1)$	$(2n-1)S=(\Vert Q\Vert^2-2n)g$ $+\left\{\frac{r^2}{2n}-(2n+1)r\right\}\eta\otimes\eta-2nS^2$
$\hat{C}(\xi,X)\cdot E=0$ *(Obtain by* $\bar{a}=\bar{b}=-\frac{1}{2n-1},\quad \bar{c}=0$ & $a=b=c=0)$	$2S=(r+2n)g+S^2$ $+(2n-r)\left\{\frac{r}{2n(2n+1)}-1\right\}\eta\otimes\eta.$
$\hat{C}(\xi,X)\cdot P=0$ *(Obtain by* $\bar{a}=\bar{b}=-\frac{1}{2n-1},\quad \bar{c}=0$ $a=-\frac{1}{2n},\quad b=0$ & $c=0)$	$2S=(r+2n)g+(\frac{r}{2n}-2n-1)\eta\otimes\eta$ $+S^2.$
$\hat{C}(\xi,X)\cdot H=0$ *(Obtain by* $\bar{a}=\bar{b}=-\frac{1}{2n-1},\quad \bar{c}=0$ $a=b=-\frac{1}{4n}$ & $c=0)$	$\frac{6n+1}{4n}S=\left(\frac{2n+1}{4n}r+n+\frac{1}{4n}\Vert Q\Vert^2\right)g$ $+\left(\frac{r}{4n}-\frac{2n+1}{2}\right)\eta\otimes\eta-\left(1+\frac{1}{4n}\right)S^2.$
$P(\xi,X)\cdot R=0$ *(Obtain by* $\bar{a}=-\frac{1}{2n},\quad \bar{b}=0,\bar{c}=0$ & $a=b=c=0)$	$\left(\frac{1}{2n}-1\right)S=\left(\frac{r}{2n}-2n\right)g$ $+\left(\frac{r}{2n}-2n-1\right)\eta\otimes\eta+\frac{1}{2n}S^2.$
$P(\xi,X)\cdot\hat{C}=0$ *(Obtain by* $\bar{a}=-\frac{1}{2n},\quad \bar{b}=0,\bar{c}=0$ $a=b=-\frac{1}{2n-1}$ & $c=0)$	$\frac{4n^2+1}{2n}S=\left\{\frac{2n+1}{2n}r-2n-\frac{\Vert Q\Vert^2}{2n}\right\}g$ $+\left\{\frac{2n+1}{2n}r-(2n+1)^2\right\}\eta\otimes\eta+S^2.$
$P(\xi,X)\cdot C=0$ *(Obtain by* $\bar{a}=-\frac{1}{2n},\quad \bar{b}=0,\bar{c}=0$ $a=b=-\frac{1}{2n-1}$ & $c=1)$	$-\frac{4n^2+1}{2n}S=\left\{\frac{r}{2n}-(2n+1)\right\}^2\eta\otimes\eta$ $-S^2+\left(\frac{r}{2n}-2n\right)\left(\frac{r}{2n}-1\right)g$ $-\frac{1}{2n}\left(\frac{r^2}{2n}-\Vert Q\Vert^2\right)g.$
$P(\xi,X)\cdot E=0$ *(Obtain by* $\bar{a}=-\frac{1}{2n},\quad \bar{b}=0,\bar{c}=0$ $a=b=0$ & $c=1)$	$\frac{1-2n}{2n}S=\left(\frac{r}{2n}-2n\right)g$ $-\frac{1}{2n+1}\left\{\frac{r}{2n}-(2n+1)\right\}^2\eta\otimes\eta-\frac{1}{2n}S^2.$
$P(\xi,X)\cdot P=0$ *(Obtain by* $\bar{a}=-\frac{1}{2n},\quad \bar{b}=0,\bar{c}=0$ $a=-\frac{1}{2n},b=0$ & $c=0)$	$(1-2n)S=(r-4n^2)\,g-S^2.$
$P(\xi,X)\cdot H=0$ *(Obtain by*$\bar{a}=-\frac{1}{2n},\quad \bar{b}=0,\bar{c}=0$ $a=b=-\frac{1}{4n}$ & $c=0)$	$S=\left(n-\frac{1}{8n^2}\Vert Q\Vert^2\right)g+$ $+\frac{1}{2n}\left(1+\frac{1}{4n}\right)S^2.$

