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$t-$BALANCING NUMBERS, PELL NUMBERS AND SQUARE TRIANGULAR NUMBERS

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Abstract. Let $t \geq 2$ be an integer. In this work we get all integer solutions of the Diophantine equation $8r^2 + 8tr + 1 = y^2$ in order to determine the general terms of all $t-$balancing numbers for which $2t^2 - 1$ is prime. Later we obtain some formulas for the sums of Pell, Pell–Lucas, balancing and Lucas–balancing numbers in terms of $t-$balancing numbers and also we deduce the general terms of all $t-$balancing numbers in terms of square triangular numbers.

1. Preliminaries.

A positive integer n is called a balancing number (see [1] and [3]) if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1.1}$$

holds for some positive integer r which is called cobalancing number (or balancer). If n is a balancing number with balancer r, then from (1.1) one has $\frac{(n-1)n}{2} = rn + \frac{r(r+1)}{2}$ and so

$$r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2} \quad \text{and} \quad n = \frac{2r+1+\sqrt{8r^2+8r+1}}{2}. \tag{1.2}$$

Let B_n denote the n^{th} balancing number, and let b_n denote the n^{th} cobalancing number. Then $B_1 = 1, B_2 = 6, B_{n+1} = 6B_n - B_{n-1}$ and $b_1 = 0, b_2 = 2$, $b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \geq 2$. The zeros of the characteristic equation $x^2 - 6x + 1 = 0$ for balancing numbers are $\alpha_1 = 3 + \sqrt{8}$ and $\beta_1 = 3 - \sqrt{8}$. Ray derived some nice results on balancing and cobalancing numbers in his Phd thesis (*Balancing and Cobalancing Numbers*, Department of Maths., National Institute of Technology, Rourkela, India, 2009). Since x is a balancing number if and only if $8x^2 + 1$ is a perfect square, he set $y^2 - 8x^2 = 1$ for some integer $y \neq 0$. The fundamental solution is $(y_1, x_1) = (3, 1)$. So $y_n + x_n\sqrt{8} = (3+\sqrt{8})^n$

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and similarly $y_n - x_n\sqrt{8} = (3-\sqrt{8})^n$ for $n \geq 1$. Thus $x_n = \frac{(3+\sqrt{8})^n - (3-\sqrt{8})^n}{2\sqrt{8}}$ which is the Binet formula for balancing numbers and is denoted by B_n. Let $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ be the roots of the characteristic equation for Pell (and also Pell–Lucas) numbers defined by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ (and $Q_0 = Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$) for $n \geq 2$. Since $\alpha^2 = 3+\sqrt{8}$ and $\beta^2 = 3-\sqrt{8}$, the Binet formula for balancing numbers is $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$. Similarly $b_n = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$.

From (1.2), we note that B_n is a balancing number if and only if $8B_n^2+1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. Thus

$$C_n = \sqrt{8B_n^2+1} \text{ and } c_n = \sqrt{8b_n^2+8b_n+1} \tag{1.3}$$

are integers called the n^{th} Lucas–balancing and n^{th} Lucas–cobalancing number. Their Binet formulas are $C_n = \frac{\alpha^{2n}+\beta^{2n}}{2}$ and $c_n = \frac{\alpha^{2n-1}+\beta^{2n-1}}{2}$ (for further details see [7, 8, 9, 10]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects (see [4, 5, 6, 12]). Recently in [2], Dash, Ota and Dash considered the $t-$balancing numbers for an integer $t \geq 2$. A positive integer n is called a $t-$balancing number if

$$1+2+\cdots+n = (n+1+t)+(n+2+t)+\cdots+(n+r+t) \tag{1.4}$$

holds for some positive integer r which is called $t-$cobalancing (or $t-$balancer) number. For example

- $2, 14, 84, 492, 2870, \cdots$ are $0-$balancing numbers with $0-$balancers $1, 6, 35, 204, 1189, \cdots$;
- $5, 34, 203, 1188, 6929, \cdots$ are $1-$balancing numbers with $1-$balancers $2, 14, 84, 492, 2870, \cdots$;
- $3, 8, 25, 54, 153, \cdots$ are $2-$balancing numbers with $2-$balancers $1, 3, 10, 22, 63, \cdots$;
- $6, 11, 45, 74, 272, \cdots$ are $3-$balancing numbers with $3-$balancers $2, 4, 18, 30, 112, \cdots$.

(Here we note that $0-$and $1-$balancing numbers can be given in terms of balancing numbers, indeed, $B_n^0 = b_{n+1}, b_n^0 = B_n, C_n^0 = c_{n+1}, c_n^0 = C_n$ and $B_n^1 = B_{n+1}-1, b_n^1 = b_{n+1}, C_n^1 = C_{n+1}, c_n^1 = c_{n+1}$, that is why it is assumed that $t \geq 2$).

