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HOMOGENEOUS IDEALIZATION AND SOME DUAL NOTIONS AROUND COMULTIPLICATION MODULES

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Abstract. Let R be a commutative ring with identity, and let M be a unital R-module. D. D. Anderson proved that a submodule N of M is multiplication if and only if $0_{(+)}N$ is a multiplication ideal of $R_{(+)}M$, the homogeneous idealization of M. In this article, we show that a similar statement holds for comultiplication modules. We develop the tool of idealization of a module particularly in the context of cocyclic modules, self-cogenerator modules, comultiplication modules (self-cogenerated modules), couniform modules, $AB5^*$ modules, direct family and inverse family of submodules.

1. Introduction

Let R be a commutative ring with identity and M a unital R-module. Then $R_{(+)}M = R \times M$ is a commutative ring with identity $(1, 0)$ under the componentwise addition and a multiplication defined by

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1).$$

Note that $(0_{(+)}M)^2 = 0$; so $0_{(+)}M \subseteq Nil(R_{(+)}M)$. We view R as a subring of $R_{(+)}M$ via $r \mapsto (r, 0)$. Homogeneous ideals of $R_{(+)}M$ have the form $I_{(+)}N$, where I is an ideal of R and N a submodule of M such that $IM \subseteq N$ (see [8, Theorem 3.1] and [12, Theorem 25(1)]). A ring whose ideals are all homogeneous is called a homogeneous ring [1]. Ideals of $R_{(+)}M$ need not have the form $I_{(+)}N$, that is, need not be homogeneous. For example, the principal ideal of $\mathbb{Z}_{(+)}\mathbb{Z}$ which is generated by $(2, 1)$ is not homogeneous. When $I_{(+)}N$ is an ideal, M/N is an R/I-module and $(R_{(+)}M)/(I_{(+)}N) \cong (R/I)_{(+)}(M/N)$. In particular, $(R_{(+)}M)/(0_{(+)}N) \cong R_{(+)}(M/N)$ and therefore $(R_{(+)}M)/(0_{(+)}M) \cong R$. So the ideals of $R_{(+)}M$ containing $0_{(+)}M$ are of the form $J_{(+)}M$ for some ideal J of R [8, Theorem 3.1]. In particular, since $(0_{(+)}M)^2 = 0$, prime (maximal)

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ideals of $R_{(+)}M$ have the form $P_{(+)}M$ where P is a prime (maximal) ideal of R, (see [8, Theorem 3.2] and [12, Theorem 25.1]).

While the idea to use idealization to extend results concerning ideals to modules is due to M. Nagata [13], it has been investigated in a wide range of topics by many of authors. In an extensive article, D. D. Anderson and M. Winders studied the ring theoretic constructions and properties of $R_{(+)}M$, especially the stability of properties for R and M to properties for $R_{(+)}M$. For example, they determined when $R_{(+)}M$ is Noetherian, Artinian, or a principal ideal ring and applied some examples using idealization to reduce questions concerning factorization in modules to factorization in commutative rings. Moreover, they covered some topics involving idealization such as Buchsbaum, Cohen-Macaulay, and Gorenstein rings, homological dimension, multiplication modules, and Boolean-like rings [8].

In a series of works, M. M. Ali developed more fully the tool of idealization of a module, particularly in the context of multiplication modules, cancellation-like modules, half (weak) cancellation modules, half join principal modules and flat modules, generalizing Anderson's theorems and discussing the behavior under idealization of some ideals and some submodules associated with a module [3, 4, 5]. Also, it has been given some necessary and sufficient conditions for a homogeneous ideal to be large, almost(generalized, weak) multiplication, projective, finitely generated flat, pure or invertible(q-invertible) [1, 2].

Some authors have taken the homogeneous idealization to examine the new notions under idealization along with other ring extensions. For example, D. F. Anderson and A. Badawi considered the stability of n-absorbing ideals under idealization of a module [9, Theorem 4.11, Example 4.12 and Example 4.13]. Also M. Axtell and J. Stickles studied zero-divisor graphs of the idealization of a module. Specifically they investigated the preservation of the diameter and girth of a zero-divisor graph under the idealizations of a ring [10].

