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ON THE MATSUMOTO CHANGE OF A FINSLER SPACE WITH MTH-ROOT METRIC

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Abstract. In the present paper, we find a condition under which a Finsler space with Matsumoto change of mth-root metric is projectively related to a mth-root metric and also we find a condition under which this Matsumoto transformed mth-root Finsler metric is locally dually flat and projectively flat.

1. Introduction

The concept of mth-root metric was introduced by Shimada [10] in 1979, applied to ecology by Antonelli [3] and studied by several authors ([8], [11], [12], [13], [14], [15]). It is regarded as a generalisation of Riemannian metric in the sense that the second root metric is a Riemannian metric. For m = 3, it is called a cubic Finsler metric and for $m = 4$, it is quatric metric. In four dimension, the special fourth root metric in the form $F = \sqrt[4]{y^1 y^2 y^3 y^4}$ is called the Berwald-Moor metric which is considered by physicists as an important subject for a possible model of space-time. Recent studies show that mth-root Finsler metrics play a very important role in physics, space-time structure and gravitation as well as in unified gauge field theories. Li and Shen [5] have studied the geometrical properties of locally projectively flat fourth root metrics in the form $F = \sqrt[4]{a_{ijkl}(x)y^i y^j y^k y^l}$ and generalised fourth root metric in the form F = $\sqrt{\sqrt{a_{ijkl}(x)y^i y^j y^k y^l} + b_{ij}(x)y^i y^j}$. In [12], Tayebi and Najafi characterize locally dually flat and Antonelli mth-root metrics and in [13] Tayebi, Peyghan and Shahbazi find a condition under which a generalized mth-root metric is projectively related to mth-root metric. In [4], Brinzei provides necessary and sufficient condition for a Finsler space with mth-root metric to be projectively flat to Berwald space.

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In this paper, we find the condition under which the transformed Finsler space is projectively related with given Finsler space. Also we find the condition under which the transformed Finsler space is locally dually flat and projectively flat.

2. Preliminaries

Let M^n be an n-dimensional C^∞-manifold, T_xM denotes the tangent space of M^n at x. The tangent bundle TM is the union of tangent spaces, $TM := \bigcup_{x\in M} T_xM$. We denote the elements of TM by (x, y), where $x = (x^i)$ is a point of M^n and $y \in T_xM$ called supporting element. We denote $TM_0 = TM \setminus \{0\}$.

Definition. A Finsler metric on M^n is a function $F : TM \to [0, \infty)$ with the following properties:
(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM and
(iii) the Hessian of F^2 with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair $(M^n, F) = F^n$ is called a Finsler space, F is called the fundamental function and g_{ij} is called the fundamental tensor of the Finsler space F^n.

The normalized supporting element l_i, angular metric tensor h_{ij} and metric tensor g_{ij} of F^n are defined respectively as:

$$l_i = \frac{\partial F}{\partial y^i}, \quad h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j} \text{ and } g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{2.1}$$

Let F be a Finsler metric defined by $F = \sqrt[m]{A}$, where A is given by $A := a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$, with $a_{i_1\ldots i_m}$ symmetric in all its indices. Then F is called an mth- root Finsler metric. Clearly, A is homogeneous of degree m in y.
Let

$$A_i = a_{i i_2 \ldots i_m}(x) y^{i_2} \ldots y^{i_m} = \frac{1}{m}\frac{\partial A}{\partial y^i}, \tag{2.2}$$

$$A_{ij} = a_{ij i_3 \ldots i_m}(x) y^{i_3} \ldots y^{i_m} = \frac{1}{m(m-1)}\frac{\partial^2 A}{\partial y^i \partial y^j}, \tag{2.3}$$

$$A_{ijk} = a_{ijk i_4 \ldots i_m}(x) y^{i_4} \ldots y^{i_m} = \frac{1}{m(m-1)(m-2)}\frac{\partial^3 A}{\partial y^i \partial y^j \partial y^k} \tag{2.4}$$

The normalized supporting element of F^n is given by

$$l_i := F_{y^i} = \frac{\partial F}{\partial y^i} = \frac{\partial \sqrt[m]{A}}{\partial y^i} = \frac{1}{m}\frac{\frac{\partial A}{\partial y^i}}{A^{\frac{m-1}{m}}} = \frac{A_i}{F^{m-1}}. \tag{2.5}$$

Let us consider the transformation

$$\overline{F} = \frac{F^2}{F - \beta}, \tag{2.6}$$

where $F = \sqrt[m]{A}$, is an mth-root metric and $\beta = b_i(x)y^i$ is a one form on the manifold M^n. Clearly $\overline{F}$ is also a Finsler metric on M^n, given by Matsumoto change of mth- root metric. Throughout the paper, we call the Finsler metric $\overline{F}$ as transformed mth-root metric and $(M^n, \overline{F}) = \overline{F}^n$ as transformed Finsler space. We restrict ourselves for $m > 2$ throughout the paper and also the quantities corresponding to the transformed Finsler space $\overline{F}^n$ will be denoted by putting bar on the top of that quantity.