From (1.4) we see that

$$r = \frac{-(2n+2t+1)+\sqrt{8n^2+8n(1+t)+(2t+1)^2}}{2} \text{ and} \tag{1.5}$$

$$n = \frac{(2r-1)+\sqrt{8r^2+8tr+1}}{2}.$$

Let B_n^t denote the n^{th} $t-$balancing number and let b_n^t denote the n^{th} $t-$cobalancing number. Then from (1.5), we see that B_n^t is a $t-$balancing number if and only if $8(B_n^t)^2 + 8B_n^t(1+t) + (2t+1)^2$ is a perfect square and b_n^t is a $t-$cobalancing number if and only if $8(b_n^t)^2 + 8tb_n^t + 1$ is a perfect square. So

$$(1.6)\quad C_n^t = \sqrt{8(B_n^t)^2 + 8B_n^t(1+t) + (2t+1)^2} \ \text{ and } \ c_n^t = \sqrt{8(b_n^t)^2 + 8tb_n^t + 1}$$

are integers which are called the n^{th} Lucas $t-$balancing and n^{th} Lucas $t-$cobalancing number.

2. Results.

In the present paper, we want to determine the general terms of all $t-$balancing numbers. But we first determine the set of all positive integer solutions of the Diophantine equation

$$8r^2 + 8tr + 1 = y^2. \tag{2.1}$$

Let us explain why? We note that for giving any $t-$cobalancing number r, $8r^2 + 8tr + 1$ is a perfect square. So we let $8r^2 + 8tr + 1 = y^2$ for some integer $y \neq 0$. Thus from (2.1), we deduce that $2(2r+t)^2 - y^2 = 2t^2 - 1$. So putting $x = 2r + t$, we get the Pell equation

$$2x^2 - y^2 = 2t^2 - 1. \tag{2.2}$$

Now let Δ be a positive non–square discriminant and let $O_\Delta = \{x + y\rho_\Delta : x, y \in \mathbb{Z}\}$, where $\rho_\Delta = \sqrt{\frac{\Delta}{4}}$ if $\Delta \equiv 0 (\text{mod } 4)$, or $\frac{1+\sqrt{\Delta}}{2}$ if $\Delta \equiv 1 (\text{mod } 4)$. So O_Δ is a subring of $\mathbb{Q}(\sqrt{\Delta}) = \{x + y\sqrt{\Delta} : x, y \in \mathbb{Q}\}$. Then the unit group O_Δ^* is defined to be the group of units of the ring O_Δ. For the quadratic form $F(x,y) = ax^2 + bxy + cy^2$ of discriminant $\Delta = b^2 - 4ac$, we can write $F(x,y) = \frac{\left(xa + y\frac{b+\sqrt{\Delta}}{2}\right)\left(xa + y\frac{b-\sqrt{\Delta}}{2}\right)}{a}$. So the module M_F of F is the $O_\Delta-$module $M_F = \{xa + y\frac{b+\sqrt{\Delta}}{2} : x, y \in \mathbb{Z}\} \subset \mathbb{Q}(\sqrt{\Delta})$. Therefore we get $(u + v\rho_\Delta)(xa + y\frac{b+\sqrt{\Delta}}{2}) = x'a + y'\frac{b+\sqrt{\Delta}}{2}$, where

$$[x' \ y'] = \begin{cases} [x \ \ y] \begin{bmatrix} u - \frac{b}{2}v & av \\ -cv & u + \frac{b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 0 (\text{mod } 4) \\ [x \ \ y] \begin{bmatrix} u + \frac{1-b}{2}v & av \\ -cv & u + \frac{1+b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 1 (\text{mod } 4). \end{cases} \tag{2.3}$$

So there is a bijection $\Psi : \{(x,y) : F(x,y) = m\} \to \{\gamma \in M_F : N(\gamma) = am\}$ for solving the Diophantine equation $F(x,y) = m$, that is, $ax^2 + bxy + cy^2 = m$. The action of $O_{\Delta,1}^* = \{\alpha \in O_\Delta^* : N(\alpha) = 1\}$ on the set $\Omega = \{(x,y) : F(x,y) = m\}$ of integral solutions of the equation $F(x,y) = m$ is most interesting when Δ is a positive non–square since $O_{\Delta,1}^*$ is infinite. Therefore the orbit of each solution will be infinite and so the set Ω is either empty or infinite. Since $O_{\Delta,1}^*$ can be explicitly determined, the set Ω is satisfactorily described by the representation of such a list, called a set of representatives of the orbits.

Let ε_Δ be the smallest unit of O_Δ that is grater than 1 and let $\tau_\Delta = \varepsilon_\Delta$ if $N(\varepsilon_\Delta) = 1$; or ε_Δ^2 if $N(\varepsilon_\Delta) = -1$. Then every $O^*_{\Delta,1}$ orbit of integral solutions of $F(x,y) = m$ contains a solution $(x,y) \in \mathbb{Z}^2$ such that $0 \leq y \leq U$, where $U = \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1 - \frac{1}{\tau_\Delta})$ if $am > 0$; or $U = \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1 + \frac{1}{\tau_\Delta})$ if $am < 0$. So for finding a set of representatives of the $O^*_{\Delta,1}$ orbits of integral solutions of $F(x,y) = m$, we must find for each integer y such that $0 \leq y \leq U$, all integers x that satisfy $F(x,y) = m$. If $F(x,y) = m$, then $\Delta y^2 + 4am = (2ax + by)^2$ and so $x = \frac{-by \pm \sqrt{\Delta y^2 + 4am}}{2a}$.