In this work, we develop the tool of homogeneous idealization in the context of some (dual) notions such as cocyclic modules, self-cogenerator modules, self cogenerated modules, couniform modules, $AB5^*$ modules, direct family and inverse family of submodules. These notions are all closely related to comultiplication modules [7].

2. Self-cogenerator and strongly self-cogenerated modules

Given a submodule L of M, a homomorphism $\beta\colon L \to M$ is called *trivial* provided there exists an $r \in R$ such that $\beta(x) = rx (x \in L)$. In particular , an endomorphism φ of the module M will be called trivial if $\varphi\colon M \to M$ is trivial in the above sense. For example if M is cyclic, then every endomorphism of M is trivial.

Lemma 2.1. *Let I be an ideal of a ring R and N be a submodule of an R-module M.*

(1) *If* $\varphi\colon R_{(+)}M \to R_{(+)}M$ *is a trivial ring homomorphism, then* $\bar{\varphi}\colon R \to R$ *given by* $\bar{\varphi}(r) = r'$ *where* $\varphi(r,0) = (r', m')$, *is a trivial ring homomorphism.*

(2) *If* $\phi\colon 0_{(+)}N \to 0_{(+)}M$ *is a trivial homomorphism of* $R_{(+)}M$*-modules, then* $\hat{\phi}\colon N \to M$ *given by* $\hat{\phi}(n) = m$ *where* $\phi(0,n) = (0,m)$ *is a trivial homomorphism of* R*-modules. Moreover* $\ker\phi = 0_{(+)}\ker\hat{\phi}$.

(3) *If* $\psi\colon I_{(+)}N \to R_{(+)}M$ *is a trivial homomorphism of* $R_{(+)}M$*-modules, then* $\bar{\psi}\colon I \to R$ *given by* $\bar{\psi}(i) = r$ *where* $\psi(i,0) = (r,m)$ *and* $\hat{\psi}\colon N \to M$ *given by* $\hat{\psi}(n) = m$ *where* $\psi(0,n) = (r,m)$ *are trivial homomorphisms of* R*-modules.*

(4) *If* $g\colon N \to M$ *is a trivial homomorphism of* R*-modules, then*

$$(0_{(+)}g)\colon 0_{(+)}N \to 0_{(+)}M$$

given by $(0_{(+)}g)(0,n) = (0, g(n))$ *is a trivial homomorphism of* $R_{(+)}M$*-modules. Moreover* $\ker(0_{(+)}g) = 0_{(+)}\ker g$.

Proof. (1) Let $r, r' \in R$. It is easily seen that

$$\bar{\varphi}(r + r') = \bar{\varphi}(r) + \bar{\varphi}(r').$$

Moreover,

$$\begin{aligned}\varphi(rr',0) &= \varphi((r,0)(r',0)) = \varphi(r,0)\varphi(r',0)\\ &= (r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + r_2m_1).\end{aligned}$$

Hence $\bar{\varphi}(rr') = \bar{\varphi}(r)\bar{\varphi}(r')$. Now, since φ is trivial, there exists $(s,m) \in R_{(+)}M$ such that $\varphi(r,0) = (s,m)(r,0) = (sr, rm)$. Therefore $\bar{\varphi}(r) = sr$, for all $r \in R$.

(2) Let $n, n' \in N, r \in R$. It is easy to check that

$$\hat{\phi}(n + n') = m + m' = \hat{\phi}(n) + \hat{\phi}(n').$$

Moreover,

$$\begin{aligned}\phi(0,rn) &= \phi((r,0)(0,n)) = (r,0)\phi(0,n)\\ &= (r,0)(0,m) = (0,rm).\end{aligned}$$

Hence $\hat{\phi}(rn) = rm = r\hat{\phi}(n)$. Now, since ϕ is trivial, there exists $(s,m) \in R_{(+)}M$ such that $\phi(0,n) = (s,m)(0,n) = (0,sn)$. Therefore $\hat{\phi}(n) = sn$, for all $n \in N$.

