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THE GEOMETRY OF TANGENT BUNDLES AND ALMOST COMPLEX STRUCTURES

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Abstract. In this paper, we study the geometry of a tangent bundle of a Riemannian manifold endowed with a Sasaki metric. Using O'Neill tensors given in [7], we prove some characteristic theorems comparing the geometries of a smooth manifold and its tangent bundle. We also show that there exists an almost complex structure on a Riemannian manifold which is not holomorphic to the canonical almost complex structure of its tangent bundle.

1. Introduction

The differential geometric properties of tangent bundle of smooth manifolds have been studied by different authors using different approaches with different notations. Many authors found interest in this topic because of its applications in many areas of Mathematics and Physics. The geometry of tangent bundle was initiated by one of Sasaki's papers [10] published in 1958. He used a given Riemannian metric g to construct a metric g^s called the Sasaki metric on the tangent bundle TM of a smooth manifold M. In [2], Dombrowski gave an explicit expression for the Lie bracket of the tangent bundle TM. Again, the Levi-Civita connection of the Sasaki metric on TM and its Riemannian curvature tensor are calculated by Kowalski in [6]. Sigmundur and Elias in [5] have written a detailed and unified presentation of some of the best known results on the geometry of tangent bundles of Riemannian manifolds.

After the study of the geometry of tangent bundle of a smooth manifold, it is important to also have a look at the submersion between the tangent bundle TM of M and the base manifold M itself.

Immersions and submersions are special tools in differential geometry, they play an important role in Riemannian geometry and other aspects of the differential geometry. The theory of Riemannian submersions was initiated by

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O'Neil [7] an Gray [4]. In [12], the Riemannian submersions were considered between almost Hermitian manifolds by Watson under the name almost Hermitian submersions. Almost Hermitian submersion have been actively studied between different kinds of subclasses of almost Hermitian manifolds, for example [11]. Most of the studies related to Riemannian or almost Hermitian submanifolds can be found in [3]. Note that Riemannian submersions are related to physics and have their applications in the Yang-Mills theory, the Kaluza-Klein theory, super-string theories, etc.

In this paper, we establish and extend some known results in [2] and [5] on the geometry of the tangent bundle of Riemannian manifold endowed with the Sasaki metric, together with a complex structure.

The paper is organized as follows. In Section 2, we recall some basic concepts on the tangent bundle and present the Sasakian metric that is used throughout the paper. In Section 3, we deal with the almost complex structure canonically obtained from the geometric structure of the tangent bundle. We also investigate the effect of the O'Neill tensors in the geometries of the tangent bundle TM and the base space M. We prove some characterization theorems linking the geometries of the tangent bundle to the one of the base space. Finally we end the paper by proving in Section 4 that there exists an almost structure complex on the base which makes the underlying Riemannian submersion a non-holomorphic map.

2. Preliminaries

Let (M, g) be an m-dimensional Riemannian manifold and ∇ be the Levi-Civita connection of g. Then, the tangent space of TM at any point $(x, u) \in TM$ splits into the horizontal and vertical subspaces with respect to the Levi-Civita connection ∇ in the Riemannian manifold (M, g) [2, 10]

$$T_{(x,u)}TM = \mathcal{H}_{(x,u)}(TM) \oplus \mathcal{V}_{(x,u)}(TM). \tag{2.1}$$

This decomposition is obtained using the two natural projections

$$TTM \to TM$$

which stem from the Riemannian manifold (M, g). The projection map $\pi\colon TM \to M$ induces a map $\pi_*\colon TTM \to TM$ whose kernel can be interpreted as those vectors $b \in T_{(x,u)}TM$ which lie tangentially to the fiber $T_{\pi(x,u)}M$ of TM. Hence vectors b are vertical and set

$$\mathcal{V}_{(x,u)}(TM) := \ker(\pi_{*|_{T_{(x,u)}TM}}). \tag{2.2}$$

The other projection is the *connection map* $K\colon TTM \to TM$ associated with the Levi-Civita ∇ on (M, g). This map K is orthogonal to the projection π_* in the sense that it geometrically assigns to $b \in T_{(x,u)}TM$ its vertical component, i.e., the component tangentially to the fiber $T_{\pi(x,u)}M$. The projection K is explicitly defined as follows.

Let

$$K\colon TTM \to TM, \quad b \mapsto Kb.$$

Let Z be a vector field in TM. Then, $Z\colon M \to TM$ induces a linear map $Z_*\colon TM \to TTM$ such that, for any $u \in T_xM$ with $x \in M$, $Z_*u \in T_{Z_x}TM$. The connection map $K\colon TTM \to TM$ is then defined by the property that if $b \in TTM$ is of the form $b = Z_*w$ for some $Z \in \Gamma(TM)$ and $w \in TM$, then [2]

$$Kb := K(Z_*w) = (\nabla_w Z)_{\pi(w)}. \tag{2.3}$$

This means that the Levi-Civita connection on (M, g) arises by first taking the differential Z_*w of Z in the direction of w and then projecting it back from TTM to the correct level TM via K in a way that extracts from Z_*w its component tangentially to $T_{\pi(w)}M$, yielding $(\nabla_w Z)_{\pi(w)}$. Hence, we can call a vector $b \in T_{(x,u)}TM \subset TTM$ with $Kb = 0$ horizontal and put

$$\mathcal{H}_{(x,u)}(TM) := \ker(K_{|T_{(x,u)}TM}). \tag{2.4}$$

The horizontal and vertical lifts of tangent vectors TM on TM are defined as follows.

Definition 2.1. [2] Let $(x, u) \in TM$ be given and $X \in T_xM$ be a tangent vector. Then, the *the horizontal lift* of X to a point (x, u) is the unique vector $X^h \in \mathcal{H}_{(x,u)}(TM)$ such that $\pi_*X^h = X$. The *vertical lift* of a vector X at $(x, u) \in TM$ is the unique vector $X^v \in \mathcal{V}_{(x,u)}(TM)$ such that $X^v(df) = X(f)$, for all functions f on M. Here df is the function defined by $(df)(x, u) = u(f)$.

This can now be extended from tangent vectors to vector fields.

