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SEMI-SLANT PSEUDO-RIEMANNIAN SUBMERSIONS FROM INDEFINITE ALMOST CONTACT 3-STRUCTURE MANIFOLDS ONTO PSEUDO-RIEMANNIAN MANIFOLDS

S. S. SHUKLA AND UMA SHANKAR VERMA

Abstract. In this paper, we introduce the notion of a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold onto a pseudo-Riemannian manifold. We investigate the geometry of foliations determined by horizontal and vertical distributions and provide a non-trivial example. We also find a necessary and sufficient condition for a semi-slant submersion to be totally geodesic. Moreover, we check the harmonicity of such submersions.

1. Introduction

The theory of Riemannian submersions was introduced by O' Neill [14] in 1966 and Gray [9] in 1967. Several geometers studied Riemannian submersions between Riemannian manifolds equipped with some additional structures such as almost complex, almost contact etc. [7, 8, 14, 15]. It is well known that Riemannian submersions are related with physics and have their applications in Kaluza-Klein theory [4, 10, 11], Yang-Mills theory [5, 22], the theory of robotics [1], supergravity and superstring theories [11, 13].

In 1976, B. Watson defined almost Hermitian submersions between almost Hermitian manifolds and gave some differential geometric properties among fibres, base manifolds and total manifolds [23]. In 2010, Sahin introduced anti-invariant and semi-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds [18, 20]. He also gave the notion of a slant submersion from an almost Hermitian manifold onto a Riemannian manifold as a generalization of Hermitian submersions and anti-invariant submersions [19]. K. S. Park also studied slant and semi-slant submersions and obtained several interesting results [16, 17].

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In the present paper, our aim is to study semi-slant pseudo-Riemannian submersions from indefinite almost contact 3-structure manifolds onto pseudo-Riemannian manifolds. The composition of the paper is as follows. In section 2, we collect some basic definitions, results on indefinite almost contact 3-structure manifolds and pseudo-Riemannian submersions. In section 3, we define semi-slant pseudo-Riemannian submersions from indefinite almost contact 3-structure manifolds onto pseudo-Riemannian manifolds giving an example. We investigate the geometry of foliations which arise from the definition of above submersions and find a necessary and sufficient condition for submersions to be totally geodesic. We also check the harmonicity of such submersions.

2. Preliminaries:

2.1. **Indefinite Almost Contact 3-structure Manifolds:** Let M be a $(4n+3)$-dimensional Riemannian manifold and for $i = 1, 2, 3$, ϕ_i be (1,1)-type tensor fields, ξ_i vector fields, called characteristic vector fields and η_i 1-forms on M. Then (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$, is called an almost contact 3-structure on M if it satisfies [2, 3, 12]

$$\phi_i\xi_j = -\phi_j\xi_i = \xi_k, \tag{2.1}$$

$$\phi_i \circ \phi_j - \eta_j \otimes \xi_i = -\phi_j \circ \phi_i + \eta_i \otimes \xi_j = \phi_k, \tag{2.2}$$

$$\eta_i(\xi_j) = 0, \tag{2.3}$$

$$\eta_i(\phi_j) = -\eta_j(\phi_i) = \eta_k, \tag{2.4}$$

for any triad (i, j, k) of cyclic permutation in symmetric group S_3.

M is said to be an almost contact 3-structure manifold, if it is equipped with an almost contact 3-structure. Again, M is called an indefinite almost contact 3-structure manifold, if it is endowed with a pseudo-Riemannian metric g such that

$$g(\phi_iX, \phi_iY) = g(X, Y) - \varepsilon_i\eta_i(X)\eta_i(Y), \tag{2.5}$$

$$g(\xi_i, \xi_i) = \varepsilon_i, \tag{2.6}$$

$$g(\xi_i, X) = \varepsilon_i\eta_i(X), \tag{2.7}$$

$$g(\phi_iX, Y) = -g(X, \phi_iY), \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$ and $\forall i = 1, 2, 3$.

An indefinite almost contact 3-structure manifold is called

(a) cosymplectic 3-structure manifold if $\nabla\phi_i = 0$,
(b) Sasakian 3-structure manifold if

$$(\nabla_X\phi_i)Y = g(X,\ Y)\xi_i - \eta_i(Y)X, \text{ and} \tag{2.9}$$

$$\nabla_X\xi = -\phi_iX, \tag{2.10}$$

for any $X, Y \in \Gamma(TM); i = 1, 2, 3$.

2.2. **Pseudo-Riemannian Submersions:** Let $(\bar{M}^m, g)$ and (M^n, g) be two connected pseudo-Riemannian manifolds of indices $\overline{s}$ $(0 \leq \overline{s} \leq m)$ and s $(0 \leq s \leq n)$ respectively, where $m \geq n$ and $\overline{s} > s$.

A pseudo-Riemannian submersion is a smooth surjective map $f\colon \bar{M}^m \to M^n$ which satisfies the following conditions [7, 8, 9, 14]:

(i) the derivative map $f_{*_p}\colon T_p\bar{M} \to T_{f(p)}M$ is surjective at each point $p \in \bar{M}$;

(ii) the fibres $f^{-1}(q)$ of f over $q \in M$ are pseudo-Riemannian submanifolds of $\bar{M}$;

(iii) f_* preserves the length of horizontal vectors.

A vector field on $\bar{M}$ is called vertical if it is always tangent to fibres and it is called horizontal if it is always orthogonal to fibres. We denote by $\mathcal{V}$ the vertical distribution and by $\mathcal{H}$ the horizontal distribution. Also, we denote vertical and horizontal projections of a vector field E on $\bar{M}$ by vE and by hE respectively. A horizontal vector field $\bar{X}$ on $\bar{M}$ is said to be basic if $\bar{X}$ is f-related to a vector field X on M i.e. $f_*\bar{X} = X \circ f$. Thus, every vector field X on M has a unique horizontal lift $\bar{X}$ on $\bar{M}$.

We recall the following lemma for later use:

Lemma 2.1 ([7, 15])**.** *If $f\colon \bar{M} \to M$ is a pseudo-Riemannian submersion and $\bar{X}, \bar{Y}$ are basic vector fields on $\bar{M}$ that are f-related to the vector fields X, Y on M respectively, then we have the following properties*

(i) $\bar{g}(\bar{X}, \bar{Y}) = g(X, Y) \circ f$,

(ii) $h[\bar{X}, \bar{Y}]$ *is a vector field and* $f_*h[\bar{X}, \bar{Y}] = [X, Y] \circ f$,

(iii) $h(\bar{\nabla}_{\bar{X}}\bar{Y})$ *is a basic vector field f-related to $\nabla_X Y$, where $\bar{\nabla}$ and ∇ are the Levi-Civita connections on $\bar{M}$ and M respectively,*

(iv) $[E, U] \in \mathcal{V}$, *for any $U \in \mathcal{V}$ and for any vector field $E \in \Gamma(T\bar{M})$.*

A pseudo-Riemannian submersion $f\colon \bar{M} \to M$ determines tensor fields $\mathcal{T}$ and $\mathcal{A}$ of type $(1, 2)$ on $\bar{M}$ defined by formulas [7, 14, 15]

$$\mathcal{T}(E, F) = \mathcal{T}_E F = h(\bar{\nabla}_{vE}vF) + v(\bar{\nabla}_{vE}hF), \tag{2.11}$$

$$\mathcal{A}(E, F) = \mathcal{A}_E F = v(\bar{\nabla}_{hE}hF) + h(\bar{\nabla}_{hE}vF), \tag{2.12}$$

for any $E, F \in \Gamma(T\bar{M})$.