(3) Let $i, i' \in I, n, n' \in N, s \in R$. It is easy to see that

$$\bar{\psi}(i + i') = \bar{\psi}(i) + \bar{\psi}(i').$$

Moreover,

$$\begin{aligned}\psi(si,0) &= \psi((s,0)(i,0)) = (s,0)\psi(i,0)\\ &= (s,0)(r,m) = (sr, sm).\end{aligned}$$

Thus $\bar{\psi}(si) = sr = s\bar{\psi}(i)$. Now since ψ is trivial, there exists $(s, m) \in R_{(+)}M$ such that $\psi(i, 0) = (s, m)(i, 0) = (si, im)$. Therefore $\bar{\psi}(i) = si$, for all $i \in I$. Also it is easy to check that

$$\hat{\psi}(n + n') = m + m' = \hat{\psi}(n) + \hat{\psi}(n').$$

Moreover,

$$\begin{aligned} \psi(0, sn) &= \psi\big((s, 0)(0, n)\big) = (s, 0)\psi(0, n) \\ &= (s, 0)(r, m) = (sr, sm). \end{aligned}$$

Hence $\hat{\psi}(sn) = sm = s\hat{\psi}(n)$. Now since ψ is trivial, there exists $(s, m) \in R_{(+)}M$ such that $\psi(0, n) = (s, m)(0, n) = (0, sn)$. Therefore $\hat{\psi}(n) = sn$, for all $n \in N$.

(4) Let $n, n' \in N, m \in M, r \in R$. It is easily seen that

$$(0_{(+)}g)((0, n) + (0, n')) = (0_{(+)}g)(0, n) + (0_{(+)}g)(0, n').$$

Moreover,

$$\begin{aligned} (0_{(+)}g)((r, m)(0, n)) &= (0_{(+)}g)(0, rn) = (0, g(rn)) \\ &= (0, rg(n)) = (r, m)(0, g(n)) \\ &= (r, m)(0_{(+)}g)(0, n). \end{aligned}$$

Now, since g is trivial, there exists $r \in R$ such that $g(n) = rn$, for all $n \in N$. Therefore

$$\begin{aligned} (0_{(+)}g)(0, n) &= (0, g(n)) = (0, rn) \\ &= (r, 0)(0, n), \end{aligned}$$

for all $(0, n) \in 0_{(+)}N$. The proof of other part is rutine. □

Example 2.2. If $f : \mathbb{Z} \to \mathbb{Z}$ is the identity map and $g : \mathbb{Z} \to \mathbb{Z}$ is a map defined by $g(x) = 2x$, then f and g are trivial homomorphisms, while it is easily seen that $f_{(+)}g$ which is defined by $(f_{(+)}g)(r, m) = (f(r), g(m))$ is not trivial. This shows that Lemma 2.1 (4) is not true in general.

An R-module M is called *self-cogenerator* provided for each submodule N of M, the factor module M/N embeds in the direct product of M^I of copies of M, for some index set I. It is easy to check that a module M is self-cogenerator if and only if for each submodule N of M there exists an index set I and endomorphisms $\varphi_i (i \in I)$ of M such that $N = \cap_{i \in I} \ker \varphi_i$. For example every vector space over a field is self-cogenerator. We shall call a module M is *strongly self-cogenerated* provided for each submodule N of M there exists a family $\varphi_i (i \in I)$ of trivial endomorphisms of M, for some index set I, such that $N = \cap_{i \in I} \ker \varphi_i$.