TABLE 2

Curvature condition	*Expression for Ricci tensor*
$H(\xi,X)\cdot R=0$ *(Obtain by* $\bar a=\bar b=-\frac{1}{4n},\quad \bar c=0$ & $a=b=c=0)$	$\frac{1-2n}{4n}S=\left(\frac{r}{4n}-n\right)g+\left(\frac{r}{4n}-\frac{1}{2}\right)\eta\otimes\eta-\frac{1}{4n}S^2.$
$H(\xi,X)\circ\hat C=0$ *(Obtain by* $\bar a=\bar b=-\frac{1}{4n},\quad \bar c=0$ $a=b=-\frac{1}{2n-1}$ & $c=0)$	$\frac{4n^2+1}{4n}S=\left\{\frac{2n+1}{4n}r-n-\frac{\Vert Q\Vert^2}{4n}\right\}g+\left\{\frac{4n+1}{4n}r-\frac{(2n+1)^2}{2}\right\}\eta\otimes\eta+\frac{1}{2n}S^2.$
$H(\xi,X)\cdot E=0$ *(Obtain by* $\bar a=\bar b=-\frac{1}{4n},\quad \bar c=0$ $a=b=0$ & $c=1)$	$\frac{1-2n}{4n}S=(\frac{r}{4n}-n)g-\frac{1}{4n}S^2+(\frac{r}{2n}-1)(\frac{1}{2}-\frac{r}{4n(2n+1)})\eta\otimes\eta.$
$H(\xi,X)\cdot P=0$ *(Obtain by* $\bar a=\bar b=-\frac{1}{4n},\quad \bar c=0,$ $a=-\frac{1}{2n},\quad b=0$ & $c=0)$	$\frac{1-2n}{4n}S=(\frac{r}{4n}-n)g+\frac{1}{2}\{2n+1-\frac{r}{2n}\}\eta\otimes\eta-\frac{1}{4n}S^2.$
$H(\xi,X)\cdot H=0$ *(Obtain by* $\bar a=\bar b=-\frac{1}{4n},\quad \bar c=0$ $a=b=-\frac{1}{4n}$ & $c=0)$	$S=\frac{1}{2n}\left(1+\frac{1}{4n}\right)S^2+\left(n-\frac{1}{8n^2}\Vert Q\Vert^2\right)g+\left(\frac{r}{4n}-\frac{2n+1}{2}\right)\eta\otimes\eta.$
$C(\xi,X)\cdot R=0$ *(Obtain by* $\bar a=\ \bar b=-\frac{1}{2n-1},\bar c=1$ $a=b=c=0)$	$\left(2-\frac{r}{2n}\right)S=2ng-S^2+(r-2n)\eta\otimes\eta.$
$C(\xi,X)\cdot\hat C=0$ *(Obtain by* $\bar a=\ \bar b=-\frac{1}{2n-1},\bar c=1$ $a=b=-\frac{1}{2n-1},c=0)$	$(2n-1-r)S=\left\{\left(r-\frac{r^2}{2n}\right)-2n+\Vert Q\Vert^2\right\}g+\left(r-\frac{r^2}{2n}\right)\eta\otimes\eta-2nS^2.$
$C(\xi,X)\cdot C=0$ *(Obtain by* $\bar a=\ \bar b=-\frac{1}{2n-1},\bar c=1$ $a=b=-\frac{1}{2n-1},c=0)$	$(2n-1-r)S=\left\{\left(r-\frac{r^2}{2n}\right)-2n+\Vert Q\Vert^2\right\}g-2nS^2.$
$C(\xi,X)\cdot P=0$ *(Obtain by* $\bar a=\ \bar b=-\frac{1}{2n-1},\bar c=1$ $a=-\frac{1}{2n},b=c=0)$	$\left(2-\frac{r}{2n}\right)S=2ng-S^2+\left(\frac{r}{2n}-1\right)\left(2n+1-\frac{r}{2n}\right)\eta\otimes\eta.$
$C(\xi,X)\cdot E=0$ *(Obtain by* $\bar a=\ \bar b=-\frac{1}{2n-1},\bar c=1$ $a=b=0,c=1)$	$\left(2-\frac{r}{2n}\right)S=2ng-S^2+\left(\frac{r}{2n}-1\right)\left\{2n(2n+1)-\frac{r}{2n+1}\right\}\eta\otimes\eta.$
$C(\xi,X)\cdot H=0$ *(Obtain by* $\bar a=\ \bar b=-\frac{1}{2n-1},\bar c=1$ $a=b=-\frac{1}{4n},c=0)$	$\frac{1}{n}\left\{\left(3n+\frac{1}{2}\right)-\left(r+\frac{r^2}{4n}\right)\right\}S=-(2+\frac{1}{2n})S^2+\left\{\frac{r}{2n}\left(1-\frac{r}{2n}\right)+2n+\frac{\Vert Q\Vert^2}{2n}\right\}g+\left(\frac{r}{2n}-2n-1\right)\left(1-\frac{r}{2n}\right)\eta\otimes\eta.$

