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SECOND ORDER PARALLEL TENSORS ON LP-SASAKIAN MANIFOLDS WITH A COEFFICIENT α

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Abstract. In 1926, Levy [3] had proved that a second order symmetric parallel nonsingular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [4] has proved that a second order parallel tensor in a Kähler space of constant holomorphic sectional curvature is a linear combination with constant coefficient of the Kählerian metric and the fundamental 2-form. In this paper, we have shown that a second order symmetric parallel tensor on Lorentzian Para Sasakian manifold (briefly LP-Sasakian) with a coefficient α (non zero Scalar function) is a constant multiple of the associated metric tensor and we have also proved that there is no non zero skew symmetric second order parallel tensor on a LP-Sasakian manifold.

1. Introduction

In 1923, Eisenhart [2] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of metric tensor is reducible. In 1926 Levy [3] obtained the necessary and sufficient conditions for the existence of such tensors. Sharma [4] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non-singular) tensor on an n-dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [4] that on a Sasakian manifold, there is no non zero parallel 2-form. In this paper we have defined LP-Sasakian manifold with a coefficient α, (non zero scalar function) and have proved the following two theorems:

Theorem 1.1. *On a LP- Sasakian manifold with a coefficient α, a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.*

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Theorem 1.2. *On a LP-Sasakian manifold with a coefficient α, there is no non zero parallel 2-form.*

Let M be an n-dimensional differentiable manifold of class c^∞ endowed with (1,1) tensor field Φ, a contravariant vector field T, a covariant vector field A and a Lorentzian metric g on M which makes T a timelike unit vector field such that the following conditions are satisfied [1].

$$A(T) = -1 \tag{1.1}$$

$$\Phi(T) = 0 \tag{1.2}$$

$$A(\Phi X) = 0 \tag{1.3}$$

$$\Phi^2 X = X + A(X)T \tag{1.4}$$

$$A(X) = g(X,T) \tag{1.5}$$

$$g(\Phi X, \Phi Y) = g(X,Y) + A(X)A(Y) \tag{1.6}$$

$$\Phi(X,Y) = g(X,\Phi Y) = g(Y,\Phi X) = \Phi(X,Y) \tag{1.7}$$

$$\Phi(X,T) = 0. \tag{1.8}$$

Then a manifold satisfying conditions (1.1)–(1.8) is called a LP-Sasakian structure (Φ, T, A, g) on M.

Definition 1.1. If in a LP-Sasakian manifold, the following relation

$$\Phi X = \frac{1}{\alpha}(\nabla_X T) \tag{1.9}$$

$$\Phi(X,Y) = \frac{1}{\alpha}(\nabla_X A(Y)) = \frac{1}{\alpha}(\nabla_X A)(Y) \tag{1.10}$$

$$\alpha(X) = \nabla_X \alpha \tag{1.11}$$

$$g(X,\overline{\alpha}) = \alpha(X) \tag{1.12}$$

$$\begin{aligned}\nabla_X \Phi(Y,Z) = {}&\\ &\alpha[\{g(X,Y) + \eta(Y)\eta(X)\}\eta(Z) + \{g(X,Z) + \eta(Z)\eta(X)\}\eta(Y)].\end{aligned} \tag{1.13}$$

hold, where ∇ denotes the Riemannian connection of the metric tensor g, then M is called a LP-Sasakian manifold with coefficient α.

2. Proofs of Theorem 1.1 and 1.2

In proving Theorems 1.1 and 1.2 we need the following theorems.

Theorem 2.1. *On a LP-Sasakian manifold with coefficient α the following holds*

$$\begin{aligned}A(R(X,Y)Z) = {}&\alpha^2[g(Y,Z)A(X) - g(X,Z)A(Y)]\\ &- [\alpha(X)\Phi(Y,Z) - \alpha(Y)\Phi(X,Z)]\end{aligned} \tag{2.1}$$

Proof. On differentiating (1.10) covariantly and using (1.11), (1.12) and (1.13) the proof follows immediately. □

Theorem 2.2. *For a LP-Sasakian manifold with coefficient α, we have:*

$$R(T,X)Y = \alpha^2 [A(Y)X + g(X,Y)T] + \alpha(Y)\Phi X - \overline{\alpha}\Phi(X,Y), \tag{2.2}$$

where $g(X,\overline{\alpha}) = \alpha(X)$.