Theorem 2.3. *Let R be a ring and N be a submodule of R-module M.*

(1) *N is a self-cogenerator submodule of M if and only if $0_{(+)}N$ is a self-cogenerator ideal of $R_{(+)}M$.*

(2) *N is a strongly self-cogenerated submodule of M if and only if $0_{(+)}N$ is a strongly self-cogenerated ideal of $R_{(+)}M$.*

Proof. (1) Let N be a self-cogenerator submodule of M and $0_{(+)}L$ be an ideal of $R_{(+)}M$ contained in $0_{(+)}N$. Since N is self-cogenerator and L is a submodule of N, there exists a family $\varphi_i (i \in I)$ of endomorphisms of N, for some index set I, such that $L = \cap_{i\in I} \ker \varphi_i$. By Lemma 2.1(4), $0_{(+)}\varphi_i (i \in I)$ is a family of endomorphisms of $0_{(+)}N$ such that $\ker(0_{(+)}\varphi_i) = 0_{(+)} \ker \varphi_i$, for all $i \in I$. Hence

$$0_{(+)}L = 0_{(+)} \cap_{i\in I} \ker \varphi_i = \cap_{i\in I} 0_{(+)} \ker \varphi_i = \cap_{i\in I} \ker(0_{(+)}\varphi_i).$$

Therefore $0_{(+)}N$ is self-cogenerator.

(2) Follows by Lemma 2.1 (4) and the proof of (1). □

An R-module M is called *comultiplication* provided for each submodule N of M there exists an ideal I of R such that $N = (0 :_M I) = \{m \in M \mid Im = 0\}$. An ideal I of R is a comultiplication ideal if it is a comultiplication R-module. It is proved that M is a comultiplication R-module if and only if M is a strongly self-cogenerated R-module ([7, Theorem 1.5]). Hence by Theorem 2.3(2), we have the following result:

Corollary 2.4. *Let M be an R-module and N a submodule of M. Then N is a comultiplication submodule of M if and only if $0_{(+)}N$ is a comultiplication ideal of $R_{(+)}M$.*

3. Large, nonsingular and small submodules

A submodule N of M is called large provided $N \cap L \neq 0$ for every nonzero submodule L of M i.e. if $N \cap L = 0$, then $L = 0$. The *socle* of M, denoted soc (M), is the intersection of all large submodules of M. M is called cocyclic provided M has a simple large socle. An R-module M is a comultiplication module if and only if for each submodule N of M such that M/N is cocyclic there exists an ideal I of R such that $N = (0 :_M I)$ [7, Proposition 1.3].

Proposition 3.1. *Let R be a ring, N a submodule of an R-module M and $I_{(+)}N$ a homogeneous ideal of $R_{(+)}M$.*

(1) *N is a minimal submodule of M if and only if $0_{(+)}N$ is a minimal ideal of $R_{(+)}M$.*
(2) *A submodule L of N is large if and only if $0_{(+)}L$ is an large submodule of $0_{(+)}N$.*
(3) *soc $(0_{(+)}N) = 0_{(+)}$ soc N.*
(4) *N is a cocyclic submodule if and only if $0_{(+)}N$ is a cocyclic submodule.*
(5) *ann $(0_{(+)}N) =$ ann $N_{(+)}M$.*
(6) *If N is faithful, then $0_{(+)}N$ is large in $I_{(+)}N$.*
(7) *If N is faithful and K is large in N, then $0_{(+)}K$ is large in $I_{(+)}N$.*

Proof. (1) The result is followed by this fact that ideals of $R_{(+)}M$ contained in $0_{(+)}N$ are exactly of the form $0_{(+)}L$ where L is a submodule of N.

(2) Suppose L is large in N and $0_{(+)}K$ is a submodule of $0_{(+)}N$ such that $(0_{(+)}L) \cap (0_{(+)}K) = 0$. Then $0 = (0_{(+)}L) \cap (0_{(+)}K) = 0_{(+)}(L \cap K)$. Thus $L \cap K = 0$. Hence $K = 0$ and hence $0_{(+)}K = 0$. This implies that $0_{(+)}L$ is large in $0_{(+)}N$. Conversely, suppose K is a submodule of N such that $L \cap K = 0$. Then $(0_{(+)}L) \cap (0_{(+)}K) = 0_{(+)}(L \cap K) = 0$. Since $0_{(+)}L$ is large in $0_{(+)}N$, $0_{(+)}K = 0$ and hence $K = 0$. This show that L is large in N.

(3) By (2), the large submodules of $0_{(+)}N$ are exactly of the form $0_{(+)}L$ where L is large in N. Hence the result follows.