Here we notice that there are one, two, three (maybe or more) sets of representatives depending on t for the Pell equation $2x^2 - y^2 = 2t^2 - 1$. For example, for $t = 3$, the set of representatives is $\{[\pm 3, 1]\}$; for $t = 5$, the set of representatives is $\{[\pm 5, 1], [\pm 7, 7]\}$; for $t = 37$, the set of representatives is $\{[\pm 37, 1], [\pm 41, 25], [\pm 43, 31], [\pm 47, 41]\}$. To determine all (positive) integer solutions of $2x^2 - y^2 = 2t^2 - 1$ we have to put some restrictions on t. From now on, we assume that t is an integer such that $2t^2 - 1$ is a prime.

Theorem 2.1. *Let $2t^2 - 1$ be a prime for an integer $t \geq 2$. Then for the Pell equation $2x^2 - y^2 = 2t^2 - 1$, we have*

(1) *The set of all positive integer solutions is $\Omega = \{(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})\}$, where*

$$[x_{2n+1} \quad y_{2n+1}] = [t \quad 1]M^n \text{ for } n \geq 0$$
$$[x_{2n} \quad y_{2n}] = [t \quad -1]M^n \text{ for } n \geq 1,$$

and $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$.

(2) *The n^{th} power of M is*

$$M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}.$$

(3) *The set of all positive integer solutions can be given in terms of balancing and Lucas–balancing numbers, that is,*

$$(x_{2n+1}, y_{2n+1}) = (tC_n + 2B_n, 4tB_n + C_n) \text{ for } n \geq 0$$
$$(x_{2n}, y_{2n}) = (tC_n - 2B_n, 4tB_n - C_n) \text{ for } n \geq 1$$

or in terms of Pell numbers, that is,

$$(x_{2n+1}, y_{2n+1}) = ((P_{2n} + P_{2n-1})t + P_{2n}, 2tP_{2n} + P_{2n} + P_{2n-1})$$
$$(x_{2n}, y_{2n}) = ((P_{2n} + P_{2n-1})t - P_{2n}, 2tP_{2n} - P_{2n} - P_{2n-1})$$

for $n \geq 1$.

Proof. (1) For the Pell equation $2x^2 - y^2 = 2t^2 - 1$, we get $\tau_8 = 3 + 2\sqrt{2}$ and $8(y^2 + 2t^2 - 1)$ is a square only for $y = 1$ in the range $0 \leq y \leq U$ and in this case $x = \pm t$. Therefore, we find that there is exactly one $O^*_{8,1}$ set of representative of the orbits and that $\{[\pm t \quad 1]\}$ is a set of representatives. $[t \quad 1]M^n$ generates the solutions (x_{2n+1}, y_{2n+1}) for $n \geq 0$ and $[t \quad -1]M^n$ generates the solutions (x_{2n}, y_{2n}) for $n \geq 1$, where $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ by (2.3). So the set of all positive integer solutions of $2x^2 - y^2 = 2t^2 - 1$ is $\Omega = \{(x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})\}$, where $[x_{2n+1} \quad y_{2n+1}] = [t \quad 1]M^n$ for $n \geq 0$ and $[x_{2n} \quad y_{2n}] = [t \quad -1]M^n$ for $n \geq 1$.

(2) We prove it by induction on n. Let us assume that $n = 1$. Then since $C_1 = 3$ and $B_1 = 1$, this relation is true. Let us assume that this relation is satisfied for $n - 1$, that is,

$$M^{n-1} = \begin{bmatrix} C_{n-1} & 4B_{n-1} \\ 2B_{n-1} & C_{n-1} \end{bmatrix}.$$

Then we get

$$M \cdot M^{n-1} = \begin{bmatrix} 3C_{n-1} + 8B_{n-1} & 12B_{n-1} + 4C_{n-1} \\ 2C_{n-1} + 6B_{n-1} & 8B_{n-1} + 3C_{n-1} \end{bmatrix}. \tag{2.4}$$

Since $3C_{n-1} + 8B_{n-1} = C_n$, $3B_{n-1} + C_{n-1} = B_n, C_{n-1} + 3B_{n-1} = B_n$ and $8B_{n-1} + 3C_{n-1} = C_n$, (2.4) becomes

$$M \cdot M^{n-1} = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix} = M^n.$$

(3) From (1) and (2), it is easily seen that $(x_{2n+1}, y_{2n+1}) = (tC_n + 2B_n, 4tB_n + C_n)$ for $n \geq 0$ and $(x_{2n}, y_{2n}) = (tC_n - 2B_n, 4tB_n - C_n)$ for $n \geq 1$. The last assertion is obvious since $P_{2n} = 2B_n$ and $P_{2n} + P_{2n-1} = C_n$. $\square$

Hence we can give the following main result.