Let $\bar{X}, \bar{Y}$ be horizontal vector fields and U, V be vertical vector fields on $\bar{M}$. Then, we have

$$\mathcal{T}_U\bar{X} = v(\bar{\nabla}_U\bar{X}), \tag{2.13}$$

$$\mathcal{T}_U V = h(\bar{\nabla}_U V), \tag{2.14}$$

$$\bar{\nabla}_U\bar{X} = \mathcal{T}_U\bar{X} + h(\bar{\nabla}_U\bar{X}), \tag{2.15}$$

$$\mathcal{T}_{\bar{X}}F = 0, \tag{2.16}$$

$$\mathcal{T}_E F = \mathcal{T}_{vE}F, \tag{2.17}$$

$$\bar{\nabla}_U V = \mathcal{T}_U V + v(\bar{\nabla}_U V), \tag{2.18}$$

$$\mathcal{A}_{\bar{X}}\bar{Y} = v(\bar{\nabla}_{\bar{X}}\bar{Y}), \tag{2.19}$$

$$\mathcal{A}_{\bar{X}}U = h(\bar{\nabla}_{\bar{X}}U), \tag{2.20}$$

$$\bar{\nabla}_{\bar{X}}U = \mathcal{A}_{\bar{X}}U + v(\bar{\nabla}_{\bar{X}}U), \tag{2.21}$$

$$\mathcal{A}_U F = 0, \tag{2.22}$$

$$\mathcal{A}_E F = \mathcal{A}_{hE}F, \tag{2.23}$$

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \mathcal{A}_{\bar{X}}\bar{Y} + h(\bar{\nabla}_{\bar{X}}\bar{Y}), \tag{2.24}$$

$$h(\bar{\nabla}_U\bar{X}) = h(\bar{\nabla}_{\bar{X}}U) = \mathcal{A}_{\bar{X}}U, \tag{2.25}$$

$$\mathcal{A}_{\bar{X}}\bar{Y} = \frac{1}{2}v[\bar{X},\bar{Y}], \tag{2.26}$$

$$\mathcal{A}_{\bar{X}}\bar{Y} = -\mathcal{A}_{\bar{Y}}\bar{X}, \tag{2.27}$$

$$\mathcal{T}_U V = \mathcal{T}_V U, \tag{2.28}$$

$\forall\ E, F \in \Gamma(T\bar{M})$.

It can be easily shown that a Riemannian submersion $f\colon \bar{M} \to M$ has totally geodesic fibres if and only if $\mathcal{T}$ vanishes identically. By lemma (2.1), the horizontal distribution $\mathcal{H}$ is integrable if and only if $\mathcal{A} = 0$. Also, in view of equations (2.27) and (2.28), $\mathcal{A}$ is alternating on the horizontal distribution and $\mathcal{T}$ is symmetric on the vertical distribution.

Now, we recall the notion of harmonic maps between pseudo-Riemannian manifolds. Let $(\bar{M},\bar{g})$ and (M,g) be pseudo-Riemannain manifolds and let $f\colon \bar{M} \to M$ be a smooth map. Then the second fundamental form of the map f is given by

$$(\bar{\nabla} f_*)(X,Y) = (\nabla^f_X f_*Y) \circ f - f_*(\bar{\nabla}_X Y) \tag{2.29}$$

for all $X, Y \in \Gamma(T\bar{M})$, where ∇^f denotes the pullback connection of ∇ with respect to f and the tension field τ of f is defined by

$$\tau(f) = \mathrm{trace}(\bar{\nabla} f_*) = \sum_{i=1}^{m}(\bar{\nabla} f_*)(e_i, e_i), \tag{2.30}$$

where $\{e_1, e_2, \cdots e_m\}$ is an orthonormal frame on $\bar{M}$.

It is known that f is harmonic if and only if $\tau(f) = 0$ [6].

In this paper, we study pseudo Riemannian submersions $f\colon \bar{M} \to M$ such that fibres $f^{-1}(q)$ over $q \in M$ be pseudo-Riemannian submanifolds admitting non-lightlike vector fields.

3. Semi-Slant Pseudo-Riemannian Submersions

Definition 3.1. Let $(\bar{M}^{4m+3}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ be an indefinite almost contact 3-structure manifold and (M^n, g) be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $f\colon \bar{M} \to M$ is called a semi-slant pseudo-Riemannian submersion if structure vector fields $\bar{\xi}_i$; $i = 1, 2, 3$ are horizontal and there exists a distribution $\bar{\mathcal{D}} \subseteq \ker f_*$ such that

(i) $\ker f_* = \bar{\mathcal{D}} \oplus \bar{\mathcal{D}}^{\perp}$;

(ii) $\bar{\phi}_i(\bar{\mathcal{D}}) \subseteq \bar{\mathcal{D}}$; and

(iii) for any non-zero vector field $\bar{X}_p \in \bar{\mathcal{D}}_p^{\perp}$, the angle θ_i between $\bar{\phi}_i\bar{X}_p$ and the space $\bar{\mathcal{D}}_p^{\perp}$ is constant.

We observe that if dimension $\bar{\mathcal{D}} = 0$, then a semi-slant pseudo-Riemannian submersion $f\colon \bar{M} \to M$ is a slant pseudo-Riemannian submersion [19].

For any vector field $U \in \mathcal{V}$, we put

$$U = PU + QU, \tag{3.1}$$

where $PU \in \bar{\mathcal{D}}$ and $QU \in \bar{\mathcal{D}}^{\perp}$.

Also, for any vector field $U \in \bar{\mathcal{D}}^{\perp}$, we set

$$\bar{\phi}_i U = \psi_i U + \omega_i U, \tag{3.2}$$

where $\psi_i U$ and $\omega_i U$ are horizontal and vertical components of $\bar{\phi}_i U$ respectively.

For any vector field $\bar{X} \in \mathcal{H}$, we put

$$\bar{\phi}_i \bar{X} = t_i \bar{X} + n_i \bar{X}, \tag{3.3}$$

where $t_i\bar{X}$ and $n_i\bar{X}$ are horizontal and vertical components of $\bar{\phi}_i\bar{X}$ respectively.