(4) follows by (1), (2) and (3).

(5) Let $(r, m) \in R_{(+)}M$. Then

$$\begin{aligned}
(r, m) \in \text{ann } (0_{(+)}N) &\Longleftrightarrow (r, m)0_{(+)}N = 0 \\
&\Longleftrightarrow 0_{(+)}rN = 0 \\
&\Longleftrightarrow rN = 0 \\
&\Longleftrightarrow r \in \text{ann } N.
\end{aligned}$$

(6) Let H be an ideal of $R_{(+)}M$ contained in $I_{(+)}N$ such that $H \cap (0_{(+)}N) = 0$. Then $H(0_{(+)}N) = 0$. Hence

$$H \subseteq \text{ann } (0_{(+)}N) = \text{ann } N_{(+)}M = 0_{(+)}M.$$

Thus $H = 0_{(+)}K$ for some submodule K of M. Therefore

$$0 = H \cap (0_{(+)}N) = (0_{(+)}K) \cap (0_{(+)}N) = 0_{(+)}(K \cap N).$$

Thus $K \cap N = 0$. Since $H = 0_{(+)}K \subseteq I_{(+)}N$, $K \subseteq N$. Hence $0 = K \cap N = K$ and hence $H = 0_{(+)}K = 0$. This implies that $0_{(+)}N$ is a large ideal of $R_{(+)}M$ contained in $I_{(+)}N$.

(7) Let H be an ideal of $R_{(+)}M$ in $I_{(+)}N$ such that $H \cap (0_{(+)}K) = 0$. Then $(H \cap (0_{(+)}N)) \cap (0_{(+)}K) = 0$. But $H \cap (0_{(+)}N) \subseteq 0_{(+)}N$ and by (2), $0_{(+)}K$ is a large ideal of $R_{(+)}M$ contained in $0_{(+)}N$. Then $H \cap (0_{(+)}N) = 0$. By (6), $0_{(+)}N$ is a large ideal of $R_{(+)}M$ contained in $I_{(+)}N$, hence $H = 0$. □

Let M be an R module, I an ideal of R and N a submodule of M. In a general case of above theorem 5, ann $(I_{(+)}N) = (\text{ann } (I) \cap \text{ann } N)_{(+)}(0 :_M I)$ [3, Lemma 1]. Using this fact we have the following characterization for comultiplication modules.

Proposition 3.2. *Let M be an R-module. Then M is a comultiplication module if and only if for each submodule N of M such that M/N is cocyclic there exists an ideal I of R such that ann $(I_{(+)}N) = (\text{ann } (I) \cap \text{ann } N)_{(+)}N$*

Proof. It immediately follows from [3, Lemma 1] and [7, Proposition 1.3]. □

An R-module M is called *nonsingular* provided $Im \neq 0$, for every large ideal I of R and non-zero element m of M. M is called a *multiplication* module provided, for each submodule N of M, there exists an ideal I of R such that

$N = IM$ [11]. Note that, if N is a submodule of a multiplication R-module M, then $I \subseteq (N : M) = \{r \in R \mid rM \subseteq N\}$ and hence $N = IM \subseteq (N : M)M \subseteq N$ so that $N = (N : M)M$. The ideal I of R is called nonsingular (multiplication) if it is a nonsingular (multiplication) R-module.

Theorem 3.3. *Let I be a ideal of R and M an R-module. If $I_{(+)}IM$ is a finitely generated nonsingular comultiplication ideal of $R_{(+)}M$, then $I_{(+)}IM$ is a multiplication ideal of $R_{(+)}M$ and ann $(I_{(+)}IM) = R(e, m)$ for some idempotent (e, m) of $R_{(+)}M$. In particular I is a multiplication ideal of R, ann $(I) = Re$ for some idempotent e of R.*

Proof. Since $I_{(+)}IM$ is a nonsingular comultiplication ideal of $R_{(+)}M$, then by [7, Corollary 1.7] it is projective, and hence is a multiplication ideal and ann $(I_{(+)}IM) = R(e, m)$ by [6, p.3899]. The in particular part follows from [3, Theorem 7]. $\square$

