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ON A TYPE OF TRANS-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study 3-dimensional trans-Sasakian manifolds admitting a W_2-curvature tensor. Trans-Sasakian manifolds satisfying the curvature condition $S(X,\xi).R = 0$ is also considered.

1. Introduction

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by Chinea and Gonzales [5] and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [8], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [12] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \bigoplus C_5$ ([10],[11]) coincides with the class of trans-Sasakian structures of type (α,β). In [11], the local nature of the two subclasses C_5 and C_6 of trans-Sasakian structures is characterized completely. In [4], some curvature identities and sectional curvatures for C_5 , C_6 and trans-Sasakian manifolds are obtained. It is known that [17] trans-Sasakian structures of type (0,0) , (0,β) , and (α,0) are cosymplectic, β - Kenmotsu and α-Sasakian respectively where α, $\beta \in \mathbb{R}$.

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by Marrero [10]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold. Hence a proper trans-Sasakian manifold exists only for three dimension. Three-dimensional trans-Sasakian manifolds have been studied by De and Tripathi [7], De and Sarkar [6], Shukla and Singh [15], and many others.

On the other hand, Pokhariyal and Mishra [14] have introduced new tensor fields, called W_2 and E-tensor fields, in a Riemannian manifold, and studied

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their properties. Then, Pokhariyal [13] has studied some properties of this tensor fields in a Sasakian manifold. Recently, De and Sarkar [10] have studied P-Sasakian manifolds admitting W_2 tensor field.

The curvature tensor W_2 is defined by

$$W_2(X,Y,U,V) = R(X,Y,U,V) + \frac{1}{n-1}[g(X,U)S(Y,V) - g(Y,U)S(X,V), \tag{1.1}$$

where S is a Ricci tensor of type (0,2). The notion of the quasi-conformal curvature tensor was introduced by Yano and Sawaki [19]. According to them a quasi-conformal curvature tensor is defined by

$$\begin{aligned}\widetilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad - g(X,Z)QY] - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.2}$$

where a and b are non-zero constants, R is the curvature tensor, S is the Ricci tensor, Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the Riemannian manifold $(M^n, g)(n \geq 3)$. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.2) takes the form

$$\begin{aligned}\widetilde{C}(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] = C(X,Y)Z,\end{aligned} \tag{1.3}$$

where C is the conformal curvature tensor [18].

On the other hand, the concircular curvature tensor $\tilde{Z}$ in a Riemannian manifold is defined by

$$\tilde{Z}(X,Y)U = R(X,Y)U - \frac{r}{n(n-1)}(g(Y,U)X - g(X,U)Y). \tag{1.4}$$

Again an trans-Sasakian manifold is called Einstein if the Ricci tensor S is of the form $S = \lambda g$, where λ is a constant.

The paper is organized as follows: In section 2, some preliminary results are recalled. After preliminaries in section 3, we construct some examples of 3-dimensional trans-Sasakian manifold. Then we have studied a 3-dimensional trans-Sasakian manifold satisfying $W_2 = 0$. In the next section, we have studied W_2-semisymmetric 3-dimensional trans-Sasakian manifolds. Also, we have classified 3-dimensional trans-Sasakian manifolds satisfying $\tilde{Z}.W_2 = 0$ and $C.W_2 = 0$. Finally we prove that a 3-dimensional trans-Sasakian manifold satisfying the condition $S(X,\xi).R = 0$ is an Einstein manifold, provided $\alpha, \beta =$ constant.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g),that is, ϕ is an (1,1) tensor field, ξ is a vector field ,η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X) \tag{2.3}$$

for all X and Y tangent to M([1],[2]).

The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.4}$$

for all X and Y tangent to M.

An almost contact metric structure (ϕ,ξ,η,g) on a connected manifold M is called a trans-Sasakian structure [12] if ($M \times \mathbb{R}$, J ,G) belongs to the class W_4 [8], where J is the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, f\frac{d}{df}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, for any vector fields X on M, f is a smooth function on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [3]

$$(\nabla_X\phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

for smooth functions α and β on M. Hence we say that the trans-Sasakian structure is of type (α,β). From (2.5) it follows that

$$\nabla_X\xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X\eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

An explicit example of a 3-dimensional proper trans-Sasakian manifold is constructed in [10]. In [7], Ricci tensor and curvature tensor for 3-dimensional trans-Sasakian manifolds are studied and their explicit formulae are given.