Example 3.2. Let $(\mathbb{R}^{15}_6, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g}, i = 1, 2, 3)$ be an indefinite almost contact 3-structure manifold such that for any $(a_1, a_2, a_3, \cdots, a_{15})^t \in \mathbb{R}^{15}_6$,

$$\begin{aligned}
&\bar{\phi}_1\left((a_1, a_2, \cdots, a_{15})^t\right) = \\
&\qquad (-a_3, a_4, a_1, -a_2, -a_7, a_8, a_5, -a_6, -a_{11}, a_{12}, a_9, -a_{10}, 0, -a_{15}, a_{14})^t, \\
&\bar{\phi}_2\left((a_1, a_2, \cdots, a_{15})^t\right) = \\
&\qquad (-a_4, -a_3, a_2, a_1, -a_8, -a_7, a_6, a_5, -a_{12}, -a_{11}, a_{10}, a_9, a_{15}, 0, -a_{13})^t, \\
&\bar{\phi}_3\left((a_1, a_2, \cdots, a_{15})^t\right) = \\
&\qquad (-a_2, a_1, -a_4, a_3, -a_6, a_5, -a_8, a_7, -a_{10}, a_9, -a_{12}, a_{11}, -a_{14}, a_{13}, 0)^t,
\end{aligned}$$

$\bar{\xi}_1 = \frac{\partial}{\partial x_{13}}$, $\bar{\xi}_2 = \frac{\partial}{\partial x_{14}}$, $\bar{\xi}_3 = \frac{\partial}{\partial x_{15}}$, $\bar{\eta}_1 = dx_{13}$, $\bar{\eta}_2 = dx_{14}$, $\bar{\eta}_3 = dx_{15}$, signature of $\bar{g} = (-, -, +, +, -, -, +, +, -, -, +, +, +, +, +)$ and let $(\mathbb{R}^7_2, g)$ be a pseudo-Riemannian manifold.

Define a submersion $f\colon \{\mathbb{R}^{15}_6; (x_1, x_2, \ldots, x_{15})^t\} \to \{\mathbb{R}^7_2; (y_1, y_2, \ldots, y_7)^t\}$ by

$$\begin{aligned}
f\left((x_1, x_2, \cdots, x_{15})^t\right) \longmapsto &(x_5 \sin\alpha - x_7\cos\alpha,\ x_6 \sin\alpha + x_8 \cos\alpha, \\
&x_9 \sin\alpha - x_{11}\cos\alpha,\ x_{10}\sin\alpha + x_{12}\cos\alpha,\ x_{13},\ x_{14},\ x_{15})^t,
\end{aligned}$$

where α is a real number.

The vertical distribution $\mathcal{V}$ is

$$\begin{aligned}
\mathrm{Span}\Big\{&\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \cos\alpha\frac{\partial}{\partial x_5} + \sin\alpha\frac{\partial}{\partial x_7}, -\cos\alpha\frac{\partial}{\partial x_6} + \sin\alpha\frac{\partial}{\partial x_8}, \\
&\cos\alpha\frac{\partial}{\partial x_9} + \sin\alpha\frac{\partial}{\partial x_{11}}, -\cos\alpha\frac{\partial}{\partial x_{10}} + \sin\alpha\frac{\partial}{\partial x_{12}}\Big\}.
\end{aligned}$$

We have $\bar{\mathcal{D}} = \operatorname{Span}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right\} \subset \mathcal{V}$ and

$$\bar{\mathcal{D}}^{\perp} = \operatorname{Span}\left\{\cos\alpha\frac{\partial}{\partial x_5} + \sin\alpha\frac{\partial}{\partial x_7},\ -\cos\alpha\frac{\partial}{\partial x_6} + \sin\alpha\frac{\partial}{\partial x_8},\right.$$
$$\left.\cos\alpha\frac{\partial}{\partial x_9} + \sin\alpha\frac{\partial}{\partial x_{11}},\ -\cos\alpha\frac{\partial}{\partial x_{10}} + \sin\alpha\frac{\partial}{\partial x_{12}}\right\}.$$

Then, $\bar{\mathcal{D}}$ is invariant with respect to $\bar{\phi}_i, i = 1, 2, 3$ and semi-slant angles $\theta_1 = \cos^{-1}(\tan 2\alpha)$, $\theta_2 = \theta_3 = \frac{\pi}{2}$.

Proposition 3.3. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). If U, V are vertical and $\bar{X}, \bar{Y}$ are horizontal vector fields on $\bar{M}$, then we have*

$$\bar{g}(\omega_i QU,\ V) = -\bar{g}(U,\ \omega_i QV), \tag{3.4}$$
$$\bar{g}(t_i\bar{X},\ \bar{Y}) = -\bar{g}(\bar{X}, t_i\bar{Y}), \tag{3.5}$$
$$\bar{g}(\psi_i QU,\ \bar{X}) = -\bar{g}(U,\ n_i\bar{X});\ i = 1, 2, 3. \tag{3.6}$$

Proof. Using equations (2.8), (3.1) and (3.2), for any vector fields $U, V \in \mathcal{V}$, we have

$$\bar{g}(\bar{\phi}_i PU + \psi_i QU + \omega_i QU,\ V) = -\bar{g}(U,\ \bar{\phi}_i PV + \psi_i QV + \omega_i QV),$$

which gives

$$\bar{g}(\omega_i QU,\ V) = -\bar{g}(U,\ \omega_i QV).$$

Similarly, for any vector fields $U, V \in \mathcal{V}$ and $\bar{X}, \bar{Y} \in \mathcal{H}$, we have equations (3.5) and (3.6). □

Theorem 3.4. *Let $f\colon \bar{M} \to M$ be a pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, f is a semi-slant Riemannian submersion if and only if there exists $\lambda \in [0, 1]$ such that*

$$(Q\omega_i)^2 = \lambda\bar{\phi}_i^2. \tag{3.7}$$

Moreover, if θ_i is semi-slant angle of the submersion, then $\lambda = \cos^2\theta_i; i = 1, 2, 3$.

Proof. Let Z be any non-zero vector field in the distribution $\bar{\mathcal{D}}^{\perp}$. Then, we have

$$\cos\theta_i = \frac{\bar{g}(\bar{\phi}_i Z, Q\omega_i Z)}{|\bar{\phi}_i Z|\,|Q\omega_i Z|}. \tag{3.8}$$

Again, we have

$$\cos\theta_i = \frac{|Q\omega_i Z|}{|\bar{\phi}_i Z|}. \tag{3.9}$$

Now, from equations (2.8), (3.2) and (3.8) we have

$$\cos\theta_i = -\frac{\bar{g}(Z,\ Q\omega_i Q\omega_i Z)}{|Q\omega_i Z|\,\left|\bar{\phi}_i Z\right|}. \tag{3.10}$$

In view of equations (3.9) and (3.10), we have

$$\cos^2\theta_i = \frac{\bar{g}(Z,(Q\omega_i)^2 Z)}{\bar{g}(Z,(\bar{\phi}_i)^2 Z)}. \tag{3.11}$$

Now, equation (3.11) implies that $\cos^2\theta_i =$ constant if and only if $(Q\omega_i)^2$ and $\bar{\phi}_i^2$ are conformally parallel.

Hence, $(Q\omega_i)^2 = \lambda\bar{\phi}_i^2$, for some $\lambda \in [0,\ \infty)$.