Theorem 3.4. *Let M be a faithful multiplication R-module and $I_{(+)}N$ be a homogeneous ideal of $R_{(+)}M$. If $I_{(+)}N$ is a nonsingular ideal of $R_{(+)}M$, then N is a nonsingular submodule of M and I is a nonsingular ideal of R. In particular if $0_{(+)}N$ is a nonsingular ideal of $R_{(+)}M$, then N is a nonsingular submodule of M.*

Proof. Let J, H be large ideals of R and $0 \neq r \in I, 0 \neq n \in N$ such that $nJ = rH = 0$. Then $(0, n)(J_{(+)}JM) = 0_{(+)}nJ = 0$ and $(r, 0)(H_{(+)}HM) = rH_{(+)}rHM = 0$. Since $J_{(+)}JM$ and $H_{(+)}HM$ are large ideals of $R_{(+)}M$ by [1, Theorem 14, (6)], it contradicts the nonsingularity of $I_{(+)}N$. $\square$

As a dual notion of a large submodule, A submodule N of an R-module M is said to be small provided for any submodule L of M, $L + N = M$ implies that $L = M$. $I_{(+)}N$ is a small ideal of $R_{(+)}M$ if and only if I is a small ideal of R [1, Proposition 17]. M is said to be couniform provided each of its non-zero submodules is small.

Theorem 3.5. *Let I be an ideal of a ring R and N be a submodule of an R-module M.*

(1) *$0_{(+)}N$ is couniform if and only if N is couniform.*
(2) *If $I_{(+)}M$ is couniform, then I is couniform.*
(3) *If $I_{(+)}IM$ is couniform, then I is couniform.*

Proof. (1) Let $0_{(+)}N$ be couniform and L, K be non-zero submodules of N such that $L+K = N$. Then $0_{(+)}N = 0_{(+)}(L+K) = (0_{(+)}L)+(0_{(+)}K)$. Hence $0_{(+)}L = 0_{(+)}N$ or $0_{(+)}K = 0_{(+)}N$ and hence $L = N$ or $K = N$. This show that N is couniform. The converse is routine.

(2) Let J, H be non-zero ideals of R in I such that $J + H = I$. Then

$$(J_{(+)}M) + (H_{(+)}M) = (J + H)_{(+)}M = I_{(+)}M.$$

Since $I_{(+)}M$ is couniform, $J_{(+)}M = I_{(+)}M$ or $H_{(+)}M = I_{(+)}M$. Thus $J = I$ or $H = I$. Hence I is couniform.

(3) Let J, H be non-zero ideals of R in I such that $J + H = I$. Then

$$(J_{(+)}JM) + (H_{(+)}HM) = (J + H)_{(+)}(J + H)M = I_{(+)}IM.$$

Since $I_{(+)}IM$ is couniform, $J_{(+)}JM = I_{(+)}IM$ or $H_{(+)}HM = I_{(+)}IM$. Thus $J = I$ or $H = I$. Hence I is couniform. □

4. Direct and inverse family of submodules

A family $L_i(i \in I)$ of submodules of an R-module M is called direct provided for each $i, j \in I$ there exists a $k \in I$ such that $L_i + L_j \subseteq L_k$ and in this case

$$N \cap (\Sigma_{i\in I} L_i) = \Sigma_{i\in I}(N \cap L_i),$$

for every submodule N of M. On the other hand, a family of submodules $H_i(i \in I)$ of M is called inverse if for each $i, j \in I$ there exists a $k \in I$ such that $H_k \subseteq H_i \cap H_j$. Also M is said to satisfy the $AB5^*$-condition and is called an $AB5^*$ module provided for every inverse family $H_i(i \in I)$ of submodules,

$$N + \cap_{i\in I} H_i = \cap_{i\in I}(N + H_i),$$

for every submodule N of M. For example multiplication modules over valuation domain are $AB5^*$-modules. Also the prüfer group $\mathbb{Z}_{p^\infty}$ is an $AB5^*$. By an $AB5^*$ submodule N of M (resp. ideal I of R), we mean that N (resp. I) is an $AB5^*$ R-module.