Theorem 2.2. (1) *For balancing numbers, we have*

$$\begin{aligned} 16B_nC_n + 32B_n^2 + 2C_n^2 + 2 &= (4b_{n+1} + 2)^2 \\ 24B_nC_n + 32B_n^2 + 4C_n^2 &= 2c_{n+1}(4b_{n+1} + 2) \\ 8B_n^2 + 2C_n^2 + 8B_nC_n - 1 &= c_{n+1}^2 \end{aligned}$$

for $n \geq 1$.

(2) *The general terms of all* $t-$*balancing numbers can be given in terms of balancing numbers as*

$$\begin{aligned} B_{2n-1}^t &= \frac{(4B_n + C_n - 1)t - (2B_n + C_n + 1)}{2} \\ b_{2n-1}^t &= \frac{(C_n - 1)t - 2B_n}{2} \\ C_{2n-1}^t &= (4b_{n+1} + 2)t - c_{n+1} \\ c_{2n-1}^t &= 4tB_n - C_n \\ B_{2n}^t &= \frac{(4B_n + C_n - 1)t + (2B_n + C_n - 1)}{2} \end{aligned}$$

$$\begin{aligned} b^t_{2n} &= \frac{(C_n-1)t+2B_n}{2} \\ C^t_{2n} &= (4b_{n+1}+2)t+c_{n+1} \\ c^t_{2n} &= 4tB_n+C_n \end{aligned}$$

for $n \geq 1$.

(3) *The general terms of balancing numbers can be given in terms of* $t-$*balancing numbers as*

$$B_n = \frac{b^t_{2n}-b^t_{2n-1}}{2} \quad \text{and} \quad C_n = \frac{c^t_{2n}-c^t_{2n-1}}{2} \text{ for } n \geq 1$$
$$b_n = \frac{C^t_{2n-2}+C^t_{2n-3}-4t}{8t} \quad \text{and} \quad c_n = \frac{C^t_{2n-2}-C^t_{2n-3}}{2} \text{ for } n \geq 2.$$

(4) *Binet formulas for* $t-$*balancing numbers are*

$$\begin{aligned} B^t_{2n-1} &= t\left(\frac{\alpha^{2n+1}+\beta^{2n+1}-2}{4}\right) - \frac{\alpha^{2n+1}-\beta^{2n+1}+2\sqrt{2}}{4\sqrt{2}} \\ b^t_{2n-1} &= t\left(\frac{\alpha^{2n}+\beta^{2n}-2}{4}\right) - \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}} \\ C^t_{2n-1} &= t\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{\sqrt{2}}\right) - \frac{\alpha^{2n+1}+\beta^{2n+1}}{2} \\ c^t_{2n-1} &= t\left(\frac{\alpha^{2n}-\beta^{2n}}{\sqrt{2}}\right) - \frac{\alpha^{2n}+\beta^{2n}}{2} \\ B^t_{2n} &= t\left(\frac{\alpha^{2n+1}+\beta^{2n+1}-2}{4}\right) + \frac{\alpha^{2n+1}-\beta^{2n+1}-2\sqrt{2}}{4\sqrt{2}} \\ b^t_{2n} &= t\left(\frac{\alpha^{2n}+\beta^{2n}-2}{4}\right) + \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}} \\ C^t_{2n} &= t\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{\sqrt{2}}\right) + \frac{\alpha^{2n+1}+\beta^{2n+1}}{2} \\ c^t_{2n} &= t\left(\frac{\alpha^{2n}-\beta^{2n}}{\sqrt{2}}\right) + \frac{\alpha^{2n}+\beta^{2n}}{2} \end{aligned}$$

for $n \geq 1$.

(5) *The general terms of Pell and Pell–Lucas numbers can be given in terms of* $t-$*balancing numbers as*

$$P_{2n} = b^t_{2n} - b^t_{2n-1}, \;\; P_{2n+1} = \frac{C^t_{2n}+C^t_{2n-1}}{4t},$$
$$Q_{2n} = \frac{b^t_{4n}-b^t_{4n-1}}{b^t_{2n}-b^t_{2n-1}} \;\; \text{and} \;\; Q_{2n+1} = C^t_{2n} - C^t_{2n-1}$$

for $n \geq 1$.

Proof. **(1)** Note that $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}, C_n = \frac{\alpha^{2n}+\beta^{2n}}{2}$ and $b_n = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$. So

$$\begin{aligned}
&16B_nC_n + 32B_n^2 + 2C_n^2 + 2\\
&= 16(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})(\frac{\alpha^{2n}+\beta^{2n}}{2}) + 32(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})^2 + 2(\frac{\alpha^{2n}+\beta^{2n}}{2})^2 + 2\\
&= \frac{\alpha^{4n+2} - 2(\alpha\beta)^{2n+1} + \beta^{4n+2}}{2}\\
&= 16(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2})^2 + 16(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}}) - 4\\
&= 16(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2})^2 + 16(\frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2}) + 4\\
&= 16b_{n+1}^2 + 16b_{n+1} + 4 = (4b_{n+1}+2)^2.
\end{aligned}$$

The others can be proved similarly.