TABLE 3

Theorem 6.1. *Let $M^{2n+1}(\phi,\ \xi,\ \eta,\ g)$ be an LP-Sasakian manifold with $\mathcal{W}\cdot S=0$. Then the Ricci tensor S admit the relation (6.6) provided $b \neq 0$.*

Now, for $b = 0$, the equation (6.6) reduces to

$$\left[\frac{cr}{2n+1}\left(\frac{1}{2n}+a\right)-1\right]\{S(Y,Z)-2ng(Y,Z)\}=0. \tag{6.7}$$

i.e,

$$S(Y,Z)=2ng(Y,Z) \quad \text{or} \quad \frac{cr}{2n+1}\left(\frac{1}{2n}+a\right)-1\neq 0 \tag{6.8}$$

Theorem 6.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$, be a LP-Sasakian manifold. Then the following conditions are equivalent:*

(a) *M is a is an Einstein space.*
(b) *M is Ricci symmetric i.e., $\nabla S = 0$.*
(c) *$P(\xi,X)\cdot S = 0$ (or $E(\xi,X)\cdot S = 0$) for all $X \in \chi(M)$.*

Example 6.3. (see [24, p. 286-287]) Let $M^3(\phi,\xi,\eta,g)$ be an LP-Sasakian manifold (M^3,g) with a ϕ-basis

$$e = e^z\frac{\partial}{\partial x},\ \phi e = e^{z-\alpha x}\frac{\partial}{\partial x},\ \xi = \frac{\partial}{\partial x},\ \text{where } \alpha \text{ is non-zero constant.}$$

Then from Koszul's formula for Lorentzian metric g, we can obtain the Levi-Civita connection as follows

$$\begin{aligned}
&\nabla_e\xi = \phi e, && \nabla_e\phi e = 0, && \nabla_e e = -\xi,\\
&\nabla_{\phi e}\xi = e, && \nabla_{\phi e}\phi e = \alpha e^z e, && \nabla_{\phi e}e = \alpha e^z\phi e,\\
&\nabla_\xi\xi = 0, && \nabla_\xi\phi e = 0, && \nabla_\xi e = 0.
\end{aligned}$$

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor R, Ricci tensor S, scalar curvature r and generalized quasi-conformal curvature tensor W as follows

$$\begin{aligned}
&R(\phi e,\xi)\xi = -\phi e, && R(e,\xi)\xi = -e, && R(e,\phi e)\phi e = (1-\alpha^2e^{2z})e,\\
&R(\phi e,\xi)\phi e = -\xi, && R(e,\xi)e = -\xi, && R(e,\phi e)e = -(1-\alpha^2e^{2z})\phi e,\\
&S(e,e) = -\alpha^2e^{2z}, && S(\phi e,\phi e) = -\alpha e^z, && S(\xi,\xi) = -2,\ r = 2(1-\alpha^2e^{2z}),\\
&\mathcal{W}(e,\xi)\xi = -\lambda_1 e, && \mathcal{W}(\phi e,\xi)\xi = -\lambda_1\phi e, &&\\
&\mathcal{W}(e,\xi)e = -\lambda_2\xi, && \mathcal{W}(\phi e,\xi)\phi e = -\lambda_2\xi, &&\\
&\mathcal{W}(e,\phi e)\phi e = \lambda_3 e, && \mathcal{W}(e,\phi e)e = -\lambda_3\phi e, &&\\
&\omega(e,\xi)\xi = -\bar{\lambda}_1 e, && \omega(\phi e,\xi)\xi = -\bar{\lambda}_1\phi e, &&\\
&\omega(e,\xi)e = -\bar{\lambda}_2\xi, && \omega(\phi e,\xi)\phi e = -\bar{\lambda}_2\xi, &&
\end{aligned}$$