Proposition 4.1. *Let M be an R-module and N be a submodule of M.*

1. *A family $L_i(i \in I)$ of submodules of N is direct (resp. inverse) if and only if a family $0_{(+)}L_i$ of ideals of $R_{(+)}M$ is direct (resp. inverse).*
2. *N is an $AB5^*$ submodule if and only if $0_{(+)}N$ is an $AB5^*$ ideal of $R_{(+)}M$.*

Proof. (1) clear.

(2) Let N be an $AB5^*$ module, $0_{(+)}H_i(i \in I)$ be any inverse family of ideals of $R_{(+)}M$ contained in $0_{(+)}N$ and $0_{(+)}L$ be a ideal of $R_{(+)}M$ contained in $0_{(+)}N$. Then

$$\begin{aligned}(0_{(+)}L) + \cap_{i\in I}(0_{(+)}H_i) &= (0_{(+)}L) + (0_{(+)}(\cap_{i\in I} H_i)) \\ &= 0_{(+)}(L + \cap_{i\in I} H_i).\end{aligned}$$

Since N is an $AB5^*$ submodule and by (1) a family $H_i(i \in I)$ is inverse in N,

$$L + \cap_{i\in I} H_i = \cap_{i\in I}(L + H_i).$$

Thus

$$\begin{aligned}0_{(+)}(L + \cap_{i\in I} H_i) &= 0_{(+)} \cap_{i\in I} (L + H_i) \\ &= \cap_{i\in I}(0_{(+)}(L + H_i)) \\ &= \cap_{i\in I}((0_{(+)}L) + (0_{(+)}H_i)).\end{aligned}$$

Hence $0_{(+)}N$ is an $AB5^*$submodule. Conversely suppose $H_i (i \in I)$ is an inverse family of submodules of N and L be a submodule of N. Then

$$\begin{aligned} 0_{(+)}(L + \cap_{i\in I} H_i) &= (0_{(+)}L) + (0_{(+)} \cap_{i\in I} H_i) \\ &= (0_{(+)}L) + \cap_{i\in I}(0_{(+)}H_i). \end{aligned}$$

Since $0_{(+)}N$ is an $AB5^*$ submodule and by (1) a family $0_{(+)}H_i (i \in I)$ is inverse in $0_{(+)}N$ and $0_{(+)}L$ is a ideal of $R_{(+)}M$ contained in $0_{(+)}N$,

$$(0_{(+)}L) + \cap_{i\in I}(0_{(+)}H_i) = \cap_{i\in I}((0_{(+)}L) + (0_{(+)}H_i)).$$

Thus

$$\begin{aligned} 0_{(+)}(L + \cap_{i\in I} H_i) &= \cap_{i\in I}((0_{(+)}L) + (0_{(+)}H_i)) \\ &= \cap_{i\in I}(0_{(+)}(L + H_i)) \\ &= 0_{(+)} \cap_{i\in I} (L + H_i). \end{aligned}$$

Hence

$$L + \cap_{i\in I} H_i = \cap_{i\in I}(L + H_i),$$

This show that N is an $AB5^*$ submodule. □

By [7, Theorem 2.9], if M is a comultiplication R-module such that $(0 :_M I \cap J) = (0 :_M I) + (0 :_M J)$ for all ideals I and J of R, then M is an $AB5^*$ module. Now by Proposition 4.1 and that every submodule of a comultiplication module is a comultiplication module [7, Lemma 2.1], we have the following result:

Corollary 4.2. *Let M be a comultiplication R-module. If N is a submodule of M such that $(0 :_N I \cap J) = (0 :_N I) + (0 :_N J)$ for all ideals I and J of R. Then N is an $AB5^*$ submodule of M, and therefore $0_{(+)}N$ is an $AB5^*$ ideal of $R_{(+)}M$.*

Theorem 4.3. *Let R be a ring and M an R-module. Let $I_{(+)}N$ be a homogeneous ideal of $R_{(+)}M$.*