From [7] we know that for a 3-dimensional trans-Sasakian manifold

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha, \tag{2.9}$$

$$\begin{aligned} S(X, Y) = {} & (\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2))g(X, Y) \\ & - (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y) \\ & - (Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y), \end{aligned} \tag{2.10}$$

$$\begin{aligned}(2.11)\quad R(X,Y)\xi &= (\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y)-\eta(Y)(X\beta)\xi+\phi(X)\alpha\xi\\ &+\eta(X)(Y\beta)\xi+\phi(Y)\alpha\xi-(Y\beta)X+(X\beta)Y-(\phi(Y)\alpha)X+(\phi(X)\alpha)Y,\end{aligned}$$

and

$$\begin{aligned}(2.12)\qquad R(X,Y)Z = (\frac{r}{2}&+2\xi\beta-2(\alpha^2-\beta^2))(g(Y,Z)X-g(X,Z)Y\\ &-g(Y,Z)[(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\xi\\ &-\eta(X)(\phi\,\mathrm{grad}\,\alpha-\mathrm{grad}\,\beta)+(X\beta+(\phi X)\alpha)\xi]\\ &+g(X,Z)[(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\xi\\ &-\eta(Y)(\phi\,\mathrm{grad}\,\alpha-\mathrm{grad}\,\beta)+(Y\beta+(\phi Y)\alpha)\xi]\\ &-[(Z\beta+(\phi Z)\alpha)\eta(Y)+(Y\beta+(\phi Y)\alpha)\eta(Z)\\ &+\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\eta(Z)]X\\ &+[(Z\beta+(\phi Z)\alpha)\eta(X)+(X\beta+(\phi X)\alpha)\eta(Z)\\ &+\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Z)]Y,\end{aligned}$$

where S is the Ricci tensor of type $(0,2)$ and R is the curvature tensor of type $(1,3)$ and r is the scalar curvature of the manifold M.
In a 3-dimensional trans-Sasakian manifold, using (2.9), (2.11) and (2.13), equation (1.3) and (1.4) reduce to

$$(2.13)\qquad \tilde{Z}(\xi,X)Y = (\alpha^2-\beta^2-\frac{r}{6})\{g(X,Y)\xi-\eta(Y)X\},$$

$$\begin{aligned}(2.14)\quad C(\xi,Y)W = \frac{(\alpha^2-\beta^2)(n-1)(n-4)+r}{(n-1)(n-2)}&\{g(Y,W)\xi-\eta(W)Y\}\\ -\frac{1}{(n-2)}&\{S(Y,W)\xi-\eta(W)QY\},\end{aligned}$$

respectively.

3. Examples of 3-dimensional trans-Sasakian manifold

Example 3.1. We consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3, z\neq 0\}$, where (x,y,z) are standard co-ordinates of $\mathbb{R}^3$.

The vector fields

$$e_1 = z(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}),\quad e_2 = z\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1,e_3)=g(e_1,e_2)=g(e_2,e_3)=0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \varepsilon \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1,\ \phi^2 Z = -Z + \eta(Z)e_3,\ g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$, the set of all smooth vector fields on M.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g and R be the curvature tensor of M. Then we have

$$[e_1, e_2] = ye_2 - z^2 e_3, \quad [e_1, e_3] = -\frac{1}{z}e_1 \quad \text{and} \quad [e_2, e_3] = -\frac{1}{z}e_2.$$

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{e_1} e_3 = -\frac{1}{z}e_1 + \frac{1}{z^2}e_2,\ \nabla_{e_1} e_2 = -\frac{1}{2}z^2 e_3,\ \nabla_{e_1} e_1 = \frac{1}{z}e_3,\ \nabla_{e_2} e_3 = -\frac{1}{z}e_2 - \frac{1}{2}z^2 e_1,$$

$$\nabla_{e_2} e_2 = ye_1 + \frac{1}{z}e_3,\ \nabla_{e_2} e_1 = \frac{1}{2}z^2 e_3 - ye_2, \nabla_{e_3} e_3 = 0,\ \nabla_{e_3} e_2 = -\frac{1}{2}z^2 e_1,$$

$$\nabla_{e_3} e_1 = \frac{1}{2}z^2 e_2.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a trans-Sasakian structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold with $\alpha = -\frac{1}{2}z^2 \neq 0$ and $\beta = -\frac{1}{z} \neq 0$.