3. Fundamental Metric Tensor of Matsumoto Transformed mth-root Metric

The Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta}$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form, is an interesting (α, β)-metric introduced by Matsumoto using gradient of slope, speed and gravity in [6]. This metric formulates the model of a Finsler space. Many authors ([1], [3]) have studied this metric by different prospectives.

The Finsler metric $\overline{F} = \dfrac{F^2}{F - \beta}$ is called the Matsumoto change of Finsler metric [7].

The differentiation of (2.6) with respect to y^i yields the normalized supporting element $\overline{l_i}$ given by

$$\overline{l_i} = \frac{A_i}{F^{m-2}}\frac{(F - 2\beta)}{(F - \beta)^2} + \frac{F^2}{(F - \beta)^2}b_i. \tag{3.1}$$

Again differentiation of (3.1) with respect to y^j yields

$$\begin{aligned}\overline{h}_{ij} = \overline{F}\Big[&\frac{(m-1)(F-2\beta)}{F^{m-2}(F-\beta)^2}A_{ij} - \Big(\frac{(m-1)(F-2\beta)}{F^{2m-4}(F-\beta)^2} - \frac{2\beta^2}{F^{2m-2}(F-\beta)^3}\Big)A_iA_j\\ &-\frac{2F\beta}{F^{m-1}(F-\beta)^3}(A_ib_j + A_jb_i) + \frac{2F^2}{(F-\beta)^3}b_ib_j\Big].\end{aligned}$$

(3.2)

From (3.1) and (3.2), the fundamental metric tensor $\overline{g}_{ij}$ of Finsler space $\overline{F}^n$ is given by

$$\overline{g}_{ij} = \overline{h}_{ij} + \overline{l}_i\overline{l}_j,$$

After simplification, we get

$$\overline{g}_{ij} = \rho A_{ij} + \rho_0 b_ib_j + \rho_1(A_ib_j + b_iA_j) + \rho_2 A_iA_j, \tag{3.3}$$

where

$$\rho = \frac{(m-1)\overline{F}(F-2\beta)}{F^{m-2}(F-\beta)^2},$$

$$\rho_0 = \frac{2F^2\overline{F}}{(F-\beta)^3} + \frac{F^4}{(F-\beta)^4},$$
$$\rho_1 = \frac{F^2(F-2\beta)}{F^{m-2}} - \frac{2\beta\overline{F}}{F^{m-2}(F-\beta)^3},$$
$$\rho_2 = \frac{2\beta^2\overline{F}}{F^{2(m-1)}(F-\beta)^3} + \frac{(F-2\beta)^2}{(F-\beta)^4F^{2(m-2)}} - \frac{(m-1)\overline{F}(F-2\beta)}{F^{2(m-4)}(F-\beta)^2}.$$

The contravariant metric tensor $\overline{g}^{ij}$ of Finsler space $\overline{F}^n$ is given by

$$\overline{g}^{ij} = \frac{A^{ij}}{\rho} - \sigma_0 b^i b^j - \sigma_2 y^i y^j - \sigma_1(y^i b^j + y^j b^i), \tag{3.4}$$

where

$$\sigma_0 = \frac{1}{\rho + \delta b^2} + \frac{{\rho_1}^2}{\rho_2 - {\rho_1}^2 d^2},$$
$$\sigma_1 = \frac{{\rho_2}^2}{\rho_2 - {\rho_1}^2 d^2},$$
$$\sigma_2 = \frac{\rho_1\rho_2\rho_4}{\rho_2 - {\rho_1}^2 d^2},$$

$$d^2 = d^i d_j = \Big[\rho_4 b^i + \frac{\rho_2}{\rho_1} y^i\Big]\Big[b_i + \frac{\rho_2}{\rho_1} A_i\Big] = \rho_4(b^2+\beta) + \frac{\rho_2}{\rho_1}\Big(\beta + \frac{\rho_2}{\rho_1} F^m\Big),$$
$$\rho_4 = \Big[\frac{1}{\rho} - \frac{1}{\rho + \delta b^2}\Big(b^2 + \frac{\rho_2}{\rho_1}\Big)\Big].$$