Definition 2.2. The *horizontal lift* of a vector field $X \in C^\infty(TM)$ on TM is the vector field $X^h \in C^\infty(TTM)$ whose value at a point (x, u) is the horizontal lift of $X(x)$ at (x, u). The *vertical lift* of a vector field is defined in the same way. More precisely, if $X \in C^\infty(TM)$, then there is exactly one vector field $X^h \in C^\infty(TTM)$ on TM called the *horizontal lift* of X such that for all $Z \in TM$:

$$\pi_*(X^h)_Z = X_{\pi(Z)} \quad \text{and} \quad KX^h_Z = 0_{\pi(Z)}. \tag{2.5}$$

The *vertical lift* X^v is the unique vector field satisfying

$$\pi_*(X^v)_Z = 0_{\pi(Z)} \quad \text{and} \quad KX^v_Z = X_{\pi(Z)}. \tag{2.6}$$

Note that the mapping $X \to X^h$ and $X \to X^v$ are isomorphisms between the vector spaces T_xM and the subspaces $\mathcal{H}_{(x,u)}(TM)$ and $\mathcal{V}_{(x,u)}(TM)$, respectively. Each tangent vector $\overline{Z} \in T_{(x,u)}TM$ can be written as

$$X^h + Y^v,$$

where $X, Y \in \Gamma(T_xM)$ are uniquely determined by $X = \pi_*(\overline{Z})$ and $Y = K(\overline{Z})$. It should also be noted that if $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth real-valued function on M, then

$$X^h(f \circ \pi) = X(f) \circ \pi \quad \text{and} \quad X^v(f \circ \pi) = 0, \tag{2.7}$$

for any vector field X on M.

Next, we present the Sasaki metric g^s on the tangent bundle TM. This metric was introduced by Sasaki in [10]. In [5], the authors calculated its Levi-Civita connection ∇^s, its Riemannian curvature tensor, and obtained interesting connections between the geometric properties of the manifold (M, g) and its tangent bundle (TM, g^s) equipped with the Sasaki metric.

Definition 2.3. Let (M, g) be a Riemannian manifold. Then, the Sasaki metric g^s on the tangent bundle TM of M is given by

(i) $g^s_{(x,u)}(X^h, Y^h) = g(X, Y)$,
(ii) $g^s_{(x,u)}(X^v, Y^h) = 0$,
(iii) $g^s_{(x,u)}(X^v, Y^v) = g(X, Y)$,

for all vector fields $X, Y \in C^\infty(TM)$ and $(x, u) \in TM$.

Let ∇ be the Levi-Civita connection on (M, g) and R be its Riemannian curvature. These geometric objects are related to its analogous in (TM, g^s), namely, ∇^s and R^s, by the following results due to Gudmundsson and Kappos in [5].

Proposition 2.4 (Gudmundsson, Kappos [5]). *Let (M, g) be a Riemannian manifold and $\hat{\nabla}$ be the Levi-Civita connection of the tangent bundle (TM, g^s) equipped with the Sasaki metric. Then*

$$(\nabla^s_{X^v} Y^v)_{(x,u)} = 0, \tag{2.8}$$

$$(\nabla^s_{X^v} Y^h)_{(x,u)} = \frac{1}{2}(R(u, X)Y)^h, \tag{2.9}$$

$$(\nabla^s_{X^h} Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} - \frac{1}{2}(R(X, Y)u)^v, \tag{2.10}$$

$$(\nabla^s_{X^h} Y^v)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + \frac{1}{2}(R(u, Y)X)^h, \tag{2.11}$$

for all vector fields $X, Y \in C^\infty(TM)$ and $(x, u) \in TM$.

As known, the Sasaki metric is a particular class of the class of natural metrics. Since a natural metric is constructed in such a way that the vertical and horizontal subbundles are orthogonal and the bundle map

$$\pi\colon (TM, g^s) \to (M, g),$$

is a Riemannian submersion, we need the following result.

Proposition 2.5 (See [3]). *Let $\pi\colon (M', g^s) \to (M, g)$ be a Riemannian submersion, and denote by ∇^s and ∇ are the Levi-Civita connections of M' and M, respectively. One has:*

(1) $g^s(X', Y') = g(X, Y)$.
(2) $\pi_*(\nabla^s_{X'} Y')^h = \nabla_X Y$,

for all vector fields X',$Y' \in \Gamma(TM')$ and X, $Y \in \Gamma(TM)$ such that $\pi_*(X') = X$ and $\pi_*(Y') = Y$.

3. Almost complex structures and O'Neill tensors on the tangent bundle

Let (M, g) be an m-dimensional Riemannian manifold and ∇ be the Levi-Civita connection of g. It is well-known that the tangent bundle TM of M has a structure of almost complex Kählerian manifold with an almost complex structure determined by the isomorphism vertical and horizontal distributions $V(TM)$ and $H(TM)$ on TM (see [2], [8] for more details and any references therein) and the Sasaki metric on TM [10]. This almost complex structure which is naturally associated with the metric g is based on the decomposition (2.1) of the tangent bundle of TM into the horizontal and vertical subbundles at the point $(x, u) \in TM$, i.e.,

$$T_{(x,u)}TM = \mathcal{H}_{(x,u)}(TM) \oplus \mathcal{V}_{(x,u)}(TM).$$

We have isomorphisms

$$\mathcal{H}_{(x,u)}(TM) \cong T_xM \cong \mathcal{V}_{(x,u)}(TM).$$

Now, taking into account the properties of the two projections π_* and K, the map $\widetilde{J}\colon TTM \to TTM$, given by $A \mapsto \widetilde{J}A$, is therefore an almost complex structure for TM characterized by [2]

$$\pi_* \circ \widetilde{J} = K, \quad K \circ \widetilde{J} = -\pi_*. \tag{3.1}$$

If the tangent bundle TM is endowed with the Sasaki metric, then

$$g^s(\widetilde{J}\,\overline{Z}_1, \widetilde{J}\,\overline{Z}_2) = g^s(\overline{Z}_1, \overline{Z}_2), \tag{3.2}$$

for any tangent vectors $\overline{Z}_1, \overline{Z}_2 \in T_{(x,u)}TM$. This means that the Sasaki metric g^s is $\widetilde{J}$-invariant, and the triple $(TM, \widetilde{J}, g^s)$ is called an *almost Hermitian manifold* (see [3] for more details).

Let us now introduce the local coordinate representations of TM. Let $(U, \varphi) = (U, (x^1, \ldots, x^m))$ be a coordinate chart on M. The bundle chart of TM associated with $(x^1, \ldots, x^m)$ is $(\pi^{-1}(U), \widetilde{\varphi}) = (\pi^{-1}(U), (\overline{x}^1, \ldots, \overline{x}^{2m}))$, where $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m$ is defined by

$$\widetilde{\varphi}(v) = \left((x^1 \circ \pi)(v), \ldots, (x^m \circ \pi)(v), v(x^1), \ldots, v(x^m)\right),$$

for any $v \in \pi^{-1}(U)$. Thus $\overline{x}^i = x^i \circ \pi$ and $\overline{x}^{m+j} = v(x^j)$, for $1 \leq i, j \leq m$. Denote by Γ^k_{ij} the Christoffel symbols of g. The two complementary distributions on TTM in (2.2) and (2.4) are defined by

$$\mathcal{V}_{(x,u)}(TM) = \{a^i \frac{\partial}{\partial \overline{x}^{m+i}}|_{(x,u)} : a^i \in \mathbb{R}\}, \tag{3.3}$$

$$\mathcal{H}_{(x,u)}(TM) = \{a^i \frac{\partial}{\partial \overline{x}^i}|_{(x,u)} + a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial \overline{x}^{m+k}}|_{(x,u)} : a^i \in \mathbb{R}\}, \tag{3.4}$$

where $(x, u) \in TM$.