(2) Since $x = 2r + t$, we get from (3) of Theorem 2.1 that

$$b_{2n-1}^t = \frac{(C_n - 1)t - 2B_n}{2}.$$

Thus from (1.6), we get

$$\begin{aligned}
c_{2n-1}^t &= \sqrt{8(b_{2n-1}^t)^2 + 8tb_{2n-1}^t + 1}\\
&= \sqrt{8(\frac{(C_n-1)t-2B_n}{2})^2 + 8t(\frac{(C_n-1)t-2B_n}{2}) + 1}\\
&= \sqrt{16t^2B_n^2 - 8tB_nC_n + C_n^2}\\
&= 4tB_n - C_n
\end{aligned}$$

since $8B_n^2 + 1 = C_n^2$. So from (1.5), we deduce that

$$B_{2n-1}^t = \frac{(4B_n + C_n - 1)t - (2B_n + C_n + 1)}{2}$$

and hence

$$\begin{aligned}
C_{2n-1}^t &= \sqrt{8(B_{2n-1}^t)^2 + 8B_{2n-1}^t(1+t) + (2t+1)^2}\\
&= \sqrt{\begin{array}{l} 8\left(\frac{(4B_n+C_n-1)t-(2B_n+C_n+1)}{2}\right)^2\\ +8\left(\frac{(4B_n+C_n-1)t-(2B_n+C_n+1)}{2}\right)(1+t)\\ +(2t+1)^2 \end{array}}\\
&= \sqrt{\begin{array}{l} t^2(16B_nC_n + 32B_n^2 + 2C_n^2 + 2)\\ -t(24B_nC_n + 32B_n^2 + 4C_n^2)\\ +(8B_n^2 + 2C_n^2 + 8B_nC_n - 1) \end{array}}\\
&= \sqrt{t^2(4b_{n+1}+2)^2 - 2tc_{n+1}(4b_{n+1}+2) + c_{n+1}^2}
\end{aligned}$$

$$= (4b_{n+1}+2)t - c_{n+1}$$

by (1). The others can be proved similarly.

(3) Since $b^t_{2n-1} = \frac{(C_n-1)t-2B_n}{2}$ and $b^t_{2n} = \frac{(C_n-1)t+2B_n}{2}$ by (2), we easily deduce that $\frac{b^t_{2n}-b^t_{2n-1}}{2} = B_n$. The others are similar.

(4) Recall that $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$ and $C_n = \frac{\alpha^{2n}+\beta^{2n}}{2}$. So we get from (2) that

$$\begin{aligned}
B^t_{2n-1} &= \frac{(4B_n+C_n-1)t-(2B_n+C_n+1)}{2} \\
&= \frac{t\left[4\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)+\frac{\alpha^{2n}+\beta^{2n}}{2}-1\right]-2\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)-\frac{\alpha^{2n}+\beta^{2n}}{2}-1}{2} \\
&= \frac{t\left(\frac{\alpha^{2n}(1+\sqrt{2})+\beta^{2n}(1-\sqrt{2})-2}{2}\right)-\frac{\alpha^{2n}(1+\sqrt{2})-\beta^{2n}(1-\sqrt{2})+2\sqrt{2}}{2\sqrt{2}}}{2} \\
&= t\left(\frac{\alpha^{2n+1}+\beta^{2n+1}-2}{4}\right)-\frac{\alpha^{2n+1}-\beta^{2n+1}+2\sqrt{2}}{4\sqrt{2}}.
\end{aligned}$$

The others can be proved similarly.

(5) Note that $b^t_{2n-1} = \frac{(C_n-1)t-2B_n}{2}$ and $b^t_{2n} = \frac{(C_n-1)t+2B_n}{2}$ by (2). So

$$b^t_{2n} - b^t_{2n-1} = \frac{(C_n-1)t+2B_n}{2} - \frac{(C_n-1)t-2B_n}{2} = P_{2n}$$

since $B_n = \frac{P_{2n}}{2}$. The others can be proved similarly. □

2.1. **Sums.** In this subsection, we consider the sums of numbers we mentioned above.