Proposition 1. *The covariant metric tensor $\overline{g}_{ij}$ and contravariant metric tensor $\overline{g}^{ij}$ of Matsumoto transformed mth-root Finsler space $\overline{F}^n$ are given by the equations (3.3) and (3.4) respectively.*

4. Spray Coefficients of Matsumoto Transformed mth-root metric

The geodesics of a Finsler space F^n are given by the following system of equations

$$\frac{d^2x^i}{dt^2} + G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where

$$G^i = \frac{1}{4} g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}$$

are called spray coefficients of F^n.

Two Finsler metrics F and $\overline{F}$ on a manifold M^n are called projectively related if there is a scalar function P(x, y) defined on TM_0 such that $\overline{G}^i = G^i + Py^i$, where $\overline{G}^i$ and G^i are the geodesics spray coefficients of $\overline{F}^n$ and F^n respectively. In other words two metrics F and $\overline{F}$ are called projectively related if any geodesic of the first is also geodesic for the second and vice versa.

In the view of equation (3.3) the metric tensor $\overline{g}_{ij}$ of $\overline{F}^n$ can be rewritten as:

$$\overline{g}_{ij} = \frac{\rho F^{m-2}}{m-1} g_{ij} + \rho_0 b_i b_j + \rho_1 (A_i b_j + A_j b_i) + \rho_3 A_i A_j, \tag{4.1}$$

where
$\rho_3 = \rho_2 + \dfrac{m-2}{(m-1)F^{m-2}}$,
and

$$g_{ij} = (m-1)\frac{A_{ij}}{F^{m-2}} - (m-2)\frac{A_i A_j}{F^{2(m-1)}}. \tag{4.2}$$

Further, in view of equation (3.4), contravariant metric tensor $\overline{g}^{ij}$ can be rewritten as:

$$\overline{g}^{ij} = \frac{m-1}{\rho F^{m-2}} g^{ij} - y^i(\sigma_1 b^j + \sigma_3 y^j) - b^i(\sigma_1 y^j + \sigma_0 b^j), \tag{4.3}$$

where
$\sigma_3 = \sigma_2 + \dfrac{m-2}{F^m \rho}$,
and

$$g^{ij} = \frac{F^{m-2}}{m-1} A^{ij} + \frac{(m-2) y^i y^j}{(m-1)F^2}. \tag{4.4}$$

The spray coefficients of Matsumoto transformed Finsler space $\overline{F}^n$ are given by-

$$\overline{G}^i = \frac{1}{4}\overline{g}^{il}\{[\overline{F}^2]_{x^k y^l} y^k - [\overline{F}^2]_{x^l}\}. \tag{4.5}$$

It can also be written as:

$$\overline{G}^i = \frac{1}{4}\overline{g}^{il}\Big(2\frac{\partial \overline{g}_{jl}}{\partial x^k} - \frac{\partial \overline{g}_{jk}}{\partial x^l}\Big) y^j y^k. \tag{4.6}$$

From (4.1) and (4.6), we have

$$\begin{aligned}\overline{G}^i = \frac{1}{4}\overline{g}^{il}\Big[&2\frac{\partial}{\partial x^k}\Big(\frac{\rho F^{m-2}}{m-1} g_{jl} + \rho_0 b_j b_l + \rho_1(A_j b_l + A_l b_j) + \rho_3 A_j A_l\Big) - \\ &\frac{\partial}{\partial x^l}\Big(\frac{\rho F^{m-2}}{m-1} g_{jk} + \rho_0 b_j b_k + \rho_1(A_j b_k + A_k b_j) + \rho_3 A_j A_k\Big)\Big] y^j y^k,\end{aligned} \tag{4.7}$$

which implies that

$$\overline{G}^i = \frac{1}{4}\overline{g}^{il}\Big[\Big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)\frac{\rho F^{m-2}}{m-1} + 2 g_{jl}\tau_k - g_{jk}\tau_l + 2\frac{\partial X_{jl}}{\partial x^k} + \frac{\partial X_{jk}}{\partial x^l}\Big] y^j y^k, \tag{4.8}$$

where

$$\tau_k = \frac{\partial}{\partial x^k}\Big(\frac{\rho F^{m-2}}{m-1}\Big),$$

and

$$X_{jl} = \rho_0 b_j b_l + \rho_1(A_j b_l + A_l b_j) + \rho_3 A_j A_k.$$

Now,

$$(4.9)\quad \overline{G}^i = \frac{1}{4}\Big[\frac{(m-1)g^{il}}{\rho F^{m-2}} - y^i(\sigma_3 y^l + \sigma_1 b^l) - b^i(\sigma_1 y^l + \sigma_0 b^l)\Big]\times$$
$$\Big[\Big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)\frac{\rho F^{m-2}}{m-1} + 2g_{jl}\tau_k - g_{jk}\tau_l + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Big]y^j y^k,$$