where,

$$\lambda_1 = \left[1+2a-b\alpha^2e^{2z}-\frac{c}{3}(1+2a+2b)(1-\alpha^2e^{2z})\right],$$

$$\lambda_2 = \left[1 - a\alpha^2 e^{2z} + 2b - \frac{c}{3}(1+2a+2b)(1-\alpha^2 e^{2z})\right],$$
$$\lambda_3 = \left[(1-\alpha^2 e^{2z})\left(1-\frac{c}{3}\right) + (a+b)\alpha^2 e^{2z}\left(\frac{2c}{3}-1\right)\right],$$
$$\bar{\lambda}_1 = \left[1 + 2\bar{a} - \bar{b}\alpha^2 e^{2z} - \frac{\bar{c}}{3}(1+2\bar{a}+2\bar{b})(1-\alpha^2 e^{2z})\right],$$
$$\bar{\lambda}_2 = \left[1 - \bar{a}\alpha^2 e^{2z} + 2\bar{b} - \frac{\bar{c}}{3}(1+2\bar{a}+2\bar{b})(1-\alpha^2 e^{2z})\right]$$

and the components which can be obtained from these by the symmetry properties. we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows:

$$(\omega(e,\xi)\cdot\mathcal{W})(\phi e,\xi)\phi e = \bar{\lambda}_1(\lambda_2-\lambda_3)e, \qquad (\omega(e,\xi)\cdot\mathcal{W})(e,\phi e)\phi e = \bar{\lambda}_2(\lambda_2-\lambda_3)\xi,$$
$$(\omega(e,\xi)\cdot\mathcal{W})(e,\phi e)e = 0, \qquad (\omega(e,\xi)\cdot\mathcal{W})(e,\phi e)\xi = (\lambda_1\bar{\lambda}_2-\bar{\lambda}_1\lambda_3)\phi e,$$
$$(\omega(\phi e,\xi)\cdot\mathcal{W})(e,\phi e)\xi = (\bar{\lambda}_1\lambda_3-\lambda_1\bar{\lambda}_2)e, \qquad (\omega(\phi e,\xi)\cdot\mathcal{W})(e,\xi)e = \bar{\lambda}_1(\lambda_2-\lambda_3),$$
$$(\omega(\phi e,\xi)\cdot\mathcal{W})(e,\phi e)\phi e = 0, \qquad (\omega(\phi e,\xi)\cdot\mathcal{W})(\phi e,\xi)\phi e = (\bar{\lambda}_1\lambda_2-\lambda_1\bar{\lambda}_2)\phi e,$$
$$(\omega(\phi e,\xi)\cdot\mathcal{W})(e,\phi e)e = \bar{\lambda}_2(\lambda_3-\lambda_2)\xi, \qquad (\omega(e,\xi)\cdot\mathcal{W})(e,\xi)e = (\bar{\lambda}_1\lambda_2-\lambda_1\bar{\lambda}_2)e,$$
$$(\omega(e,\xi)\cdot\mathcal{W})(\phi e,\xi)e = (\bar{\lambda}_1\lambda_3-\lambda_1\bar{\lambda}_2)\xi, \qquad (\omega(\phi e,\xi)\cdot\mathcal{W})(\phi e,\xi)e = 0,$$
$$(\omega(e,\xi)\cdot\mathcal{W})(e,\xi)\phi e = 0, \qquad (\omega(\phi e,\xi)\cdot\mathcal{W})(e,\xi)\phi e = (\bar{\lambda}_1\lambda_3-\lambda_1\bar{\lambda}_2)e.$$

From the above it is clear that the manifold (M^3, g) under consideration is a semi-symmetry type LP-Sasakian manifold if $\bar{\lambda}_1 = \lambda_2 = \lambda_3$.

Lemma 6.4. *There exists a semi-symmetry type LP-Sasakian manifold provided* $\bar{\lambda}_1 = \lambda_1$, $\bar{\lambda}_2 = \lambda_2 = \lambda_3 = \bar{\lambda}_3$.

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