Example 3.2. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq 0\}$, where (x, y, z) are standard co-ordinates of $\mathbb{R}^3$.

The vector fields

$$e_1 = \frac{\partial}{\partial z} - y\frac{\partial}{\partial x},\ \ e_2 = \frac{\partial}{\partial y},\ \ e_3 = 2\frac{\partial}{\partial x}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1,\ \phi^2 Z = -Z + \eta(Z)e_3,\ g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = e_1e_2 - e_2e_1 = (\frac{\partial}{\partial z} - y\frac{\partial}{\partial x})\frac{\partial}{\partial y} - \frac{\partial}{\partial y}(\frac{\partial}{\partial z} - y\frac{\partial}{\partial x}) = \frac{\partial}{\partial x} = \frac{1}{2}e_3.$$

Similarly,

$$[e_1, e_3] = 0 \quad \text{and} \quad [e_2, e_3] = 0.$$

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{e_1}e_3 = \frac{1}{4}e_2, \quad \nabla_{e_1}e_2 = -\frac{1}{4}e_3, \quad \nabla_{e_1}e_1 = 0,$$
$$\nabla_{e_2}e_3 = -\frac{1}{4}e_1, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_1 = \frac{1}{4}e_3,$$
$$\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = -\frac{1}{4}e_1, \quad \nabla_{e_3}e_1 = \frac{1}{4}e_2.$$

We see that the structure (ϕ, ξ, η, g) satisfies the formula (2.6) for $\alpha = \frac{1}{4}$ and $\beta = 0$. Hence the manifold is a trans-Sasakian manifold of type $(\frac{1}{4}, 0)$.

4. 3-dimensional trans-Sasakian manifolds satisfying $W_2 = 0$

In this section we consider a 3-dimensional trans-Sasakian manifolds satisfying $W_2 = 0$. Then we have from (1.1)

$$R(X, Y, U, V) = \frac{1}{n-1}[g(Y, U)S(X, V) - g(X, U)S(Y, V)]. \tag{4.1}$$

Using $X = U = \xi$ in (4.1), we have

$$R(\xi, Y, \xi, V) = \frac{1}{n-1}[g(Y, \xi)S(\xi, V) - g(\xi, \xi)S(Y, V)]. \tag{4.2}$$

From (2.1), (2.9) and (2.11), we get

$$\begin{aligned} S(Y, V) = {} & 2(\alpha^2 - \beta^2 - \xi\beta)g(Y, V) + (\xi\beta)\eta(Y)\eta(V) \\ & - \{(\phi V)\alpha\}\eta(Y) - (V\beta)\eta(Y). \end{aligned} \tag{4.3}$$

If α and β are constant, then we have

$$S(Y, V) = 2(\alpha^2 - \beta^2)g(Y, V). \tag{4.4}$$

Thus we have the following:

Theorem 4.1. *A 3-dimensional trans-Sasakian manifold satisfying $W_2 = 0$ is an Einstein manifold, provided $\alpha, \beta =$ constant.*

Now using (4.4) in (4.1), we get

$$R(X, Y, U, V) = (\alpha^2 - \beta^2)[g(Y, U)g(X, V) - g(X, U)g(Y, V)]. \tag{4.5}$$

Corollary 4.1. *A 3-dimensional trans-Sasakian manifold satisfying $W_2 = 0$ is a manifold of constant curvature $(\alpha^2 - \beta^2)$, provided $\alpha, \beta=$ constant.*

5. W_2-semisymmetric 3-dimensional trans-Sasakian manifolds

A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric ([16],[9]) if $R(X,Y).R = 0$, where R is the Riemannian curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. If a Riemannian manifold satisfies

$$R(X,Y).W_2 = 0, \tag{5.1}$$

then the manifold is said to be W_2 semi-symmetric manifold.