Proof. The proof follows in an obvious manner after making use of (1.12) and (2.1). □

Theorem 2.3. *For a LP-Sasakian manifold, with a coefficient α the following holds:*

$$R(T,X)T = \beta\phi x + \alpha^2[X + A(X)T] \tag{2.3}$$

Proof. In view of equation (3.2), the proof follows immediately. □

Proof of Theorem 1.1. Let J denote a $(0,2)$–tensor field on a LP-Sasakian manifold M with a coefficient α such that $\nabla J = 0$, then it follows that

$$J(R(W,X)Y,Z) + J(Y,R(W,X)Z) = 0 \tag{2.4}$$

holds for arbitrary vector fields X, Y, Z, W on M. Substituting $W = Y = Z = T$ in (2.4) we get

$$J(R(T,X)T,T) + J(T,R(T,X)T) = 0. \tag{2.5}$$

On using Theorem 3.3, the equation (2.5) becomes

$$2\beta J(\Phi X,T) + 2\alpha^2 J(X,T) + 2\alpha^2 g(X,T)J(T,T) = 0. \tag{2.6}$$

On simplifying (2.6), we get

$$-g(X,T)J(T,T) - J(X,T) - \frac{\beta}{\alpha^2}J(\Phi X,T) = 0 \tag{2.7}$$

Replacing X by ΦY in (2.7) we get

$$J(\Phi Y,T) = g(\Phi Y,T)J(T,T) + \frac{\beta}{\alpha^2}J\left(\Phi^2 Y,T\right) \tag{2.8}$$

Using (1.4) and (1.5) in the above equation we get

$$J(\Phi Y,T) = -\frac{\beta}{\alpha^2}\left[J(T,T)A(Y) + J(Y,T)\right] \tag{2.9}$$

Using (2.7) and (2.9) we get

$$J(T,T)A(Y) + J(Y,T) = 0 \text{ if } \alpha^4 + \beta^2 \neq 0 \tag{2.10}$$

Differentiating (2.10) covariantly with respect to y we get

$$J(T,T)g(X,\Phi Y) + 2g(X,T)J(\Phi Y,T) + J(X,\Phi Y) = 0 \tag{2.11}$$

From the above equation and (1.9) we obtain

$$J(T,T)g(X,\Phi Y) = -J(X,\Phi Y) \tag{2.12}$$

Replacing Φy by y in (2.12) we get

$$J(X,Y) = -J(T,T)g(X,Y) \tag{2.13}$$

In view of the fact that $J(T,T)$ is constant which can be checked by differentiating it along any vector field on M. Thus we have proved the theorem. $\square$

Proof of Theorem 1.2. Let J be a parallel 2-form on a LP-Sasakian manifold M with a coefficient α. Then putting $W = Y = T$ in (2.4) and using Theorem 3.3 and equations (1.1)–(1.6) we get

$$\beta J\left(\Phi X, Z\right) + \alpha^2 J\left(X, Z\right) + \alpha^2 J\left(T, Z\right) A\left(X\right) + \alpha^2 J\left(T, X\right) A\left(Z\right) + J\left(T, \Phi X\right) \alpha\left(Z\right) - J\left(\overline{\alpha}, T\right) \Phi\left(X, Z\right) = 0 \tag{2.14}$$

Let us define Φ^* to be a $(2,0)$ tensor field metrically equivalent to Φ then contracting (2.14) with Φ^* and using the antisymmetry property of J and the symmetry property of Φ^*, we obtain in view of equations (1.3)–(1.6) and after simplifying the following:

$$J\left(\overline{\alpha}, T\right) = 0. \tag{2.15}$$

Substituting (2.15) in (2.14) we get

$$\beta J\left(\Phi X, Z\right) + \alpha^2 \left[J\left(X, Z\right) + J\left(T, Z\right) A\left(X\right) + J\left(T, X\right) A(Z)\right] + J\left(T, \Phi X\right) \alpha\left(Z\right) = 0. \tag{2.16}$$