On simplifying, we get

$$\overline{G}^i = \frac{1}{4}g^{il}\Big(\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)y^j y^k + \frac{1}{4}\frac{(m-1)g^{il}}{\rho F^{m-2}}\Big(2g_{jl}\tau_k - g_{jk}\tau_l + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Big)y^j y^k$$
$$-y^i(\sigma_3 y^l + \sigma_1 b^l)\Big[(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l})\frac{\rho F^{m-2}}{m-1} + 2g_{jl}\tau_k - g_{jk}\tau_l + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Big]y^j y^k$$
$$-b^i(\sigma_1 y^l + \sigma_0 b^l)\Big[(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l})\frac{\rho F^{m-2}}{m-1} + 2g_{jl}\tau_k - g_{jk}\tau_l + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Big]y^j y^k$$

Or

$$(4.10)\quad \overline{G}^i = G^i + \frac{m-1}{\rho F^{m-2}}R_{jkl}\Big(\frac{F^{m-2}}{m-1}A^{il} + \frac{m-2}{m-1}y^i y^l\Big)y^j y^k$$
$$- y^i(\sigma_3 y^l + \sigma_1 b^l)S_{jkl}y^j y^k - b^i(\sigma_1 y^l + \sigma_0 b^l)S_{jkl}y^j y^k,$$

where

$$R_{jkl} = 2g_{jl}\tau_k - g_{jk}\tau_l + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l},$$

and

$$S_{jkl} = \Big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)\frac{\rho F^{m-2}}{m-1} + 2g_{jl}\tau_k - g_{jk}\tau_l + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}.$$

The above equation may be written as

$$(4.11)\qquad \overline{G}^i = G^i + Y^i P + Q^i,$$

where

$$P = \frac{m-2}{\rho F^{m-2}}R_{jkl}y^l y^j y^k - (\sigma_3 y^l + \sigma_1 b^l)S_{jkl}y^j y^k,$$

$$Q^i = \frac{R_{jkl}A^{il}}{\rho}y^j y^k - b^i(\sigma_1 y^l + \sigma_0 b^l)S_{jkl}y^j y^k.$$

Now, $\overline{F}$ and F are projectively related if $Q^i = 0$, which implies

$$(4.12)\qquad R_{jkl}A^{il} = \rho b^i(\sigma_1 y^l + \sigma_0 b^l)S_{jkl}.$$

Thus, we have the following

Theorem 1. *The Matsumoto transformed mth-root metric $\overline{F}$ and mth-root metric F, on an open subset U of M^n, are projectively related if eq. (4.12) is satisfied.*

5. Locally Dually Flatness of Matsumoto Transformed mth-root metric

The notion of dually flat Riemannian metrics was introduced by Amari and Nagaoka [2], when they studied the information geometry on Riemannian manifolds. In Finsler geometry, Shen [9] extended the notion of locally dually flatness for Finsler metrics. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which plays a very important role in studying flat Finsler information structure. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions.

A Matsumoto transformed Finsler metric $\overline{F} = \overline{F}(x, y)$ on a manifold M^n is said to be locally dually flat, if at any point there is a standard co-ordinate system (x^i, y^i) in TM such that $[\overline{F}^2]_{x^k y^l} y^k = 2[\overline{F}^2]_{x^l}$. In this case the co-ordinate (x^i) is called an adapted local co-ordinate system. Every locally Minkowskian metric is locally dually flat.

Consider the Matsumoto transformation $\overline{F} = \dfrac{F^2}{F-\beta}$, where F is an mth-root metric.