Theorem 2.3. (1) *For the sums of $t-$balancing numbers, we have*

$$\sum_{i=1}^{n} B^t_i = (B_{\frac{n+2}{2}} + b_{\frac{n+2}{2}} - \frac{n+2}{2})t - \frac{n}{2}$$

$$\sum_{i=1}^{n} b^t_i = (B_{\frac{n}{2}} + b_{\frac{n+2}{2}} - \frac{n}{2})t$$

$$\sum_{i=1}^{n} C^t_i = (3B_{\frac{n+2}{2}} + B_{\frac{n}{2}} + 2b_{\frac{n+2}{2}} - 3)t$$

$$\sum_{i=1}^{n} c^t_i = 4b_{\frac{n+2}{2}}t$$

for even $n \geq 2$ or

$$\sum_{i=1}^{n} B^t_i = (B_{\frac{n+3}{2}} - B_{\frac{n+1}{2}} - \frac{n+3}{2})t - b_{\frac{n+3}{2}} - \frac{n+1}{2}$$

$$\sum_{i=1}^{n} b_i^t = (2B_{\frac{n+1}{2}} - \frac{n+1}{2})t - B_{\frac{n+1}{2}}$$
$$\sum_{i=1}^{n} C_i^t = (b_{\frac{n+5}{2}} - b_{\frac{n+1}{2}} - 4)t - 2B_{\frac{n+3}{2}} + 2b_{\frac{n+3}{2}} + 1$$
$$\sum_{i=1}^{n} c_i^t = 4(B_{\frac{n+1}{2}} + b_{\frac{n+1}{2}})t - 2B_{\frac{n+1}{2}} - 2b_{\frac{n+1}{2}} - 1$$

for odd $n \geq 1$.

(2) *For the sums of Pell numbers, we have*

$$\sum_{i=1}^{n} P_{2i-1} = \frac{b_{2n}^t - b_{2n-1}^t}{2}$$
$$\sum_{i=1}^{n} P_{2i} = \frac{C_{2n}^t + C_{2n-1}^t - 4t}{8t}$$
$$\sum_{i=0}^{2n} P_{2i+1} = \frac{(C_{2n}^t + C_{2n-1}^t)(C_{2n}^t - C_{2n-1}^t)}{8t}$$
$$\sum_{i=1}^{2n} P_{2i} = \frac{(b_{2n}^t - b_{2n-1}^t)(C_{2n}^t - C_{2n-1}^t)}{2}$$
$$\sum_{i=0}^{2n}(P_{2i+1} + P_{2i+2}) = \frac{(c_{2n+2}^t - c_{2n+1}^t)(C_{2n}^t - C_{2n-1}^t)}{4}.$$

(3) *For the sums of Pell–Lucas numbers, we have*

$$\sum_{i=0}^{2n} Q_i = \frac{2(b_{4n+2}^t - b_{4n+1}^t)}{C_{2n}^t - C_{2n-1}^t}$$
$$\sum_{i=1}^{2n} Q_{2i} = \frac{(b_{2n}^t - b_{2n-1}^t)(C_{2n}^t + C_{2n-1}^t)}{t}.$$

(4) *For the sums of balancing numbers, we have*

$$\sum_{i=1}^{2n} B_i = \frac{(b_{2n}^t - b_{2n-1}^t)(C_{2n}^t - C_{2n-1}^t)}{4}$$
$$\sum_{i=1}^{2n} B_{2i} = \frac{(b_{2n}^t - b_{2n-1}^t)(c_{2n}^t - c_{2n-1}^t)(b_{4n+2}^t - b_{4n+1}^t)}{4}$$
$$\sum_{i=1}^{2n}(B_i + B_{i+1}) = 2(b_{2n}^t - b_{2n-1}^t)(b_{2n+2}^t - b_{2n+1}^t)$$

$$\sum_{i=0}^{2n} B_{2i+1} = \frac{(C^t_{2n}+C^t_{2n-1})(C^t_{2n}-C^t_{2n-1})(b^t_{4n+2}-b^t_{4n+1})}{16t}$$
$$\sum_{i=0}^{2n}(B_{2i+1}+B_{2i+2}) = \frac{(C^t_{4n+2}-C^t_{4n+1})(b^t_{4n+2}-b^t_{4n+1})}{4}.$$

(5) *For the sums of Lucas–cobalancing numbers, we have*

$$\sum_{i=1}^{2n+1} c_{i+1} = \frac{(C^t_{2n}-C^t_{2n-1})(C^t_{2n+2}-C^t_{2n+1})}{4}$$
$$\sum_{i=1}^{2n+1} c_{2i+1} = \frac{(C^t_{2n}-C^t_{2n-1})(C^t_{2n}+C^t_{2n-1})(C^t_{4n+4}-C^t_{4n+3})}{16t}.$$

Proof. **(1)** Let n be even, say $n=2k$ for an integer $k\geq 1$. Then from (4) of Theorem 2.2, we easily get

$$\begin{aligned}\sum_{i=1}^{2k} B^t_i &= B^t_1+B^t_2+\cdots+B^t_{2k}\\
&= \left[\left(\frac{\alpha^3+\beta^3-2}{4}\right)t-\frac{\alpha^3-\beta^3+2\sqrt{2}}{4\sqrt{2}}\right]\\
&\quad+\left[\left(\frac{\alpha^3+\beta^3-2}{4}\right)t+\frac{\alpha^3-\beta^3-2\sqrt{2}}{4\sqrt{2}}\right]\\
&\quad+\cdots+\left[\left(\frac{\alpha^{2k+1}+\beta^{2k+1}-2}{4}\right)t-\frac{\alpha^{2k+1}-\beta^{2k+1}-2\sqrt{2}}{4\sqrt{2}}\right]\\
&= \left(\frac{\alpha^3+\alpha^5+\cdots+\alpha^{2k+1}+\beta^3+\beta^5+\cdots+\beta^{2k+1}}{2}-k\right)t-k\\
&= \left(\frac{\alpha^{2k+2}-\beta^{2k+2}}{4\sqrt{2}}+\frac{\alpha^{2k+1}-\beta^{2k+1}}{4\sqrt{2}}-\frac{1}{2}-\frac{2k+2}{2}\right)t-k\\
&= \left(B_{\frac{n+2}{2}}+b_{\frac{n+2}{2}}-\frac{n+2}{2}\right)t-\frac{n}{2}.\end{aligned}$$