Again, from equations (3.7) and (3.11), we have $\lambda = \cos^2\theta_i$. Consequently, we have $\lambda \in [0,\ 1]$. □

Corollary 3.5. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M},\bar{\phi}_i,\bar{\xi}_i,\bar{\eta}_i,\bar{g})$ onto a pseudo-Riemannian manifold (M,g). If θ_i is a semi-slant angle of the submersion, then for $U,V \in \bar{\mathcal{D}}^\perp; i = 1,2,3,$ we have*

$$\bar{g}(Q\omega_i U,\ Q\omega_i V) = (\bar{g}(U,V) - \varepsilon_i\bar{\eta}_i(U)\bar{\eta}_i(V))\cos^2\theta_i, \tag{3.12}$$

$$\bar{g}(\psi_i U,\ \psi_i V) = (\bar{g}(U,V) - \varepsilon_i\bar{\eta}_i(U)\bar{\eta}_i(V))\sin^2\theta_i - \bar{g}(P\omega_i U,\ P\omega_i V). \tag{3.13}$$

Proof. For any $U,V \in \bar{\mathcal{D}}^\perp$, using equations (3.1) and (3.2), we have

$$\bar{g}(Q\omega_i U,\ V) = -\bar{g}(U,\ Q\omega_i V).$$

On replacing V by $Q\omega_i V$ and using equation (3.7), above equation implies

$$\bar{g}(Q\omega_i U,\ Q\omega_i V) = -\lambda\bar{g}(U,\ (\bar{\phi}_i)^2 V).$$

Now, in view of equation (2.5), we have equation (3.12).

Similarly, we can obtain equation (3.13). □

Lemma 3.6. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M},\bar{\phi}_i,\bar{\xi}_i,\bar{\eta}_i,\bar{g})$ onto a pseudo-Riemannian manifold (M,g). Then, we have*

$$\bar{\phi}_i\bar{\mathcal{D}}^\perp \subseteq \mathcal{H} \oplus_{\text{ortho}} (\psi_i\bar{\mathcal{D}}^\perp)^\perp, \tag{3.14}$$

$$\begin{aligned}\bar{\phi}_i^2\bar{\mathcal{D}}^\perp \subseteq{}& t_i\psi_i\bar{\mathcal{D}}^\perp \oplus_{\text{ortho}} n_i\psi_i\bar{\mathcal{D}}^\perp \oplus_{\text{ortho}} \psi_i\omega_i\bar{\mathcal{D}}^\perp \oplus_{\text{ortho}} P^2(\omega_i\bar{\mathcal{D}}^\perp)\\ &\oplus_{\text{ortho}} Q^2(\omega_i\bar{\mathcal{D}}^\perp) \oplus_{\text{ortho}} Q(P\omega_i\bar{\mathcal{D}}^\perp) \oplus_{\text{ortho}} P(Q\omega_i\bar{\mathcal{D}}^\perp);\end{aligned} \tag{3.15}$$

for $i = 1,2,3$.

Proof. The first result is obvious as $\psi_i\bar{\mathcal{D}}^\perp \subset \mathcal{H}$.

From equations (3.1), (3.2) and (3.3), we have

$$\bar{\phi}_i\bar{\mathcal{D}}^\perp \subseteq \psi_i\bar{\mathcal{D}}^\perp \oplus P\omega_i\bar{\mathcal{D}}^\perp \oplus Q\omega_i\bar{\mathcal{D}}^\perp.$$

Now, by using equations (3.1), (3.2) and (3.3), we get (3.15). □

Theorem 3.7. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, $(\psi_i \bar{\mathcal{D}}^{\perp})^{\perp}$ is invariant with respect to $\bar{\phi}_i; i = 1, 2, 3$.*

Proof. Let $X \in (\psi_i \bar{\mathcal{D}}^{\perp})^{\perp}, i = 1, 2, 3$. Then, for any $U \in \bar{\mathcal{D}}^{\perp}$, we have

$$\begin{aligned}\bar{g}(\bar{\phi}_i X,\ Q\omega_i U) &= -\bar{g}(X,\ \bar{\phi}_i(Q\omega_i U))\\ &= -\bar{g}(X,\ P\omega_i(Q\omega_i U) + (Q\omega_i)^2 U + \psi_i(Q\omega_i U))\\ &= 0.\end{aligned}$$

Again, for any $U \in \bar{\mathcal{D}}^{\perp}$,

$$\begin{aligned}\bar{g}(\bar{\phi}_i X,\ U) &= -\bar{g}(X,\ \phi_i U)\\ &= -\bar{g}(X,\ \psi_i U + P\omega_i U + Q\omega_i U)\\ &= 0.\end{aligned}$$

Thus, $\bar{\phi}_i X \in (\psi_i \bar{\mathcal{D}}^{\perp})^{\perp}$. Hence, $\bar{\phi}_i(\psi_i \bar{\mathcal{D}}^{\perp})^{\perp} \subseteq (\psi_i \bar{\mathcal{D}}^{\perp})^{\perp}$. $\square$

Lemma 3.8. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, for any $U, V \in \mathcal{V}$, $\bar{X}, \bar{Y} \in \mathcal{H}$ and $i = 1, 2, 3$, we have*

$$\begin{aligned}& h(\bar{\nabla}_{PU}(\bar{\phi}_i PV)) + v(\bar{\nabla}_{PU}(\bar{\phi}_i PV)) + h(\bar{\nabla}_{QU}(\bar{\phi}_i PV)) + v(\bar{\nabla}_{QU}(\bar{\phi}_i PV))\\ &\quad + h(\bar{\nabla}_{PU}(\psi_i QV)) + \mathcal{T}_{PU}(\psi_i QV) + \mathcal{T}_{PU}(\omega_i QV) + v(\bar{\nabla}_{PU}(\omega_i QV))\\ &\quad + h(\bar{\nabla}_{QU}(\psi_i QV)) + \mathcal{T}_{QU}(\psi_i QV) + h(\bar{\nabla}_{QU}(\omega_i QV)) + \mathcal{T}_{QU}(\omega_i QV)\\ &= (\bar{\nabla}_U \bar{\phi}_i)V + t_i(\mathcal{T}_U V) + n_i(\mathcal{T}_U V) + \bar{\phi}_i(P(v\bar{\nabla}_U V))\\ &\quad + \psi_i(Q(v\bar{\nabla}_U V)) + \omega_i(Q(v\bar{\nabla}_U V)),\end{aligned} \tag{3.16}$$

$$\begin{aligned}& h\bar{\nabla}_{\bar{X}}(t_i\bar{Y}) + \mathcal{A}_{\bar{X}}(t_i\bar{Y}) + \mathcal{A}_{\bar{X}}(n_i\bar{Y}) + v(\bar{\nabla}_{\bar{X}}(n_i\bar{Y}))\\ &= (\bar{\nabla}_{\bar{X}}\bar{\phi}_i)\bar{Y} + t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i(P(\mathcal{A}_{\bar{X}}\bar{Y}))\\ &\quad + \psi(Q(\mathcal{A}_{\bar{X}}\bar{Y})) + \omega_i(Q(\mathcal{A}_{\bar{X}}\bar{Y})),\end{aligned} \tag{3.17}$$