It is easy to see that

$$\pi_*((\frac{\partial}{\partial \overline{x}^i})_Z) = (\frac{\partial}{\partial x^i})_{\pi(Z)} \quad \text{and} \quad \pi_*(\frac{\partial}{\partial \overline{x}^{m+i}})_Z = 0,$$

for any $Z \in TM$ and $i = 1, 2, \ldots, m$, since

$$\begin{aligned} \pi_*(\frac{\partial}{\partial \overline{x}^i})(f) &= \frac{\partial}{\partial \overline{x}^i}(f \circ \pi) = \frac{\partial}{\partial x^i}(f), \\ \text{and} \quad \pi_*(\frac{\partial}{\partial \overline{x}^{m+i}})(f) &= \frac{\partial}{\partial \overline{x}^{m+i}}(f \circ \pi) = 0, \end{aligned}$$

for any smooth function f defined on M. This leads to the following result for the horizontal and vertical lifts of a vector field on M.

Let X, $Z \in C^\infty(TM)$ be vector vectors on M which locally are represented by

$$X = \sum_{i=1}^m \xi^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Z = \sum_{i=1}^m \eta^i \frac{\partial}{\partial x^i}.$$

In local coordinates the map $Z\colon M \to TM$ is given by

$$Z\colon M \to TM, \quad (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^m, \eta^1, \ldots, \eta^m).$$

We have

$$Z_* X = \sum_{k=1}^{2m} X(\overline{x}^k \circ Z) \frac{\partial}{\partial \overline{x}^k}.$$

Since $\overline{x}^k \circ Z = x^k \circ \pi \circ Z = x^k$ and $\overline{x}^{m+k} \circ Z = Z(x^k) = \eta^k$, for $1 \leq k \leq m$, one obtains

$$\begin{aligned} Z_* X &= \sum_{i=1}^m X(\overline{x}^i \circ Z) \frac{\partial}{\partial \overline{x}^i} + \sum_{i=1}^m X(\overline{x}^{m+i} \circ Z) \frac{\partial}{\partial \overline{x}^{m+i}} \\ &= \sum_{i=1}^m \xi^i \frac{\partial}{\partial \overline{x}^i} + \sum_{i,j=1}^m \xi^j \frac{\partial \eta^i}{\partial x^j} \frac{\partial}{\partial \overline{x}^{m+i}}. \end{aligned} \tag{3.5}$$

On the other hand, using the properties of a linear connection, we get

$$\begin{aligned} \nabla_X Z &= \sum_{i=1}^m \nabla_X (\eta^i \frac{\partial}{\partial x^i}) \\ &= \sum_{i=1}^m \sum_{j=1}^m \xi^j \{ \frac{\partial \eta^i}{\partial x^j} + \sum_{k=1}^m \eta^k \Gamma^i_{jk} \} \frac{\partial}{\partial x^i}. \end{aligned} \tag{3.6}$$

Now, by (2.3), we have

$$K(Z_* X) = \sum_{i=1}^m \sum_{j=1}^m \xi^j \{ \frac{\partial \eta^i}{\partial x^j} + \sum_{k=1}^m \eta^k \Gamma^i_{jk} \} \frac{\partial}{\partial x^i}. \tag{3.7}$$

This relation implies that $K(Y_*v) = 0$, if and only if

$$\sum_{j=1}^{m} \xi^j \frac{\partial \eta^i}{\partial x^j} = -\sum_{j,k=1}^{m} \xi^j \eta^k \Gamma^i_{jk}.$$

This means that $K(Y_*v) = 0$ if and only if Z_*X is in the kernel of K and therefore Z_*X is of the form

$$Z_*X = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial \overline{x}^i} - \sum_{i=1}^{m}\sum_{j,k=1}^{m} \xi^j \eta^k \Gamma^i_{jk} \frac{\partial}{\partial \overline{x}^{m+i}}.$$

Hence, we have

$$X^h{}_{|Z} = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial \overline{x}^i} - \sum_{i=1}^{m}\sum_{j,k=1}^{m} \xi^j \eta^k \Gamma^i_{jk} \frac{\partial}{\partial \overline{x}^{m+i}}, \tag{3.8}$$

$$\text{and} \quad X^v{}_{|Z} = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial \overline{x}^{m+i}}. \tag{3.9}$$

Consequently, we have

$$(\frac{\partial}{\partial \overline{x}^i})^H = \frac{\partial}{\partial \overline{x}^i} - \sum_{j,k=1}^{m} \eta^k \Gamma^j_{ik} \frac{\partial}{\partial \overline{x}^{m+j}}, \tag{3.10}$$

$$\text{and} \quad (\frac{\partial}{\partial \overline{x}^i})^V = \frac{\partial}{\partial \overline{x}^{m+i}}. \tag{3.11}$$

Using the relations in (3.1), it follows that

$$\widetilde{J}X^h = X^v, \quad \text{and} \quad \widetilde{J}X^v = -X^h. \tag{3.12}$$

Now, let $w \in T_{(x,u)}TM$. Then

$$w = \sum_{i=1}^{m} w^i \frac{\partial}{\partial \overline{x}^i} + \sum_{j=1}^{m} w^{m+j} \frac{\partial}{\partial \overline{x}^{m+j}}. \tag{3.13}$$

Assume that $Kw = 0$. Then $w \in \mathcal{H}_{(x,u)}(TM)$ and

$$\sum_{i=1}^{m} w^i K(\frac{\partial}{\partial \overline{x}^i}) = -\sum_{j=1}^{m} w^{m+j} K(\frac{\partial}{\partial \overline{x}^{m+j}}). \tag{3.14}$$

Using (3.10) and (3.11), one has

$$K(\frac{\partial}{\partial \overline{x}^{m+i}}) = \frac{\partial}{\partial \overline{x}^i} \quad \text{and} \quad K(\frac{\partial}{\partial \overline{x}^i}) = \sum_{j,k=1}^{m} \eta^k \Gamma^j_{ik} \frac{\partial}{\partial \overline{x}^j}. \tag{3.15}$$

Putting the pieces (3.15) into (3.14), we have

$$w^{m+j} = -\sum_{i,k=1}^{m} w^i \eta^k \Gamma^j_{ik}. \tag{3.16}$$

It follows that w can be written as a sum

$$w = w^H + w^V,$$

where

$$w^H = \sum_{i=1}^{m} w^i \frac{\partial}{\partial \overline{x}^i} - \sum_{j=1}^{m}\left(\sum_{i,k=1}^{m} w^i \eta^k \Gamma_{ik}^j\right)\frac{\partial}{\partial \overline{x}^{m+j}}, \tag{3.17}$$

$$\text{and} \quad w^V = \sum_{j=1}^{m}\left(w^{m+j} + \sum_{i,k=1}^{m} w^i \eta^k \Gamma_{ik}^j\right)\frac{\partial}{\partial \overline{x}^{m+j}}. \tag{3.18}$$

Next, we define anti-invariant Riemannian submersions from an almost Hermitian manifold onto a Riemannian manifold.