On simplifying (2.16) we get

$$\beta J\left(\Phi Z, X\right) + \alpha^2 \left[J\left(Z, X\right) + J\left(T, X\right) A\left(Z\right) + J\left(T, Z\right) A(X)\right] + J\left(T, \Phi Z\right) \alpha\left(X\right) = 0. \tag{2.17}$$

On simplifying (2.16) and (2.17) we get

$$-\beta[J\left(Z, \Phi X\right) + J(X, \Phi Z)] - \alpha\left(X\right)\ J\left(\Phi Z, T\right) - \alpha\left(Z\right) J\left(\Phi X, T\right) = 0. \tag{2.18}$$

On replacing X by ΦY in (2.18) we get

$$-\beta[J\left(Z, \Phi^2 Y\right) + J\left(\Phi Y, \Phi Z\right)] - \alpha\left(\Phi Y\right) J\left(\Phi Z, T\right) - \alpha\left(Z\right) J\left(\Phi^2 Y, T\right) = 0. \tag{2.19}$$

On making use of (1.4) in the above equation, we get the following equation:

$$-\beta[J\left(Z, Y\right) + J\left(Z, T\right) A\left(Y\right) + J(\Phi Y, \Phi Z)] - \alpha\left(Z\right) J\left(Y, T\right) - \alpha\left(\Phi Y\right) J\left(\Phi Z, T\right) = 0. \tag{2.20}$$

On simplifying (2.20) we get

$$-\beta[J\left(Y, Z\right) + J\left(Y, T\right) A\left(Z\right) + J(\Phi Z, \Phi Y)] - \alpha\left(Y\right) J\left(Z, T\right) - \alpha\left(\Phi Z\right) J\left(\Phi Y, T\right) = 0. \tag{2.21}$$

In view of (2.20) and (2.21) and after simplifying we obtain

$$\beta[J\left(T, Z\right) A\left(Y\right) + J\left(T, Y\right) A(Y)] + \alpha(Z) J(T, Y) + J\left(T, \Phi Z\right) \alpha\left(\Phi Y\right) + \alpha\left(Y\right) J\left(Z, T\right) + \alpha\left(\Phi Z\right) J\left(T, \Phi Y\right) = 0. \tag{2.22}$$

Putting $Y = \overline{\alpha}$ in (2.22) and using (2.15) we get

$$\beta J(T,Z) A(\overline{\alpha}) + J(T,\Phi Z)\alpha(\Phi\overline{\alpha}) + \alpha(\overline{\alpha}) J(Z,T) = 0 \tag{2.23}$$

Let us put $\alpha\overline{\alpha} = \widehat{\alpha}$ and $\widehat{\beta} = \alpha(\Phi,\overline{\alpha})$ in (2.23) we get

$$J(Z,T)\left[\beta A(\overline{\alpha}) - \alpha(\overline{\alpha})\right] = J(T,\Phi Z)\widehat{\beta}. \tag{2.24}$$

Replacing Z by ΦZ in (2.24) we get

$$J(\Phi Z,T)\left[\beta^2 - \overline{\alpha}\right] = \widehat{\beta} J(T,Z). \tag{2.25}$$

Replacing Z by ΦZ in (2.25) we get

$$J\left(\Phi^2 Z,T\right) = \frac{\widehat{\beta}}{\overline{\alpha} - \beta^2} J(\Phi Z,T). \tag{2.26}$$

On making use of (2.25) and (1.4) in (2.26) we get

$$\frac{\overline{\alpha} - \beta^2}{\widehat{\beta}} J(Z,T) = \frac{\widehat{\beta}}{\overline{\alpha} - \beta^2}\ J(Z,T). \tag{2.27}$$

From (2.27) it follows immediately that

$$J(Z,T) = 0 \text{ unless } \left(\overline{\alpha} - \beta^2\right)^2 - \left(\widehat{\beta}\right)^2 \neq 0. \tag{2.28}$$

Using (2.28) in (2.28) we get

$$\beta J(Z,\Phi X) + \alpha^2 J(Z,X) = 0 \tag{2.29}$$

Differentiating (2.28) covariantly along Y and using the fact that $\nabla J = 0$ we get

$$J(Z,\Phi Y) = 0. \tag{2.30}$$

In view of (2.30) and (2.29), we see that $J(Y,Z) = 0$. □

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