1. *A family $I_i (i \in I)$ of ideals of R contained in I is direct (resp. inverse) if and only if the family $I_{i(+)}N (i \in I)$ of ideals of $R_{(+)}M$ contained in $I_{(+)}N$ is direct (resp. inverse).*
2. *If a family $I_{(+)}N_i (i \in I)$ of homogeneous ideals of $R_{(+)}M$ contained in $I_{(+)}N$ is direct (resp. inverse), then the family $N_i (i \in I)$ of submodules of N is direct (resp. inverse). The converse is true if $IM \subseteq N_i$ for all $i \in I$.*
3. *If $I_{(+)}N$ is an $AB5^*$ ideal of $R_{(+)}M$, then I is an $AB5^*$ ideal of R.*

Proof. (1) Since $I_iM \subseteq IM \subseteq N$, $I_{i(+)}N$'s are ideals of $R_{(+)}M$. Hence for any $i, j, k \in I$,

$$\begin{aligned} I_i + I_j \subseteq I_k &\Longleftrightarrow (I_i + I_j)_{(+)}N \subseteq I_{k(+)}N \\ &\Longleftrightarrow (I_{i(+)}N) + (I_{j(+)}N) \subseteq I_{k(+)}N \end{aligned}$$

Also

$$\begin{aligned} I_k \subseteq I_i \cap I_j &\iff I_{k(+)}N \subseteq (I_i \cap I_j)_{(+)}N \\ &\iff I_{k(+)}N \subseteq (I_{i(+)}N) \cap (I_{j(+)}N). \end{aligned}$$

It follows that a family $I_i (i \in I)$ of ideals of R contained in I is direct if and only if the family $I_{i(+)}N (i \in I)$ of ideals of $R_{(+)}M$ contained in $I_{(+)}N$ is direct.

(2) Let for all $i \in I$, $IM \subseteq N_i$. Then for any $i, j, k \in I$,

$$\begin{aligned} N_i + N_j \subseteq N_k &\iff I_{(+)}(N_i + N_j) \subseteq I_{(+)}N_k \\ &\iff (I_{(+)}N_i) + (I_{(+)}N_j) \subseteq I_{(+)}N_k \end{aligned}$$

Also

$$\begin{aligned} N_k \subseteq N_i \cap N_j &\iff I_{(+)}N_k \subseteq I_{(+)}(N_i \cap N_j) \\ &\iff I_{(+)}N_k \subseteq (I_{(+)}N_i) \cap (I_{(+)}N_j). \end{aligned}$$

(3) Suppose that $I_i (i \in I)$ is an inverse family of ideals of R contained in I and J is an ideal of R contained in I. Then

$$\begin{aligned} (J + \cap_{i \in I} I_i)_{(+)}N &= (J_{(+)}N) + \cap_{i \in I}(I_{i(+)}N) \\ &= (J_{(+)}N) + \cap_{i \in I}(I_{i(+)}N) \end{aligned}$$

By (1) a family $I_{i(+)}N (i \in I)$ of ideals of $R_{(+)}M$ contained in $I_{(+)}N$ is inverse and $J_{(+)}N$ is an ideal $R_{(+)}M$ contained in $I_{(+)}N$. By assumption $I_{(+)}N$ is an $AB5^*$ ideal, thus

$$\begin{aligned} (J_{(+)}N) + \cap_{i \in I}(I_{i(+)}N) &= \cap_{i \in I}\big((J_{(+)}N) + (I_{i(+)}N)\big) \\ &= \cap_{i \in I}\big((J + I_i)_{(+)}N\big) \\ &= \big(\cap_{i \in I}(J + I_i)\big)_{(+)}N. \end{aligned}$$

Then

$$(J + \cap_{i \in I} I_i)_{(+)}N = \big(\cap_{i \in I}(J + I_i)\big)_{(+)}N.$$

Thus

$$J + \cap_{i \in I} I_i = \cap_{i \in I}(J + I_i).$$

Hence I is an $AB5^*$ ideal of R. □

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