Definition 3.1 ([9])**.** Let N be a complex $2m$-dimensional almost Hermitian manifold with Hermitian metric g_N and almost complex structure $\widetilde{J}$ and N' be a Riemannian manifold with Riemannian metric $g_{N'}$. Suppose that there exists a Riemannian submersion $F\colon N \to N'$ such that $\ker F_*$ is anti-invariant with respect to $\widetilde{J}$, i.e., $\widetilde{J}(\ker F_*) \subseteq (\ker F_*)^\perp$. Then, we say that F is an *anti-invariant Riemannian submersion.* Moreover, if $\widetilde{J}(\ker F_*) = (\ker F_*)^\perp$, we say that F is a *Lagrangian submersion.*

Next, applying $\widetilde{J}$ to (3.17) and (3.18), one has,

$$\begin{aligned}\widetilde{J}w^H &= \sum_{i=1}^{m} w^i \widetilde{J}(\frac{\partial}{\partial \overline{x}^i}) - \sum_{j=1}^{m}(\sum_{i,k=1}^{m} w^i \eta^k \Gamma_{ik}^j)\widetilde{J}(\frac{\partial}{\partial \overline{x}^{m+j}})\\ &= \sum_{i=1}^{m} w^i \frac{\partial}{\partial \overline{x}^{m+i}},\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\widetilde{J}w^V &= \sum_{j=1}^{m}\left(w^{m+j} + \sum_{i,k=1}^{m} w^i \eta^k \Gamma_{ik}^j\right)\widetilde{J}(\frac{\partial}{\partial \overline{x}^{m+j}})\\ &= -\sum_{j=1}^{m}\left(w^{m+j} + \sum_{i,k=1}^{m} w^i \eta^k \Gamma_{ik}^j\right)\left(\frac{\partial}{\partial \overline{x}^j} - \sum_{l,k=1}^{m} \eta^k \Gamma_{jk}^l \frac{\partial}{\partial \overline{x}^{m+l}}\right).\end{aligned}$$

Letting

$$\widetilde{w}^j = w^{m+j} + \sum_{i,k=1}^{m} w^i \eta^k \Gamma_{ik}^j,$$

we have

$$\widetilde{J}w^V = -\sum_{j=1}^{m} \widetilde{w}^j \frac{\partial}{\partial \overline{x}^j} + \sum_{l,j,k=1}^{m} \widetilde{w}^j \eta^k \Gamma_{jk}^l \frac{\partial}{\partial \overline{x}^{m+l}}. \tag{3.20}$$

Using the relations (3.8), (3.9), (3.19) and (3.20), we conclude that

$$\widetilde{J}\mathcal{V}_{(x,u)}(TM) = \mathcal{H}_{(x,u)}(TM). \tag{3.21}$$

We have the following lemma.

Lemma 3.2. *The Riemannian submersion $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ is Lagrangian.*

As an example, we have the following.

Example 3.3. Let $\pi\colon \mathbb{R}^4 \to \mathbb{R}^2$ be a submersion defined by

$$\pi(x_1, x_2, x_3, x_4) = \left(\frac{x_1 - x_4}{\sqrt{2}}, \frac{x_2 - x_3}{\sqrt{2}}\right).$$

Then, by a straightforward calculation, we have

$$\mathcal{V}(\mathbb{R}^4) = \ker \pi_* = \mathrm{Span}\{Z_1 = \partial x_1 + \partial x_4, Z_2 = \partial x_2 + \partial x_3\}$$

and

$$\mathcal{H}(\mathbb{R}^4) = (\ker \pi_*)^{\perp} = \mathrm{Span}\{X_1 = \partial x_1 - \partial x_4, X_2 = \partial x_2 - \partial x_3\}.$$

It is easy to see that π is a Riemannian submersion. Moreover, $\widetilde{J}Z_1 = X_2$ and $\widetilde{J}Z_2 = -X_1$ imply that $\widetilde{J}\mathcal{V}(\mathbb{R}^4) = \mathcal{H}(\mathbb{R}^4)$. As a result, π is a Lagrangian Riemannian submersion.

The next two theorems confirm the results on the tangent bundle endowed with a Sasakian metric given in [3, Example 1.3]. They shall be proved using the O'Neill's tensors.

In general, the geometry of Riemannian submersions is characterized by O'Neil's tensors $\mathcal{T}$ and $\mathcal{A}$ defined

$$\mathcal{T}_{\overline{Z}}\overline{W} = (\nabla^s_{\overline{Z}^v}\overline{W}^v)^h + (\nabla^s_{\overline{Z}^v}\overline{W}^h)^v, \tag{3.22}$$

$$\text{and } \mathcal{A}_{\overline{Z}}\overline{W} = (\nabla^s_{\overline{Z}^h}\overline{W}^v)^h + (\nabla^s_{\overline{Z}^h}\overline{W}^h)^v, \tag{3.23}$$

for any vector fields $\overline{Z}$ and $\overline{W}$ on TM at (x,u).