Proposition 5.1. *Let M be an 3-dimensional trans-Sasakian manifold. Then the W_2-curvature tensor on M satisfies the condition*

$$W_2(X,Y,U,\xi) = 0. \tag{5.2}$$

From (5.1)we have

$$\begin{aligned} R(X,Y)W_2(Z,U)V - W_2(R(X,Y)Z,U)V \\ - W_2(Z,R(X,Y)U)V - W_2(Z,U)R(X,Y)V = 0. \end{aligned} \tag{5.3}$$

This equation implies

$$\begin{aligned} g(R(X,Y)W_2(Z,U)V,\xi) - g(W_2(R(X,Y)Z,U)V,\xi) \\ - g(W_2(Z,R(X,Y)U)V,\xi) - g(W_2(Z,U)R(X,Y)V,\xi) = 0. \end{aligned} \tag{5.4}$$

Putting $X = \xi$ in(5.4) we obtain

$$\begin{aligned} g(R(\xi,Y)W_2(Z,U)V,\xi) - g(W_2(R(\xi,Y)Z,U)V,\xi) \\ - g(W_2(Z,R(\xi,Y)U)V,\xi) - g(W_2(Z,U)R(\xi,Y)V,\xi) = 0. \end{aligned} \tag{5.5}$$

Using (5.4) in (5.5), we get

$$\begin{aligned} &- g(Y,W_2(Z,U)V)\xi + \eta(W_2(Z,U)V)Y + g(Y,Z)g(W_2(\xi,U)V,\xi) \\ &- \eta(Z)g(W_2(Y,U)V,\xi) + g(Y,U)g(W_2(Z,\xi)V,\xi) - \eta(U)g(W_2(Z,Y)V,\xi) \\ &+ g(Y,V)g(W_2(Z,U)\xi,\xi) - \eta(V)g(W_2(Z,U)Y,\xi) = 0. \end{aligned} \tag{5.6}$$

Taking the inner product with ξ and using (5.2) in (5.7), we obtain

$$W_2(Z,U,V,Y) = 0.$$

Then from previous Theorem and Corollary we have

Theorem 5.1. *A W_2-semisymmetric 3-dimensional trans-Sasakian manifold is an Einstein manifold and hence a manifold of constant curvature $(\alpha^2 - \beta^2)$, provided $\alpha, \beta=$ constant.*

6. 3-dimensional trans-Sasakian manifolds satisfying $\tilde{Z}(X,Y).W_2 = 0$

In this section we consider a 3-dimensional trans-Sasakian manifolds satisfying the condition

$$\tilde{Z}(X,Y).W_2 = 0. \tag{6.1}$$

This equation implies

$$\tilde{Z}(X,Y)W_2(Z,U)V - W_2(\tilde{Z}(X,Y)Z,U)V - W_2(Z,\tilde{Z}(X,Y)U)V - W_2(Z,U)\tilde{Z}(X,Y)V = 0. \tag{6.2}$$

Putting $X = \xi$ in(6.2) we obtain

$$\tilde{Z}(\xi,Y)W_2(Z,U)V - W_2(\tilde{Z}(\xi,Y)Z,U)V - W_2(Z,\tilde{Z}(\xi,Y)U)V - W_2(Z,U)\tilde{Z}(\xi,Y)V = 0. \tag{6.3}$$

Using (2.13) in (6.3), we obtain

$$\begin{aligned}(\alpha^2-\beta^2-\frac{r}{6})\{&g(Y,W_2(Z,U)V)\xi - g(W_2(Z,U)V,\xi)Y\\ &-g(Y,Z)W_2(\xi,U)V + \eta(Z)W_2(Y,U)V - g(Y,U)W_2(Z,\xi)V\\ &\eta(U)W_2(Z,U)V - g(Y,V)W_2(Z,U)\xi + \eta(V)W_2(Z,U)Y\} = 0.\end{aligned} \tag{6.4}$$

Taking the inner product with ξ and using (4.2)in (6.5), we have

$$(\alpha^2-\beta^2-\frac{r}{6})g(Y,W_2(Z,U)V) = 0. \tag{6.5}$$

Again from (2.13) we have $(\alpha^2-\beta^2-\frac{r}{6}) \neq 0$. Hence we have

$$W_2(Z,U,V,Y) = 0. \tag{6.6}$$

From the proof of Theorem 4.1 and Corollary 4.1 we have

Theorem 6.1. *A 3-dimensional trans-Sasakian manifold satisfying the condition $\tilde{Z}(X,Y).W_2 = 0$ is an Einstein manifold and hence a manifold of constant curvature $(\alpha^2-\beta^2)$, provided $\alpha,\beta=$ constant.*