The others can be proved similarly. □

In [11], Santana and Diaz–Barrero proved that the sum of first nonzero $4n+1$ terms of Pell numbers is a perfect square, that is,

$$\sum_{i=1}^{4n+1} P_i = \left(\sum_{i=0}^{n}\binom{2n+1}{2i}2^i\right)^2. \tag{2.5}$$

Later in [13, Theorem 2.1], Tekcan and Tayat proved that the sum of first nonzero $2n+1$ terms of Pell numbers is a perfect square if n is even or half of a perfect square if n is odd, that is,

$$\sum_{i=1}^{2n+1} P_i = \begin{cases} \left(\frac{\alpha^{n+1}+\beta^{n+1}}{2}\right)^2 & \text{for even } n \\ \frac{\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{2}}\right)^2}{2} & \text{for odd } n. \end{cases}$$

They set $X_n = \frac{\alpha^{n+1}+\beta^{n+1}}{2}$ and $Y_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{2}}$ for $n \geq 0$ and proved that the right hand side of (2.5) is $(2X_n^2 - 2X_nY_{n-1} + (-1)^{n+1})^2$. Similarly, we can give the following theorem.

Theorem 2.4. *Let P_n denote the n^{th} Pell number, let Q_n denote the n^{th} Pell–Lucas number and let B_n denote the n^{th} balancing number. Then*

(1) *The sum of Pell numbers from 1 to $4n-3$ is a perfect square and is*

$$\sum_{i=1}^{4n-3} P_i = \left(\frac{C_{2n-2}^t - C_{2n-3}^t}{2}\right)^2$$

for $n \geq 2$.

(2) *The sum of Pell numbers from 1 to $4n-1$ and adding 1 is a perfect square and is*

$$1 + \sum_{i=1}^{4n-1} P_i = \left(\frac{c_{2n}^t - c_{2n-1}^t}{2}\right)^2$$

for $n \geq 1$.

(3) *The sum of Pell numbers from 1 to $2n-1$ is a perfect square and is*

$$\sum_{i=1}^{2n-1} P_i = \left(\frac{C_{n-1}^t - C_{n-2}^t}{2}\right)^2$$

for odd $n \geq 3$, and the half of the sum of Pell numbers from 1 to $2n-1$ is a perfect square and is

$$\frac{\sum_{i=1}^{2n-1} P_i}{2} = \left(b_n^t - b_{n-1}^t\right)^2$$

for even $n \geq 2$.

(4) *The sum of $(2i-1)^{st}$ Pell–Lucas numbers from 1 to $2n$ is a perfect square and is*

$$\sum_{i=1}^{2n} Q_{2i-1} = \left(2(b_{2n}^t - b_{2n-1}^t)\right)^2$$

for $n \geq 1$.

(5) *The half of the sum of* $(2i+1)^{st}$ *Pell–Lucas numbers from* 0 *to* $2n$ *is a perfect square and is*

$$\frac{\sum_{i=0}^{2n} Q_{2i+1}}{2} = \left(\frac{C^t_{2n} - C^t_{2n-1}}{2}\right)^2$$

for $n \geq 1$.

(6) *The sum of* $(2i-1)^{st}$ *balancing numbers from* 1 *to* $2n$ *is a perfect square and is*

$$\sum_{i=1}^{2n} B_{2i-1} = \left(\frac{b^t_{4n} - b^t_{4n-1}}{2}\right)^2$$

for $n \geq 1$.

Proof. (1) Note that $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $\sum_{i=1}^{n} P_i = \frac{P_n + P_{n+1} - 1}{2}$. So

$$\begin{aligned}
\sum_{i=1}^{4n-3} P_{2i-1} &= \frac{P_{4n-3} + P_{4n-2} - 1}{2} \\
&= \frac{\frac{\alpha^{4n-3} - \beta^{4n-3}}{2\sqrt{2}} + \frac{\alpha^{4n-2} - \beta^{4n-2}}{2\sqrt{2}} - 1}{2} \\
&= \frac{\alpha^{4n-2} + \beta^{4n-2} - 2}{4} \\
&= \left(\frac{(4b_n + 2)t + c_n - (4b_n + 2)t + c_n}{2}\right)^2 \\
&= \left(\frac{C^t_{2n-2} - C^t_{2n-3}}{2}\right)^2
\end{aligned}$$

by (2) of Theorem 2.2. The other cases can be proved similarly. □

2.2. **Relationship with Triangular Numbers.** In this subsection, we consider the relationship between $t-$balancing numbers and triangular numbers which are the numbers of the form $T_n = \frac{n(n+1)}{2}$ for $n \geq 0$. There are infinitely many triangular numbers that are also square numbers which are called square triangular numbers and is denoted by S_n. For the n^{th} square triangular number S_n, we can write

$$S_n = s_n^2 = \frac{t_n(t_n + 1)}{2},$$

where s_n and t_n are the sides of the corresponding square and triangle.