For any $\overline{Z} \in \Gamma(T_{(x,u)}TM)$, $\mathcal{T}_{\overline{Z}}$ and $\mathcal{A}_{\overline{Z}}$ are skew-symmetric operators on (TTM, g^s) reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_{\overline{Z}} = \mathcal{T}_{\overline{Z}^v}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_{\overline{Z}} = \mathcal{A}_{\overline{Z}^h}$. We note that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T}_{\overline{Z}^v}\overline{W}^v = \mathcal{T}_{\overline{W}^v}\overline{Z}^v, \tag{3.24}$$

$$\text{and } \quad \mathcal{A}_{\overline{Z}^h}\overline{W}^h = -\mathcal{A}_{\overline{W}^h}\overline{Z}^h = \frac{1}{2}[\overline{Z}^h, \overline{W}^h]^v. \tag{3.25}$$

Using (3.24) and (3.25), one obtains

$$\nabla^s_{\overline{Z}^v}\overline{W}^v = \mathcal{T}_{\overline{Z}^v}\overline{W}^v + (\nabla^s_{\overline{Z}^v}\overline{W}^v)^v, \tag{3.26}$$

$$\nabla^s_{\overline{Z}^v}\overline{W}^h = \mathcal{T}_{\overline{Z}^v}\overline{W}^h + (\nabla^s_{\overline{Z}^v}\overline{W}^h)^h, \tag{3.27}$$

$$\nabla^s_{\overline{Z}^h}\overline{W}^v = \mathcal{A}_{\overline{Z}^h}\overline{W}^v + (\nabla^s_{\overline{Z}^h}\overline{W}^v)^v, \tag{3.28}$$

$$\nabla^s_{\overline{Z}^h}\overline{W}^h = \mathcal{A}_{\overline{Z}^h}\overline{W}^h + (\nabla^s_{\overline{Z}^h}\overline{W}^h)^h. \tag{3.29}$$

Now, for any $\overline{Z},\ \overline{W} \in \Gamma(T_{(x,u)}TM)$, we have $\overline{Z} = X_1^h + Y_1^v$ with $X_1 = \pi_*(\overline{Z})$ and $Y_1 = K(\overline{Z})$, and $\overline{W} = X_2^h + Y_2^v$ with $X_2 = \pi_*(\overline{W})$ and $Y_2 = K(\overline{W})$. Using this, (2.10) and (2.11), one has,

$$\begin{aligned}(\mathcal{A}_{\overline{Z}}\overline{W})_{(x,u)} &= (\nabla^s_{\overline{Z}^h}\overline{W}^v)^h_{(x,u)} + (\nabla^s_{\overline{Z}^h}\overline{W}^h)^v_{(x,u)}\\ &= (\nabla^s_{X_1^h}Y_2^v)^h_{(x,u)} + (\nabla^s_{X_1^h}X_2^h)^v_{(x,u)}.\end{aligned} \tag{3.30}$$

Since

$$(\nabla^s_{X_1^h}Y_2^v)_{(x,u)} = (\nabla_{X_1}Y_2)^v_{(x,u)} + \frac{1}{2}(R(u,Y_2)X_1)^h, \tag{3.31}$$

$$\text{and}\quad \nabla^s_{X_1^h}X_2^h)_{(x,u)} = (\nabla_{X_1}X_2)^h_{(x,u)} - \frac{1}{2}(R(X_1,X_2)u)^v. \tag{3.32}$$

The relation (3.30) becomes

$$(\mathcal{A}_{\overline{Z}}\overline{W})_{(x,u)} = \frac{1}{2}(R(u,Y_2)X_1)^h - \frac{1}{2}(R(X_1,X_2)u)^v. \tag{3.33}$$

This implies that

$$\pi_*(\mathcal{A}_{\overline{Z}}\overline{W})_{(x,u)} = \frac{1}{2}R(u,Y_2)X_1, \quad \text{and} \quad K(\mathcal{A}_{\overline{Z}}\overline{W})_{(x,u)} = -\frac{1}{2}R(X_1,X_2)u. \tag{3.34}$$

We have the following theorem.

Theorem 3.4. *Let $\pi\colon (TM,\widetilde{J},g^s) \to (M,g)$ be a Riemannian submersion from an almost Kähler manifold $(TM,\widetilde{J},g^s)$ onto a Riemannian manifold (M,g). Then the following assertions are equivalent:*

(i) *$\mathcal{H}(TM) = \ker K$ is integrable.*
(ii) *M is flat.*
(iii) *The almost complex structure $\widetilde{J}$ is Kähler.*

Proof. The equivalence of (i) and (ii) follows from (3.33) and (3.34). The one between (ii) and (iii) is given in [2]. □

Theorem 3.4 can be extended to more comparisons between the geometries of the manifold (M,g) and its tangent bundle TM equipped with the Sasaki metric g^s and the almost complex structure $\widetilde{J}$. Therefore, we have the following theorem.

Theorem 3.5. *Let* $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ *be a Riemannian submersion from a Kähler manifold* $(TM, \widetilde{J}, g^s)$ *onto a Riemannian manifold* (M, g)*. Then the following assertions hold:*

(i) $(TM, g^s, \widetilde{J})$ *is flat.*
(ii) $(TM, g^s, \widetilde{J})$ *is Einstein.*
(iii) $(TM, g^s, \widetilde{J})$ *is locally symmetric.*
(iv) $(TM, g^s, \widetilde{J})$ *is locally homogeneous.*
(v) $(TM, g^s, \widetilde{J})$ *has constant scalar curvature.*

Likewise (2.8) and (2.9), we have

$$\begin{aligned}\mathcal{T}_{\overline{Z}}\overline{W} &= (\nabla^s_{\overline{Z}^v}\overline{W}^v)^h + (\nabla^s_{\overline{Z}^v}\overline{W}^h)^v\\ &= (\nabla^s_{Y_1^v}Y_2^v)^h + (\nabla^s_{Y_1^v}X_2^h)^v\\ &= 0. \end{aligned} \tag{3.35}$$

Therefore, we have the following theorem.

Theorem 3.6. *Let* $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ *be a Riemannian submersion from an almost Kähler manifold* $(TM, \widetilde{J}, g^s)$ *onto a Riemannian manifold* (M, g)*. Then, the distribution* $\mathcal{V}(TM) = \ker \pi_*$ *defines a totally geodesic foliation on* TM*.*

Finally, let us recall the notion of harmonic maps between Riemannian manifolds TM and M (see [1] for more details). Then, the differential π_* of π can be viewed as a section of the bundle $\mathcal{H}om(TTM, \pi^{-1}TM) \to TM$, where $\pi^{-1}TM$ is the pullback bundle which has fibres $(\pi^{-1}TM)_x = T_{\pi(x)}M$, $x \in M$. $\mathcal{H}om(TTM, \pi^{-1}TM)$ has a connection ∇^s induced from the Levi-Civita connection ∇ on M and the pullback connection ∇^π. Then, the second fundamental form of π is given by

$$(\nabla^s\pi_*)(\overline{Z}, \overline{W}) = \nabla^\pi_{\overline{Z}}\pi_*\overline{W} - \pi_*(\nabla^s_{\overline{Z}}\overline{W}), \tag{3.36}$$

for any $\overline{Z}$, $\overline{W} \in \Gamma(T_{(x,u)}TM)$.

Note that a differentiable map F Riemannian manifolds is called *totally geodesic* if $\nabla^s F_* = 0$.