$$\begin{aligned}& \mathcal{A}_{\bar{X}}(\bar{\phi}_i PU) + v(\bar{\nabla}_{\bar{X}}(\bar{\phi}_i PU)) + h(\bar{\nabla}_{\bar{X}}(\psi_i QU))\\ &+ \mathcal{A}_{\bar{X}}(\psi_i QU) + \mathcal{A}_{\bar{X}}(\omega_i QU) + v(\bar{\nabla}_{\bar{X}}(\omega_i QU))\\ &= (\bar{\nabla}_{\bar{X}}\bar{\phi}_i)U + t_i(\mathcal{A}_{\bar{X}}U) + n_i(\mathcal{A}_{\bar{X}}U) + \bar{\phi}_i(P(v\bar{\nabla}_{\bar{X}}U))\\ &+ \psi_i(Q(v\bar{\nabla}_{\bar{X}}U)) + \omega_i(Q(v\bar{\nabla}_{\bar{X}}U)),\end{aligned} \tag{3.18}$$

$$\begin{aligned}& h(\bar{\nabla}_U(t_i\bar{X})) + \mathcal{T}_U(t_i\bar{X}) + \mathcal{T}_U(n_i\bar{X}) + v(\bar{\nabla}_U(n_i\bar{X}))\\ &= (\bar{\nabla}_U\bar{\phi}_i)\bar{X} + t_i(h\bar{\nabla}_U\bar{X}) + n_i(h\bar{\nabla}_U\bar{X}) + \bar{\phi}_i(P(\mathcal{T}_U\bar{X}))\\ &\quad + \psi_i(Q(\mathcal{T}_U\bar{X})) + \omega_i(Q(\mathcal{T}_U\bar{X})).\end{aligned} \tag{3.19}$$

Proof. For $U, V \in \mathcal{V}$, we have

$$(\bar{\nabla}_U \bar{\phi}_i)V + \bar{\phi}_i(\bar{\nabla}_U V) = \bar{\nabla}_{PU+QU}(\bar{\phi}_i(PV + QV)),$$

which gives equation (3.16). Similarly, we can obtain other equations. □

Lemma 3.9. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite cosymplectic 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, for any $U, V \in \mathcal{V}$ $\bar{X}, \bar{Y} \in \mathcal{H}$ and $i = 1, 2, 3$, we have*

$$\begin{aligned} &\mathcal{T}_{PU}(\bar{\phi}_i PV) + h(\bar{\nabla}_{QU}(\bar{\phi}_i PV)) + h(\bar{\nabla}_{PU}(\psi_i QV)) \\ &\quad + \mathcal{T}_{PU}(\omega_i QV) + h(\bar{\nabla}_{QU}(\psi_i QV)) + h(\bar{\nabla}_{QV}(\omega_i QV) \\ &= t_i(\mathcal{T}_U V) + \psi_i(Q(v\bar{\nabla}_U V)); \end{aligned} \tag{3.20}$$

$$\begin{aligned} &v(\bar{\nabla}_{PU}(\bar{\phi}_i PV)) + v(\bar{\nabla}_{QU}(\bar{\phi}_i PV)) + \mathcal{T}_{PU}(\psi_i QV) \\ &\quad + v(\bar{\nabla}_{PU}(\omega_i QV)) + \mathcal{T}_{QU}(\psi_i QV) + \mathcal{T}_{QU}(\omega_i QV) \\ &= n_i(\mathcal{T}_U V) + \bar{\phi}_i P(v\bar{\nabla}_U V) + \omega_i(Q(v\bar{\nabla}_U V)). \end{aligned} \tag{3.21}$$

$$h(\bar{\nabla}_{\bar{X}}(t_i\bar{Y})) + \mathcal{A}_{\bar{X}}(n_i\bar{Y}) = t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \psi_i(Q(\mathcal{A}_{\bar{X}}\bar{Y})); \tag{3.22}$$

$$\mathcal{A}_{\bar{X}}(t_i\bar{Y}) + v(\bar{\nabla}_{\bar{X}}(n_i\bar{Y})) = n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_i(Q(\mathcal{A}_{\bar{X}}\bar{Y})) + \bar{\phi}_i P(\mathcal{A}_{\bar{X}}\bar{Y}). \tag{3.23}$$

$$\begin{aligned} \mathcal{A}_{\bar{X}}(\bar{\phi}_i PU) + h(\bar{\nabla}_{\bar{X}}(\psi_i QU)) + \mathcal{A}_{\bar{X}}(\omega_i QU) = t_i(\mathcal{A}_{\bar{X}}U) \\ + \psi_i(Q(v\bar{\nabla}_{\bar{X}}U)); \end{aligned} \tag{3.24}$$

$$\begin{aligned} &v(\bar{\nabla}_{\bar{X}}(\bar{\phi}_i PU)) + \mathcal{A}_{\bar{X}}(\psi_i QU) + v(\bar{\nabla}_{\bar{X}}(\omega_i QU)) \\ &\qquad = n_i(\mathcal{A}_{\bar{X}}U) + \bar{\phi}_i P(v\bar{\nabla}_{\bar{X}}U) + \omega_i Q(v\bar{\nabla}_{\bar{X}}U). \end{aligned} \tag{3.25}$$

$$h(\bar{\nabla}_U(t_i\bar{X})) + \mathcal{T}_U(n_i\bar{X}) = t_i(h\bar{\nabla}_U\bar{X}) + \psi_i(Q\mathcal{T}_U\bar{X}); \tag{3.26}$$

$$\mathcal{T}_U(t_i\bar{X}) + v(\bar{\nabla}_U(n_i\bar{X})) = n_i(h\bar{\nabla}_U\bar{X}) + \bar{\phi}_i(P\mathcal{T}_U\bar{X}) + \omega_i Q(\mathcal{T}_U\bar{X}). \tag{3.27}$$

Proof. For $U, V \in \mathcal{V}$, we have

$$(\bar{\nabla}_U \bar{\phi}_i)V + \bar{\phi}_i(\bar{\nabla}_U V) = \bar{\nabla}_{PU+QU}(\bar{\phi}_i(PV + QV)),$$

which gives

$$\begin{aligned} &t_i(\mathcal{T}_U V) + n_i(\mathcal{T}_U V) + \bar{\phi}_i(P(v\bar{\nabla}_U V)) + \psi_i(Q(v\bar{\nabla}_U V)) + \omega_i(Q(v\bar{\nabla}_U V)) \\ &= \mathcal{T}_{PU}(\bar{\phi}_i PV) + h(\bar{\nabla}_{QU}(\bar{\phi}_i PV)) + h(\bar{\nabla}_{PU}(\psi_i QV)) \\ &\quad + \mathcal{T}_{PU}(\omega_i QV) + h(\bar{\nabla}_{QU}(\psi_i QV)) + h(\bar{\nabla}_{QV}(\omega_i QV) \\ &\quad + v(\bar{\nabla}_{PU}(\bar{\phi}_i PV)) + v(\bar{\nabla}_{QU}(\bar{\phi}_i PV)) + \mathcal{T}_{PU}(\psi_i QV) \\ &\quad + v(\bar{\nabla}_{PU}(\omega_i QV)) + \mathcal{T}_{QU}(\psi_i QV) + \mathcal{T}_{QU}(\omega_i QV). \end{aligned}$$

On equating horizontal and vertical components in above equation, we get equations (3.20) and (3.21).
Similarly, we can obtain other equations. $\square$