We have,

$$[\overline{F}^2]_{x^k} = \frac{\frac{4}{m} A^{\frac{4-m}{m}} A_{x^k}}{(F-\beta)^2} - \frac{2F^4(\frac{1}{m} A^{\frac{1-m}{m}} A_{x^k} - \beta_k)}{(F-\beta)^3}. \tag{5.1}$$

From (5.1), we get

$$\begin{aligned}[\overline{F}^2]_{x^k y^l} = {} & \Big[\frac{4(4-m)}{m^2}\frac{A^{\frac{2(2-m)}{m}}}{(F-\beta)^2} - \frac{8}{m^2}\frac{A^{\frac{5-2m}{m}}}{(F-\beta)^3} - \frac{2(5-m)}{m^2}\frac{A^{\frac{5-2m}{m}}}{(F-\beta)^3}\Big] A_{y^l} A_{x^k} + {} \\ & \Big[\frac{4}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^2} - \frac{2}{m}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^3}\Big] A_{x^k y^l} + \Big[\frac{6}{m^2}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^4} + \frac{8}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^3} - {} \\ & \frac{6}{m}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^4}\Big] \beta_k A_{y^l} + A^{\frac{5}{m}} b_{lk} - A^{\frac{4}{m}} \beta b_{lk} + 3A^{\frac{4}{m}} \beta_k b_l + \frac{8}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^3} A_{x^k b_l}.\end{aligned}$$

and

(5.2)

$$[\overline{F}^2]_{x^k y^l} y^k = \Big[\frac{4(4-m)}{m^2}\frac{A^{\frac{2(2-m)}{m}}}{(F-\beta)^2} - \frac{8}{m^2}\frac{A^{\frac{5-2m}{m}}}{(F-\beta)^3} - \frac{2(5-m)}{m^2}\frac{A^{\frac{5-2m}{m}}}{(F-\beta)^3}\Big]A_0A_{y^l} + \Big[\frac{4}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^2} - \frac{2}{m}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^3}\Big]A_{0l} + \Big[\frac{6}{m^2}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^4} + \frac{8}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^3} - \frac{6}{m}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^4}\Big]\beta_k A_{y^l} y^k + A^{\frac{5}{m}}\beta_l - A^{\frac{4}{m}}\beta\beta_l + 3A^{\frac{4}{m}}\beta_k b_l y^k + \frac{8}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^3}A_0 b_l.$$

Let the Finsler metric $\overline{F}$ is locally dually flat. Then, we have

$$[\overline{F}^2]_{x^k y^l} y^k - 2[\overline{F}^2]_{x^l} = 0. \tag{5.3}$$

Therefore from equations (5.1), (5.2) and (5.3), we obtain

(5.4)

$$[\overline{F}^2]_{x^k y^l} y^k - 2[\overline{F}^2]_{x^l} = \Big[\frac{4(4-m)}{m^2}\frac{A^{\frac{2(2-m)}{m}}}{(F-\beta)^2} - \frac{8}{m^2}\frac{A^{\frac{5-2m}{m}}}{(F-\beta)^3} - \frac{2(5-m)}{m^2}\frac{A^{\frac{5-2m}{m}}}{(F-\beta)^3}\Big]A_0A_{y^l} + \Big[\frac{4}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^2} - \frac{2}{m}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^3}\Big]A_{0l} + \Big[\frac{6}{m^2}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^4} + \frac{8}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^3} - \frac{6}{m}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^4}\Big]\beta_k A_{y^l} y^k + A^{\frac{5}{m}}\beta_l - A^{\frac{4}{m}}\beta\beta_l + 3A^{\frac{4}{m}}\beta_k b_l y^k + \frac{8}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^3}A_0 b_l - \frac{2}{(F-\beta)^3}\Big[\frac{4}{m}(F-\beta)A^{\frac{4-m}{m}}A_{x^l} - \frac{2}{m}F^4A^{\frac{1-m}{m}}A_{x^l} + 2F^4\beta_l\Big] = 0.$$

From eq. (5.4), we get

$$\frac{4}{m}(F-2\beta)F^3A_{x^l} = \frac{4}{m^2}\{(m-4\beta)-5F\}A^{\frac{3-m}{m}}A_0A_{y^l} + \frac{2}{m}(F-2\beta)F^3A_{0l} + \frac{2F^3\{3F(1-m)+4m\}}{m^2(F-\beta)}\beta_k A_{y^l} y^k + \frac{8}{m}F^3A_0b_l + F^5\beta_l - F^4\beta\beta_l + 3F^4\beta_k b_l y^k - 4F^4A^{\frac{m-1}{m}}\beta_l,$$