Let $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ be a Riemannian submersion from an almost Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g). We know that the second fundamental form of a Riemannian submersion satisfies

$$(\nabla^s\pi_*)(\overline{Z}^h, \overline{W}^h) = 0, \tag{3.37}$$

for any $\overline{Z}$, $\overline{W} \in \Gamma(T_{(x,u)}TM)$ and using (3.27) and (3.36), we have

$$\begin{aligned}(\nabla^s\pi_*)(\overline{Z}^v, \overline{W}^v) &= \nabla^{\pi}_{\overline{Z}^v}\pi_*\overline{W}^v - \pi_*(\nabla^s_{\overline{Z}^v}\overline{W}^v)\\ &= -\pi_*(\nabla^s_{\overline{Z}^v}\overline{W}^v)\\ &= \pi_*(\widetilde{J}(\widetilde{J}\nabla^s_{\overline{Z}^v}\overline{W}^v))\\ &= \pi_*(\widetilde{J}(\mathcal{T}_{\overline{Z}^v}\widetilde{J}\overline{W}^v - (\nabla^s_{\overline{Z}^v}\widetilde{J})\overline{W}^v)).\end{aligned} \tag{3.38}$$

On the other hand, for any $\overline{X} \in \Gamma(T_{(x,u)}TM)$, using (3.29) and (3.36), we get

$$\begin{aligned}(\nabla^s\pi_*)(\overline{X}^h, \overline{W}^v) &= \nabla^{\pi}_{\overline{X}^h}\pi_*\overline{W}^v - \pi_*(\nabla^s_{\overline{X}^h}\overline{W}^v)\\ &= \pi_*(\widetilde{J}(\widetilde{J}\nabla^s_{\overline{X}^h}\overline{W}^v))\\ &= \pi_*(\widetilde{J}(\mathcal{A}_{\overline{X}^h}\widetilde{J}\overline{W}^v - (\nabla^s_{\overline{X}^h}\widetilde{J})\overline{W}^v)).\end{aligned} \tag{3.39}$$

Therefore, we have the following theorem.

Theorem 3.7. *Let $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ be a Riemannian submersion from an almost Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g). If the structure $\widetilde{J}$ is Kähler, then the Riemannian submersion π is a totally geodesic map.*

Proof. Assume that the structure $\widetilde{J}$ is Kähler. Then $\nabla^s\widetilde{J} = 0$ on TM and using the Theorem 3.6, the relations (3.38) and (3.39)

$$(\nabla^s\pi_*)(\overline{Z}^v, \overline{W}^v) = \pi_*(\widetilde{J}(\nabla^s_{\overline{Z}^v}\widetilde{J})\overline{W}^v) = 0,$$

$$\text{and}\quad (\nabla^s\pi_*)(\overline{X}^h, \overline{W}^v) = \pi_*(\widetilde{J}(\mathcal{A}_{\overline{X}^h}\widetilde{J}\overline{W}^v - (\nabla^s_{\overline{X}^h}\widetilde{J})\overline{W}^v)) = 0,$$

which completes the proof. □

Now, for any $\overline{X}$, $\overline{Z}$, $\overline{W} \in \Gamma(T_{(x,u)}TM)$,

$$\begin{aligned}g^s(\nabla^s_{\overline{X}^h}\overline{W}^h, \overline{Z}^v) &= g^s(\nabla^s_{\overline{X}^h}\widetilde{J}\overline{W}^h - (\nabla^s_{\overline{X}^h}\widetilde{J})\overline{W}^h, \widetilde{J}\overline{Z}^v)\\ &= g^s(\mathcal{A}_{\overline{X}^h}\widetilde{J}\overline{W}^h - (\nabla^s_{\overline{X}^h}\widetilde{J})\overline{W}^h, \widetilde{J}\overline{Z}^v).\end{aligned} \tag{3.40}$$

But

$$g^s(\mathcal{A}_{\overline{X}^h}\widetilde{J}\overline{W}^h, \widetilde{J}\overline{Z}^v) = -g^s((\nabla^s\pi_*(\overline{X}^h, \widetilde{J}\overline{W}^h), \widetilde{J}\overline{Z}^v). \tag{3.41}$$

Therefore we have the following theorem.

Theorem 3.8. *Let $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ be a Riemannian submersion from an almost Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g). Then the following assertions are equivalent:*

(i) $\mathcal{H}(TM) = \ker K$ *defines a totally geodesic foliation on* TM.

(ii) $\mathcal{A}_{\overline{X}^h}\widetilde{J}\overline{W}^h = 0$.

(iii) $\nabla^s\pi_*(\overline{X}^h, \widetilde{J}\overline{W}^h) = 0$,

for any $\overline{X}$, $\overline{W} \in \Gamma(T_{(x,u)}TM)$.

If π is a Riemannian submersion from a Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g), then we have the following corollary.

Corollary 3.9. *Let $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ be a Riemannian submersion from a Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g). Then $\mathcal{H}(TM) = \ker K$ defines a totally geodesic foliation on TM.*

Now, we obtain some decomposition theorems for the Riemannian submersion π from a Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g).

Definition 3.10 ([9])**.** Let G be metric be a Riemannian metric tensor on the manifold $N = M \times B$ and assume that the canonical foliations $\mathcal{D}_M$ and $\mathcal{D}_B$ intersect perpendicularly everywhere. Then G is a metric tensor of

(i) a usual product of Riemannian manifolds if and only if $\mathcal{D}_M$ and $\mathcal{D}_B$ are totally geodesic foliations.
(ii) a twisted product if and only if $\mathcal{D}_M$ is a totally geodesic foliation and $\mathcal{D}_B$ is a totally umbilical foliation.

We have the following decomposition theorem for the Riemannian submersion π which follows from Theorem 3.6 and Theorem 3.8.

Theorem 3.11. *Let $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ be a Riemannian submersion from an almost Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g). Then the tangent bundle TM is a locally product manifold if and only if $\mathcal{A}_{\overline{X}^h} \widetilde{J}\overline{W}^h = 0$, for any $\overline{X}$, $\overline{W} \in \Gamma(T_{(x,u)}TM)$.*

If π is a Riemannian submersion from a Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g), then we have the following corollary.