By using similar steps as in Lemma 3.9, we have

Lemma 3.10. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite Sasakian 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, for any $U, V \in \mathcal{V}$ $\bar{X}, \bar{Y} \in \mathcal{H}$ and $i = 1, 2, 3$, we have*

$$\begin{aligned}(3.28)\quad & \bar{g}(U,V)\bar{\xi}_i + t_i(\mathcal{T}_U V) + n_i(h\bar{\nabla}_U V) + \bar{\phi}_i P(v(\bar{\nabla}_U V)) \\ &\quad + \psi_i Q(v\bar{\nabla}_U V) + \omega_i Q(v\bar{\nabla}_U V) \\ &= h(\bar{\nabla}_{PU}(\bar{\phi}_i PV)) + v(\bar{\nabla}_{PU}(\bar{\phi}_i PV)) \\ &\quad + h(\bar{\nabla}_{QU}(\bar{\phi}_i PV)) + v(\bar{\nabla}_{QU}(\bar{\phi}_i PV)) \\ &\quad + h(\bar{\nabla}_{PU}(\psi_i QV)) + \mathcal{T}_{PU}(\psi_i QV) + \mathcal{T}_{PU}(\omega_i QV) + v(\bar{\nabla}_{PU}(\omega_i QV)) \\ &\quad + h(\bar{\nabla}_{QU}(\psi_i QV)) + \mathcal{T}_{QU}(\psi_i QV) + h(\bar{\nabla}_{QU}(\omega_i QV)) + \mathcal{T}_{QU}(\omega_i QV),\end{aligned}$$

$$\begin{aligned}(3.29)\quad & h(\bar{\nabla}_{\bar{X}}(t_i\bar{Y})) + \mathcal{A}_{\bar{X}}(t_i\bar{Y}) + \mathcal{A}_{\bar{X}}(n_i\bar{Y}) + v(\bar{\nabla}_{\bar{X}}(n_i\bar{Y})) \\ &= \bar{g}(\bar{X},\ \bar{Y})\bar{\xi}_i - \varepsilon_i\bar{\eta}_i(\bar{Y})\bar{X} + t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) \\ &\quad + \bar{\phi}_i P(\mathcal{A}_{\bar{X}}\bar{Y}) + \psi_i Q(\mathcal{A}_{\bar{X}}\bar{Y}) + \omega_i Q(\mathcal{A}_{\bar{X}}\bar{Y}),\end{aligned}$$

$$\begin{aligned}(3.30)\quad & \mathcal{A}_{\bar{X}}(\bar{\phi}_i PU) + v(\bar{\nabla}_{\bar{X}}(\bar{\phi}_i PU)) + h(\bar{\nabla}_{\bar{X}}(\psi_i QU)) \\ &\quad + \mathcal{A}_{\bar{X}}(\psi_i QU) + \mathcal{A}_{\bar{X}}(\omega_i QU) + v(\bar{\nabla}_{\bar{X}}(\omega_i QU)) \\ &= \bar{g}(\bar{X},\ U)\bar{\xi}_i + t_i(\mathcal{A}_{\bar{X}}U) + n_i(\mathcal{A}_{\bar{X}}U) \\ &\quad + \bar{\phi}_i P(v\bar{\nabla}_{\bar{X}}U) + \psi Q(v\bar{\nabla}_{\bar{X}}U) + \Omega_i Q(v\bar{\nabla}_{\bar{X}}U),\end{aligned}$$

$$\begin{aligned}(3.31)\quad & h(\bar{\nabla}_U(t_i\bar{X})) + \mathcal{T}_U(t_i\bar{X}) + \mathcal{T}_U(n_i\bar{X}) + v(\bar{\nabla}_U(n_i\bar{X})) \\ &= \bar{g}(U,\ \bar{X})\bar{\xi}_i - \varepsilon\bar{\eta}_i(\bar{X})U + t_i(h\bar{\nabla}_U\bar{X}) + n_i(h\bar{\nabla}_U\bar{X}) \\ &\quad + \bar{\phi}_i P(\mathcal{T}_U\bar{X}) + \psi_i Q(\mathcal{T}_U\bar{X}) + \omega_i Q(v\bar{\nabla}_U\bar{X}).\end{aligned}$$

Theorem 3.11. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite cosymplectic 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the fibres of f are totally geodesic if and only if*

$$\bar{\nabla}_U(\bar{\phi}_i V) = \bar{\phi}_i(v\bar{\nabla}_U V), \tag{3.32}$$

for any $U, V \in \mathcal{V}$ and $i = 1, 2, 3$.

Proof. Let $U, V \in \mathcal{V}$. For $i = 1, 2, 3$, using equation (3.1), we have

$$\bar{\nabla}_U(\bar{\phi}_i V) = (\bar{\nabla}_U\bar{\phi}_i)V + \bar{\phi}_i(\bar{\nabla}_{PU}PV + \bar{\nabla}_{PU}QV + \bar{\nabla}_{QU}PV + \bar{\nabla}_{QU}QV).$$

Again, using equations (2.14) and (3.1), above equation gives

$$\bar{\nabla}_U(\bar{\phi}_i V) = (\bar{\nabla}_U \bar{\phi}_i)V + \bar{\phi}_i(\mathcal{T}_U V) + \bar{\phi}_i(v\bar{\nabla}_U V).$$

As the manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ is cosymplectic, from above equation, we have

$$\bar{\nabla}_U(\bar{\phi}_i V) = \bar{\phi}_i(\mathcal{T}_U V) + \bar{\phi}_i(v\bar{\nabla}_U V). \tag{3.33}$$

Now, fibres are totally geodesic if and only if $\mathcal{T}_U V = 0$. So the proof follows from equation (3.33). ☐

Theorem 3.12. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the horizontal distribution $\mathcal{H}$ defines a totally geodesic foliation if and only if*

$$\begin{aligned} &\bar{g}(t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \psi_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU) \\ &+\bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i P(v\bar{\nabla}_X Y) + \omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) \\ &+\bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i P(v\bar{\nabla}_X Y) + \omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) = 0, \end{aligned} \tag{3.34}$$

for any $\bar{X}, \bar{Y} \in \mathcal{H}$, $U \in \mathcal{V}$ and $i = 1, 2, 3$.