Therefore, $\overline{F}$ is locally dually flat metric if and only if

$$A_{x^l} = \frac{\{(m-4\beta)-5F\}}{mA(F-2\beta)}A_0A_{y^l} + \frac{A_{0l}}{2} + \frac{\{3F(1-m)+4m\}}{2m(F-\beta)(F-2\beta)}\beta_k A_{y^l} y^k + \frac{2A_0b_l}{(F-2\beta)} + \frac{mF^2\beta_l}{4(F-2\beta)} - \frac{mF\beta\beta_l}{4(F-2\beta)} + \frac{3mFb_l\beta_k y^k}{4(F-2\beta)} - \frac{mF^m\beta_l}{(F-2\beta)}. \tag{5.5}$$

Thus, we have the following,

Theorem 2. *Let $\overline{F}$ be a Matsumoto transformed mth-root Finsler metric on a manifold M^n. Then, $\overline{F}$ is locally dually flat if and only if equation (5.5) is satisfied.*

6. Projectively flatness of Matsumoto Transformed mth-root metric

A Matsumoto transformed Finsler Space with metric $\overline{F} = \dfrac{F^2}{F-\beta}$, on an open subset U of manifold M^n, is projectively flat [5] if and only if it satisfies the following equations :

$$\left[\overline{F}^2\right]_{x^k y^l} y^k = \left[\overline{F}^2\right]_{x^l}. \tag{6.1}$$

Therefore, from eq. (5.1), (5.2) and (5.3), we obtain

$$[\overline{F}^2]_{x^k y^l} y^k - [\overline{F}^2]_{x^l} = \Big[\frac{4(4-m)}{m^2}\frac{A^{\frac{2(2-m)}{m}}}{(F-\beta)^2} - \frac{8}{m^2}\frac{A^{\frac{5-2m}{m}}}{(F-\beta)^3} - \frac{2(5-m)}{m^2}\frac{A^{\frac{5-2m}{m}}}{(F-\beta)^3}\Big]A_0 A_{y^l} +$$
$$\Big[\frac{4}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^2} - \frac{2}{m}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^3}\Big]A_{0l} + \Big[\frac{6}{m^2}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^4} + \frac{8}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^3} -$$
$$\frac{6}{m}\frac{A^{\frac{5-m}{m}}}{(F-\beta)^4}\Big]\beta_k A_{y^l} y^k + A^{\frac{5}{m}}\beta_l - A^{\frac{4}{m}}\beta\beta_l + 3A^{\frac{4}{m}}\beta_k b_l y^k + \frac{8}{m}\frac{A^{\frac{4-m}{m}}}{(F-\beta)^3}A_0 b_l$$
$$-\frac{1}{(F-\beta)^3}\Big[\frac{4}{m}(F-\beta)A^{\frac{4-m}{m}}A_{x^l} - \frac{2}{m}F^4 A^{\frac{1-m}{m}}A_{x^l} + 2F^4\beta_l\Big] = 0.$$

Above equation can be written as,

$$\frac{4}{m}(F-2\beta)F^3 A_{x^l} = \frac{8}{m^2}\{(m-4\beta)-5F\}A^{\frac{3-m}{m}}A_0 A_{y^l} +$$
$$\frac{4}{m}(F-2\beta)F^3 A_{0l} + \frac{4F^3\{3F(1-m)+4m\}}{m^2(F-\beta)}\beta_k A_{y^l} y^k +$$
$$\frac{16}{m}F^3 A_0 b_l + F^5\beta_l - 2F^4\beta\beta_l + 6F^4\beta_k b_l y^k - 8F^4 A^{\frac{m-1}{m}}\beta_l,$$

Therefore, $\overline{F}$ is projectively flat metric if and only if

$$A_{x^l} = \frac{2\{(m-4\beta)-5F\}}{mA(F-2\beta)}A_0 A_{y^l} + A_{0l} + \frac{\{3F(1-m)+4m\}}{m(F-\beta)(F-2\beta)}\beta_k A_{y^l} y^k +$$
$$\frac{4A_0 b_l}{(F-2\beta)} + \frac{mF^2\beta_l}{2(F-2\beta)} - \frac{mF\beta\beta_l}{2(F-2\beta)} + \frac{3mFb_l\beta_k y^k}{2(F-2\beta)} - \frac{2mF^m\beta_l}{(F-2\beta)}. \tag{6.2}$$

Thus, we have the following:

Theorem 3. *A Finsler space $F^n = (M^n, \bar{F})$ with metric $\bar{F}$ on an open subset U of manifold M^n is projectively flat if and only if it satisfies the equation (6.2).*

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