Corollary 3.12. *Let $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ be a Riemannian submersion from a Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g). Then the tangent bundle TM is a locally product manifold.*

Let $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ be a Riemannian submersion from an almost Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g). Let α be the second fundamental form of $\mathcal{H}(TM)$. Then we have

$$\begin{aligned} g^s(\nabla^s_{\overline{Z}^h}\overline{W}^h, \overline{X}^v) &= g^s((\nabla^s_{\overline{Z}^h}\overline{W}^h)^h + (\nabla^s_{\overline{Z}^h}\overline{W}^h)^v, \overline{X}^v) \\ &= g^s(\alpha(\overline{Z}^h, \overline{W}^h), \overline{X}^v), \end{aligned} \tag{3.42}$$

for any $\overline{X}$, $\overline{W} \in \Gamma(T_{(x,u)}TM)$. If $\mathcal{H}(TM)$ is a totally umbilical foliation, we have

$$g^s(\nabla^s_{\overline{Z}^h}\overline{W}^h, \overline{X}^v) = g^s(H, \overline{X}^v) g^s(\overline{Z}^h, \overline{W}^h), \tag{3.43}$$

where H is the mean curvature vector field of $\mathcal{H}(TM)$.

On the other hand, we have

$$g^s(\nabla^s_{\overline{Z}^h}\overline{W}^h, \overline{X}^v) = -g^s(\widetilde{J}\overline{W}^h, \widetilde{J}\nabla^s_{\overline{Z}^h}\overline{X}^v)$$
$$= -g^s(\widetilde{J}\overline{W}^h, \mathcal{A}_{\overline{Z}^h}\widetilde{J}\overline{X}^v - (\nabla^s_{\overline{Z}^h}\widetilde{J})\overline{X}^v). \tag{3.44}$$

Thus from (3.42) and (3.44), we have

$$-g^s(H, \overline{X}^v)\widetilde{J}\overline{Z}^h = \mathcal{A}_{\overline{Z}^h}\widetilde{J}\overline{X}^v - (\nabla^s_{\overline{Z}^h}\widetilde{J})\overline{X}^v,$$

which implies, using (3.25), that

$$g^s(H, \overline{X}^v)||\widetilde{J}\overline{Z}^h||^2 = g^s(\overline{X}^v, \mathcal{A}_{\overline{Z}^h}\overline{Z}^h) = 0.$$

Since g^s is a Riemannian metric and $H \in \mathcal{V}(TM)$, we obtain $H = 0$, that is, $\mathcal{H}(TM)$ is totally geodesic. Therefore, we have the following theorem.

Theorem 3.13. *There exist no Riemannian submersions from an almost Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g) such that TM is a locally proper twisted product manifold of the form $TM_{\mathcal{V}(TM)} \times_f TM_{\mathcal{H}(TM)}$.*

4. An almost complex structure in the base space

Let $\pi\colon (TM, \widetilde{J}, g^s) \to (M, g)$ be a Riemannian submersion from an almost Kähler manifold $(TM, \widetilde{J}, g^s)$ onto a Riemannian manifold (M, g).

Let $J\colon TM \to TM$ be a smooth tensor field of (1,1)-type on M defined by

$$J = \pi_* \circ \widetilde{J} \circ \pi_*^{-1}. \tag{4.1}$$

Let $X \in \Gamma(T_xM)$. Then, there exists a tangent vector $\overline{Z} \in T_{(x,u)}TM$ such that $X = \pi_*(\overline{Z})$ or $X = K(\overline{Z})$. Now, if $X = \pi_*(\overline{Z})$, we have

$$J^2X = (\pi_* \circ \widetilde{J}) \circ \widetilde{J}(\overline{Z}) = -X. \tag{4.2}$$

Likewise, if $X = K(\overline{Z})$, using relations in (3.1), one obtains

$$J^2X = (\pi_* \circ \widetilde{J} \circ \pi_*^{-1}) \circ (\pi_* \circ \widetilde{J} \circ \pi_*^{-1})(K(\overline{Z})) = -X. \tag{4.3}$$

From (4.2) and (4.3), we conclude that,

$$J^2X = -X,$$

for any $X \in \Gamma(T_xM)$.

Next we investigate whether the metric g on M is Hermitian with respect to the structure J defined in (4.1).

Let TM be the tangent bundle of (M, g) endowed with the Sasaki metric g^s. For any X, $Y \in \Gamma(T_xM)$, there are tangent vectors $\overline{Z}_1$, $\overline{Z}_2 \in T_{(x,u)}TM$ such that $X = \pi_*(\overline{Z}_1)$ or $X = K(\overline{Z}_1)$, and $Y = \pi_*(\overline{Z}_2)$ or $Y = K(\overline{Z}_2)$. Then, for $X = \pi_*(\overline{Z}_1)$ and $Y = \pi_*(\overline{Z}_2)$, using (3.2), (4.1) and Proposition 2.5, we have

$$g(JX, JY) = g(J \circ \pi_*(\overline{Z}_1), J \circ \pi_*(\overline{Z}_2))$$
$$= g(\pi_*(\overline{Z}_1), \pi_*(\overline{Z}_2)) = g(X, Y). \tag{4.4}$$

For $X=\pi_*(\overline{Z}_1)$ and $Y=K(\overline{Z}_2)$, using Proposition 2.5, one has,

$$\begin{aligned} g(JX,JY) &= g((J\circ\pi_*)(\overline{Z}_1),(J\circ K)(\overline{Z}_2)) \\ &= g(\pi_*(\overline{Z}_1),\pi_*\circ\widetilde{J}(\overline{Z}_2)) = g(X,Y). \end{aligned} \tag{4.5}$$

For $X=K(\overline{Z}_1)$ and $Y=\pi_*(\overline{Z}_2)$, we similarly obtain

$$g(JX,JY)=g(X,Y). \tag{4.6}$$

Now, for $X=K(\overline{Z}_1)$ and $Y=K(\overline{Z}_2)$, using (3.2), (4.1) and Proposition 2.5, we have

$$\begin{aligned} g(JX,JY) &= g(J\circ K(\overline{Z}_1),J\circ K(\overline{Z}_2)) \\ &= g(\pi_*\circ\widetilde{J}\circ\pi_*^{-1}\circ K(\overline{Z}_1),\pi_*\circ\widetilde{J}\circ\pi_*^{-1}\circ K(\overline{Z}_2)) \\ &= g(K(\overline{Z}_1),K(\overline{Z}_2)) = g(X,Y). \end{aligned} \tag{4.7}$$

From the pieces (4.4), (4.5), (4.6) and (4.7), we conclude that

$$g(JX,JY)=g(X,Y), \tag{4.8}$$

for any X, $Y\in\Gamma(T_xM)$ with $x\in M$. Therefore, we have the following theorem.

Theorem 4.1. *Let $(TM,\widetilde{J},g^s)$ be a tangent bundle, over an m-dimensional smooth Riemannian manifold (M,g), endowed with am Sasakian metric and an almost complex structure $\widetilde{J}$. Then, the tensor field of (1,1)-type $J: TM\longrightarrow TM$ defined by*

$$J=\pi_*\circ\widetilde{J}\circ\pi_*^{-1},$$

is an almost complex structure on the smooth manifold M. Moreover, the dimension of M is even.