Proof. Let $\bar{X}, \bar{Y} \in \mathcal{H}$. Then, for any $U \in \mathcal{V}$, from equation (2.5), we have

$$\bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y},\ U) = \bar{g}(\bar{\phi}_i(\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i U).$$

By splitting horizontal and vertical components and using equations (3.1), (3.2), and (3.3), we have

$$\begin{aligned} \bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y},\ U) &= \bar{g}(t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) + \bar{g}(t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU) \\ &\quad +\bar{g}(t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) + \bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) \\ &\quad +\bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU) + \bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) \\ &\quad +\bar{g}(\bar{\phi}_i P(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) + \bar{g}(\bar{\phi}_i P(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU) \\ &\quad +\bar{g}(\bar{\phi}_i P(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) + \bar{g}(\psi_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) \\ &\quad +\bar{g}(\psi_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU) + \bar{g}(\psi_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) \\ &\quad +\bar{g}(\omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) + \bar{g}(\omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU) \\ &\quad +\bar{g}(\omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) \\ &= \bar{g}(t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU) + \bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) \\ &\quad +\bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) + \bar{g}(\bar{\phi}_i P(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) \\ &\quad +\bar{g}(\bar{\phi}_i P(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) + \bar{g}(\psi_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU) \\ &\quad +\bar{g}(\omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) + \bar{g}(\omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) \\ &= \bar{g}(t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \psi_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU) \\ &\quad +\bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i P(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU) \\ &\quad +\bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i P(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU), \end{aligned}$$

which implies $\bar{\nabla}_{\bar{X}}\bar{Y} \in \mathcal{H}$ if and only if right side of above equation vanishes. So the proof follows from above equation. ☐

Corollary 3.13. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, following statements are equivalent:*

(a) *The horizontal distribution $\mathcal{H}$ defines a totally geodesic foliation,*

(b)
$$\begin{aligned}&\bar{g}(t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \psi_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_i QU)\\ &\quad + \bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i P(v\bar{\nabla}_X Y) + \omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{\phi}_i PU)\\ &\quad + \bar{g}(n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i P(v\bar{\nabla}_X Y) + \omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_i QU) = 0,\end{aligned}$$

(c)
$$\bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y},\ t_i\psi_i QU + \psi_i Q\omega_i QU) = 0,$$

(d)
$$\begin{aligned}&\bar{g}(n_i t_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i P n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_i Q n_i(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i^2 P(v\bar{\nabla}_{\bar{X}}\bar{Y})\\ &\quad + n_i\psi_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{\phi}_i P\omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_i Q\omega_i Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \ U) = 0,\end{aligned}$$

for all $\bar{X}, \bar{Y} \in \mathcal{H}, U \in \mathcal{V}$ and $i = 1, 2, 3$.

Proof. In theorem (3.12), we have proved $(a) \iff (b)$. Similarly, we can prove $(b) \iff (c)$, $(c) \iff (d)$ and $(d) \iff (a)$. □

Theorem 3.14. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite cosymplectic 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the horizontal distribution $\mathcal{H}$ defines a totally geodesic foliation if and only if*

$$\begin{aligned}\bar{g}(h\bar{\nabla}_{\bar{X}}(t_i\bar{Y}) + h\bar{\nabla}_{\bar{X}}(n_i\bar{Y}),\ \psi_i QU) + \bar{g}(v\bar{\nabla}_{\bar{X}}(t_i\bar{Y}),\ \bar{\phi}_i PU + \omega_i QU)&\\ + \bar{g}(v\bar{\nabla}_{\bar{X}}(n_i\bar{Y}),\ \bar{\phi}_i PU + \omega_i QU) = 0,&\end{aligned} \tag{3.35}$$

for all $\bar{X}, \bar{Y} \in \mathcal{H}$ and $i = 1, 2, 3$.

Proof. Let $\bar{X}, \bar{Y} \in \mathcal{H}, U \in \mathcal{V}$. Then, we have
$$\bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y},\ U) = \bar{g}(\bar{\nabla}_{\bar{X}}(\bar{\phi}_i\bar{Y}),\ \bar{\phi}_i U).$$
By using equations (3.2) and (3.3), above equation implies

$$\begin{aligned}\bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y},\ U) &= \bar{g}(h\bar{\nabla}_{\bar{X}}(t_i\bar{Y}) + h\bar{\nabla}_{\bar{X}}(n_i\bar{Y}),\ \psi_i QU)\\ &\quad + \bar{g}(v\bar{\nabla}_{\bar{X}}(t_i\bar{Y}),\ \bar{\phi}_i PU + \omega_i QU) + \bar{g}(v\bar{\nabla}_{\bar{X}}(n_i\bar{Y}),\ \bar{\phi}_i PU + \omega_i QU).\end{aligned} \tag{3.36}$$

The distribution $\mathcal{H}$ defines a totally geodesic foliation if and only if $\bar{\nabla}_{\bar{X}}\bar{Y} \in \mathcal{H}$. So the proof follows from equation (3.36). □

Theorem 3.15. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the vertical distribution $\mathcal{V}$ defines a totally geodesic foliation if and only if*

$$\bar{g}(t_i(h\bar{\nabla}_U V) + \psi_i Q(v\bar{\nabla}_U V),\ t_i\bar{X}) + \bar{g}(n_i(h\bar{\nabla}_U V) + \bar{\phi}_i P(v\bar{\nabla}_U V) + \omega_i Q(v\bar{\nabla}_U V),\ n_i\bar{X}) = 0, \tag{3.37}$$

for all $U, V \in \mathcal{V},\ \bar{X} \in \mathcal{H}$ *and* $i = 1, 2, 3$.

Proof. Let $U, V \in \mathcal{V}, \bar{X} \in \mathcal{H}$. Using equation (2.5), we have

$$\bar{g}(\bar{\nabla}_U V,\ \bar{X}) = \bar{g}(\bar{\phi}_i(\bar{\nabla}_U V),\ \bar{\phi}_i\bar{X}) + \varepsilon_i\bar{\eta}_i(\bar{\nabla}_U V)\bar{\eta}_i(\bar{X}).$$

By splitting vertical and horizontal components in above equation and using equations (3.2) and (3.3), we get

$$\bar{g}(\bar{\nabla}_U V,\ \bar{X}) = \bar{g}(t_i(h\bar{\nabla}_U V) + \psi_i Q(v\bar{\nabla}_U V),\ t_i\bar{X}) + \bar{g}(n_i(h\bar{\nabla}_U V) + \bar{\phi}_i P(v\bar{\nabla}_U V) + \omega_i Q(v\bar{\nabla}_U V),\ n_i\bar{X}) + \varepsilon_i\bar{\eta}_i(\bar{\nabla}_U V)\bar{\eta}_i(\bar{X}). \tag{3.38}$$

The vertical distribution $\mathcal{V}$ defines a totally geodesic foliation if and only if $\bar{\nabla}_U V \in \mathcal{V}$. This completes the proof. ☐

Theorem 3.16. *Let* $f\colon \bar{M} \to M$ *be a semi-slant pseudo-Riemannian submersion from an indefinite cosymplectic 3-structure manifold* $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ *onto a pseudo-Riemannian manifold* (M, g)*. Then, the vertical distribution* $\mathcal{V}$ *defines a totally geodesic foliation if and only if*

$$\bar{g}(h\bar{\nabla}_U(\bar{\phi}_i PV) + h\bar{\nabla}_U(\psi_i QV) + h\bar{\nabla}_U(\omega_i QV),\ t_i\bar{X}) + \bar{g}(v\bar{\nabla}_U(\bar{\phi}_i PV) + v\bar{\nabla}_U(\psi_i QV) + v\bar{\nabla}_U(\omega_i QV),\ n_i\bar{X}) = 0, \tag{3.39}$$

for all $U, V \in \mathcal{V}, \bar{X} \in \mathcal{H}, i = 1, 2, 3$.