In the following theorem, we can give the general terms of s_n, t_n and S_n in terms of $t-$balancing numbers and contrary, we can give the general terms of all $t-$balancing numbers in terms of squares and triangles.

Theorem 2.5. (1) *The general terms of s_n, t_n and S_n are*

$$s_n = \frac{b^t_{2n} - b^t_{2n-1}}{2},\ t_n = \frac{c^t_{2n} - c^t_{2n-1} - 2}{4},\ S_n = (\frac{C^t_{2n} - C^t_{2n-1} - b^t_{2n+2} + b^t_{2n+1}}{2})^2$$

for $n \geq 1$.

(2) *The general terms of all $t-$balancing numbers are*

$$\begin{aligned}
B^t_{2n-1} &= (t_n + 2s_n)t - (s_n + t_n + 1)\\
b^t_{2n-1} &= t_n t - s_n\\
C^t_{2n-1} &= (4s_n + 4t_n + 2)t - (s_n + s_{n+1})\\
c^t_{2n-1} &= 4s_n t - (2t_n + 1)\\
B^t_{2n} &= (t_n + 2s_n)t + (s_n + t_n)\\
b^t_{2n} &= t_n t + s_n\\
C^t_{2n} &= (4s_n + 4t_n + 2)t + (s_n + s_{n+1})\\
c^t_{2n} &= 4s_n t + (2t_n + 1)
\end{aligned}$$

for $n \geq 1$.

Proof. **(1)** Since $s_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}, t_n = \frac{\alpha^{2n}+\beta^{2n}-2}{4}$ and $S_n = (\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})^2$, we deduce from (4) of Theorem 2.2 that

$$\begin{aligned}
s_n &= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}\\
&= \frac{\left(t\left(\frac{\alpha^{2n}+\beta^{2n}-2}{4}\right) + \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right) - \left(t\left(\frac{\alpha^{2n}+\beta^{2n}-2}{4}\right) - \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)}{2}\\
&= \frac{b^t_{2n} - b^t_{2n-1}}{2}
\end{aligned}$$

and

$$\begin{aligned}
t_n &= \frac{\alpha^{2n} + \beta^{2n} - 2}{4}\\
&= \frac{\left(t\left(\frac{\alpha^{2n}-\beta^{2n}}{\sqrt{2}}\right) + \frac{\alpha^{2n}+\beta^{2n}}{2}\right) - \left(t\left(\frac{\alpha^{2n}-\beta^{2n}}{\sqrt{2}}\right) - \frac{\alpha^{2n}+\beta^{2n}}{2}\right) - 2}{4}\\
&= \frac{c^t_{2n} - c^t_{2n-1} - 2}{4}.
\end{aligned}$$

Similarly it can be showed that $S_n = \left(\frac{C^t_{2n}-C^t_{2n-1}-b^t_{2n+2}+b^t_{2n+1}}{2}\right)^2$.

(2) We get from (2) and (4) of Theorem 2.2 that

$$\begin{aligned}
B^t_{2n-1} &= \frac{(4B_n + C_n - 1)t - (2B_n + C_n + 1)}{2}\\
&= \frac{\left[4\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right) + \frac{\alpha^{2n}+\beta^{2n}}{2} - 1\right]t - \left[2\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right) + \frac{\alpha^{2n}+\beta^{2n}}{2} + 1\right]}{2}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\alpha^{2n+1}+\beta^{2n+1}-2}{4}\right)t - \left(\frac{\alpha^{2n+1}-\beta^{2n+1}+2\sqrt{2}}{4\sqrt{2}}\right) \\
&= \frac{\left(\alpha^{2n}+\beta^{2n}-2+\sqrt{2}\alpha^{2n}-\sqrt{2}\beta^{2n}\right)t}{4} - \frac{\alpha^{2n}-\beta^{2n}+\sqrt{2}\alpha^{2n}+\sqrt{2}\beta^{2n}+2\sqrt{2}}{4\sqrt{2}} \\
&= \left(\frac{\alpha^{2n}+\beta^{2n}-2}{4}+\frac{\alpha^{2n}-\beta^{2n}}{2\sqrt{2}}\right)t - \left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}+\frac{\alpha^{2n}+\beta^{2n}-2}{4}+1\right) \\
&= (t_n+2s_n)t-(s_n+t_n+1).
\end{aligned}$$

The others can be proved similarly. □

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