Note that, since $(TM,\widetilde{J},g^s)$ and (M,J,g) with J defined in (4.1) are almost Hermitian manifolds, the Riemannian submersion $\pi\colon (TM,\widetilde{J},g^s)\to(M,J,g)$ *does not satisfy*, in general, the following equality

$$J\circ\pi_*=\pi_*\circ\widetilde{J}. \tag{4.9}$$

Any Riemannian submersions satisfy the relation (4.9) is called *an almost complex map.* The details of the latter can be found in [3] and references therein, in which characterizations are given for P-manifolds are given.

The non satisfaction of the structure under study can obviously be observed through the following two evaluations. For any $X\in\Gamma(TM)$, using (2.5), (2.6) and (3.12), one obtains

$$\begin{aligned} &J\circ\pi_*(X^v)=J(\pi_*X^v)=0, \\ \text{and}\quad &\pi_*\circ\widetilde{J}(X^v)=\pi_*(\widetilde{J}X^v)=-\pi_*X^h=-X. \end{aligned} \tag{4.10}$$

Therefore, we have the following proposition.

Proposition 4.2. *The Riemannian submersion* $\pi\colon (TM, \widetilde{J}, g^s) \to (M, J, g)$ *is not an almost complex map.*

Now, we want to introduce the integrability condition on the almost complex structure $\widetilde{J}$ on the tangent bundle (TM, g^s) over (M, g), endowed with the Sasaki metric g^s, and its effect of the almost complex structure J on M.

Definition 4.3. An almost Hermitian manifold $(TM, \widetilde{J}, g^s)$ (respectively, (M, J, g)) is called *Kähler manifold* if $\widetilde{J}$ is parallel with respect to the Levi-Civita connection ∇^s on TM (respectively, if J is parallel with respect to the Levi-Civita connection ∇ on M).

Now assume that $(TM, \widetilde{J}, g^s)$ is a Kähler manifold. Then

$$\nabla^s \widetilde{J} = 0. \tag{4.11}$$

Using Proposition 2.5 and for any $\overline{Z}, \overline{W} \in \Gamma(T_{(x,u)}TM)$, one has $\overline{Z} = X_1^h + Y_1^v$, $\overline{W} = X_2^h + Y_2^v$ with $X_1 = \pi_*(\overline{Z})$, $Y_1 = K(\overline{Z})$, $X_2 = \pi_*(\overline{W})$ and $Y_2 = K(\overline{W})$,

$$\nabla^s_{\overline{Z}} \overline{W} = \nabla^s_{X_1^h} X_2^h + \nabla^s_{X_1^h} Y_2^v + \nabla^s_{Y_1^v} X_2^h + \nabla^s_{Y_1^v} Y_2^v. \tag{4.12}$$

For any X, $Y \in \Gamma(TM)$, there exist $\overline{Z}$ and $\overline{W} \in \Gamma(T_{(x,u)}TM)$ such that $X = \pi_*(\overline{Z})$ or $X = K(\overline{Z})$ and $Y = \pi_*(\overline{W})$ or $Y = K(\overline{W})$.

Now, if $X = \pi_*(\overline{Z})$ and $Y = \pi_*(\overline{W})$, by Proposition 2.5, (2.5) and (4.1), we have,

$$\begin{aligned}
(\nabla_X J)Y &= \nabla_X JY - J(\nabla_X Y) \\
&= \nabla_{\pi_*(\overline{Z})} J \circ \pi_*(\overline{W}) - J(\nabla_{\pi_*(\overline{Z})} \pi_*(\overline{W})) \\
&= \nabla_{\pi_*(\overline{Z})} \pi_* \circ \widetilde{J}(\overline{W}) - \pi_* \circ \widetilde{J} \circ \pi_*^{-1}(\nabla_{\pi_*(\overline{Z})} \pi_*(\overline{W})) \\
&= (\nabla^s_{\overline{Z}} \widetilde{J})\overline{W} = 0. && (4.13)
\end{aligned}$$

If $X = \pi_*(\overline{Z})$ and $Y = K(\overline{W})$, then in this case $K(\overline{Z}) = 0$ and $\pi_*(\overline{W}) = 0$, and we have

$$\begin{aligned}
(\nabla_X J)Y &= \nabla_X JY - J(\nabla_X Y) \\
&= \nabla_{\pi_*(\overline{Z})} J \circ K(\overline{W}) - J(\nabla_{\pi_*(\overline{Z})} K(\overline{W})) \\
&= -\nabla_{\pi_*(\overline{Z})} \pi_*(\overline{W}) = 0. && (4.14)
\end{aligned}$$

If $X = K(\overline{Z})$ and $Y = \pi_*(\overline{W})$, then $\pi_*(\overline{Z}) = K(\overline{W}) = 0$, and we have

$$\begin{aligned}
(\nabla_X J)Y &= \nabla_X JY - J(\nabla_X Y) \\
&= \nabla_{K(\overline{Z})} J \circ \pi_*(\overline{W}) - J(\nabla_{K(\overline{Z})} \pi_*(\overline{W})) \\
&= \nabla_{K(\overline{Z})} K(\overline{W}) - K(\nabla^s_{\widetilde{J}\overline{Z}} \overline{W})^h = 0. && (4.15)
\end{aligned}$$

Lastly, if $X = K(\overline{Z})$ and $Y = K(\overline{W})$, then $\pi_*(\overline{Z}) = \pi_*(\overline{W}) = 0$, we have

$$\begin{aligned}(\nabla_X J)Y &= \nabla_X JY - J(\nabla_X Y)\\ &= -\nabla_{\pi_*\circ\widetilde{J}(\overline{Z})}\pi_*(\overline{W}) - J(\nabla_{\pi_*\circ\widetilde{J}(\overline{Z})}\pi_*\circ\widetilde{J}(\overline{W}))\\ &= -\nabla_{\pi_*\circ\widetilde{J}(\overline{Z})}\pi_*(\overline{W}) - K(\nabla^s_{\widetilde{J}(\overline{Z})}\widetilde{J}\overline{W})^h = 0. \end{aligned}\tag{4.16}$$

From (4.13), (4.14), (4.15) and (4.16), we have the following theorem.

Theorem 4.4. *Let $(TM, \widetilde{J}, g^s)$ be a tangent bundle endowed with a Sasakian metric and an almost complex structure $\widetilde{J}$ over an even-dimensional smooth Riemannian manifold (M, J, g) with J an almost complex structure defined in (4.1). If the almost structure $\widetilde{J}$ is complex, so is J.*

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