Proof. Let $U, V \in \mathcal{V}$ and $\bar{X} \in \mathcal{H}$.
Using equation (2.5), we have

$$\bar{g}(\bar{\nabla}_U V,\ \bar{X}) = \bar{g}(\bar{\phi}_i(\bar{\nabla}_U V),\ \bar{\phi}_i\bar{X}) + \varepsilon_i\bar{\eta}_i(\bar{\nabla}_U V)\bar{\eta}_i(\bar{X}).$$

As the manifold $\bar{M}$ is cosymplectic, we have $(\bar{\nabla}_U\bar{\phi}_i)V = 0$. So, by using equations (3.2) and (3.3), we get

$$\bar{g}(\bar{\nabla}_U V,\ \bar{X}) = \bar{g}(h\bar{\nabla}_U(\bar{\phi}_i PV) + h\bar{\nabla}_U(\psi_i QV) + h\bar{\nabla}_U(\omega_i QV),\ t_i\bar{X}) + \bar{g}(v\bar{\nabla}_U(\bar{\phi}_i PV) + v\bar{\nabla}_U(\psi_i QV) + v\bar{\nabla}_U(\omega_i QV),\ n_i\bar{X}) + \varepsilon_i\bar{\eta}_i(\bar{\nabla}_U V)\bar{\eta}_i(\bar{X}). \tag{3.40}$$

The equation (3.40) implies that $\bar{\nabla}_U V \in \mathcal{V}$ if and only if equation (3.39) is satisfied. ☐

Now, using similar steps as in theorem 22 and theorem 24 of [21], we have

Theorem 3.17. *Let* $f\colon \bar{M} \to M$ *be a semi-slant pseudo-Riemannian submersion from indefinite cosymplectic 3-structure manifold* $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ *onto a pseudo-Riemannian manifold* (M, g)*. Then, the submersion* f *is an affine map on* $\mathcal{H}; i = 1, 2, 3$.

Theorem 3.18. *Let* $f\colon \bar{M} \to M$ *be a semi-slant pseudo-Riemannian submersion from indefinite cosymplectic 3-structure manifold* $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ *onto a pseudo-Riemannian manifold* (M, g)*. Then, the submersion* f *is an affine map if and only if* $h(\bar{\nabla}_E hF) + \mathcal{A}_{hE}vF + \mathcal{T}_{vE}vF$ *is* f*-related to* $\nabla_X Y$*, for any* $E, F \in \Gamma(T\bar{M}); i = 1, 2, 3$.

Theorem 3.19. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the submersion map f is totally geodesic if and only if*

$$\mathcal{T}_U V + \mathcal{A}_{\bar{X}} V + h\bar{\nabla}_U \bar{Y} = 0, \tag{3.41}$$

for any $U, V \in \mathcal{V}, \bar{X}, \bar{Y} \in \mathcal{H}$ and $\forall\, i = 1, 2, 3$.

Proof. Let $E = \bar{X} + U,\ F = \bar{Y} + V \in \Gamma(T\bar{M})$. In view of equation (2.29) and theorem (3.17), by splitting E, F in horizontal and vertical components, we have

$$\begin{aligned}(\bar{\nabla} f_*)(E,\ F) &= (\bar{\nabla} f_*)(U, V) + (\bar{\nabla} f_*)(\bar{X}, V) + (\bar{\nabla} f_*)(U, \bar{Y})\\ &= -f_*(h(\bar{\nabla}_U V + \bar{\nabla}_{\bar{X}} V + \bar{\nabla}_U \bar{Y})),\end{aligned}$$

which gives

$$(\bar{\nabla} f_*)(E,\ F) = -f_*(\mathcal{T}_U V + \mathcal{A}_{\bar{X}} V + h\bar{\nabla}_U \bar{Y}). \tag{3.42}$$

The submersion map f is totally geodesic if and only if $\bar{\nabla} f_* = 0$. So the proof follows from equation (3.42). □

Theorem 3.20. *Let $f\colon \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from indefinite almost contact 3-structure manifold $(\bar{M}^{4m+3}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold (M^n, g). If the fibres $f^{-1}(q)$ of f over $q \in M$ are totally geodesic, then f is a harmonic map.*

Proof. The tension field $\tau(f)$ of the map $f\colon \bar{M} \to M$ is defined as

$$\tau(f) = \mathrm{trace}(\bar{\nabla} f_*). \tag{3.43}$$

Let $\{e_1, e_2, \ldots, e_{4m+3-n}, e_{4m+3-n+1} = \bar{e}_1, \bar{e}_2, \ldots\ e_{4m+3} = \bar{e}_n\}$ be an orthonormal basis of $\Gamma(T\bar{M})$, where $\{e_1, e_2, \ldots, e_{4m+3-n}\}$ is an orthonormal basis of $\mathcal{V}$ and $\{\bar{e}_1, \bar{e}_2, \ldots, e_{4m+3} = \bar{e}_n\}$ is an orthonormal basis of $\mathcal{H}$. Then, we have

$$\tau(f) = \sum_{i=1}^{4m+3-n} \bar{g}(e_i,\ e_i)(\bar{\nabla} f_*)(e_i,\ e_i) + \sum_{j=1}^{n} \bar{g}(\bar{e}_j,\ \bar{e}_j)(\bar{\nabla} f_*)(\bar{e}_j,\ \bar{e}_j). \tag{3.44}$$

For any vertical vector fields $U, V \in \mathcal{V}$, using equation (2.14), we have

$$(\bar{\nabla} f_*)(U, V) = (\nabla^f_U(f_* V))\circ f - f_*(\bar{\nabla}_U V) = -f_*(h\bar{\nabla}_U V) = -f_*(\mathcal{T}_U V), \tag{3.45}$$

where ∇^f is the pullback connection of ∇ with respect to f. For any horizontal vector fields $\bar{X}, \bar{Y} \in \mathcal{H}$, which are f-related to $X, Y \in \Gamma(TM)$ respectively, Lemma 2.1 and Theorem 3.17 imply

$$\begin{aligned}(\bar{\nabla} f_*)(\bar{X}, \bar{Y}) &= (\nabla^f_{\bar{X}}(f_* \bar{Y})) \circ f - f_*(\bar{\nabla}_{\bar{X}} \bar{Y})\\ &= (\nabla_{f_* \bar{X}}(f_* \bar{Y})) \circ f - f_*(h\bar{\nabla}_{\bar{X}} \bar{Y}) = 0.\end{aligned} \tag{3.46}$$

In view of equations (3.44), (3.45), (3.46) and Theorem 3.18, we get

$$\tau(f) = -\sum_{i=1}^{4m+3-n} \bar{g}(e_i,\ e_i) f_*(\mathcal{T}_{e_i} e_i). \tag{3.47}$$

Now, if the fibres $f^{-1}(q)$ of f over $q \in M$ are totally geodesic, then $\mathcal{T} = 0$. So the proof of the theorem follows from equation (3.47). □

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S. S. Shukla,
Department of Mathematics,
University of Allahabad,
Allahabad, PIN-211002,
India
E-mail address: ssshukla_au@rediffmail.com

Uma Shankar Verma,
Department of Mathematics,
University of Allahabad,
Allahabad, PIN-211002,
India
E-mail address: umashankarverma7@gmail.com