7. 3-dimensional trans-Sasakian manifolds satisfying $C(X,Y).W_2 = 0$

In this section we characterize the 3-dimensional trans-Sasakian manifold satisfying the condition

$$C(X,Y).W_2 = 0. \tag{7.1}$$

This equation implies

$$C(X,Y)W_2(Z,U)V - W_2(C(X,Y)Z,U)V - W_2(Z,C(X,Y)U)V - W_2(Z,U)C(X,Y)V = 0. \tag{7.2}$$

Putting $X = \xi$ in(7.2) we obtain

$$C(\xi, Y)W_2(Z, U)V - W_2(C(\xi, Y)Z, U)V - W_2(Z, C(\xi, Y)U)V - W_2(Z, U)C(\xi, Y)V = 0. \tag{7.3}$$

Using (2.14) in (7.3), we obtain

$$\begin{aligned}(\frac{(\alpha^2-\beta^2)(n-1)(n-4)+r}{(n-1)(n-2)})\{&g(Y, W_2(Z, U)V)\xi - g(W_2(Z, U)V, \xi)Y \\ &-g(Y, Z)W_2(\xi, U)V + \eta(Z)W_2(Y, U)V - g(Y, U)W_2(Z, \xi)V \\ &\eta(U)W_2(Z, U)V - g(Y, V)W_2(Z, U)\xi + \eta(V)W_2(Z, U)Y\} = 0.\end{aligned} \tag{7.4}$$

Taking the inner product with ξ and using (4.2)in (7.5), we have

$$(\frac{(\alpha^2-\beta^2)(n-1)(n-4)+r}{(n-1)(n-2)})g(Y, W_2(Z, U)V) = 0. \tag{7.5}$$

Let U_1 and U_2 be a part of M satisfying $(\alpha^2 - \beta^2)(n-1)(n-4)+r) = 0$ and

$$W_2(Z, U, V, Y) = 0. \tag{7.6}$$

This leads to the following:

Theorem 7.1. *Let M be a 3-dimensional trans-Sasakian manifold satisfying the condition $C(X, Y).W_2 = 0$. Then either $(\alpha^2 - \beta^2)(n-1)(n-4)+r) = 0$, or M is a manifold of constant curvature $(\alpha^2 - \beta^2)$, provided $\alpha, \beta =$ constant.*

8. 3-dimensional trans-Sasakian manifolds satisfying $S(X, \xi).R = 0$

We now consider a 3-dimensional trans-Sasakian manifold satisfying the condition

$$S(X, \xi).R = 0. \tag{8.1}$$

By definition we have

$$\begin{aligned}(S(X, \xi).R)(U, V)Z &= ((X \wedge_S \xi).R)(U, V)Z \\ &= (X \wedge_S \xi)R(U, V)Z + R((X \wedge_S \xi)U, V)Z \\ &\quad + R(U, (X \wedge_S \xi)V)Z + R(U, V)(X \wedge_S \xi)Z,\end{aligned} \tag{8.2}$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y. \tag{8.3}$$

Using the definition of (8.3) in (8.2) we get by virtue of (8.1)

$$\begin{aligned}&S(\xi, R(U, V)Z)X - S(X, R(U, V)Z)\xi + R(S(\xi, U)X - S(X, U)\xi, V)Z \\ &+ R(U, S(\xi, V)X - S(X, V)\xi)Z + R(U, V)\{S(\xi, Z)X - S(X, Z)\xi\} = 0.\end{aligned} \tag{8.4}$$

Taking the inner product of (8.4) by ξ we obtain

$$S(\xi, R(U, V)Z)\eta(X) - S(X, R(U, V)Z) + S(\xi, U)\eta R(X, V)Z \tag{8.5}$$

$$- S(X,U)\eta(R(\xi,V)Z) + S(\xi,V)\eta(R(U,X)Z - S(X,V)\eta(R(U,\xi)Z)$$
$$+ S(\xi,Z)\eta(R(U,V)X) - S(X,Z)\eta(R(U,V)\xi) = 0.$$

Putting $U = Z = \xi$ in (8.5) and using (2.9) and (2.11) we get

$$S(X,V) = 2(\alpha^2 - \beta^2)^2 g(X,V) + 4(\alpha^2 - \beta^2)^2 \eta(X)\eta(V), \tag{8.6}$$

provided $\alpha, \beta =$ constant. This leads to the following:

Theorem 8.1. *3-dimensional trans-Sasakian manifold satisfying the condition $S(X,\xi).R = 0$ is an Einstein manifold, provided $\alpha, \beta =$ constant.*

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