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ON $N(k)$ MIXED QUASI EINSTEIN MANIFOLDS AND SOME GLOBAL PROPERTIES

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This paper has been dedicated to Prof. M. Majumdar and Prof. A. Bhattacharyya

Abstract. In this paper we have studied quasi conformally flat, conharmonically flat and projectively flat $N(k)$-mixed quasi Einstein manifold, Ricci-semi symmetric $N(k)$-mixed quasi Einstein manifold $N(k)-(MQE)_n$, $(n > 3)$ and studied some properties on it.

1. Introduction

The notion of quasi Einstein manifold was introduced in a paper [8] by M. C. Chaki and R. K. Maity. According to them a non-flat Riemannian manifold (M^n, g),$(n \geq 3)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{1}$$

and is not identically zero, where a, b are scalars $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, U) = A(X), \ \forall X \in TM. \tag{2}$$

U being a unit vector field. In such a case a, b are called the associated scalars. A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold is denoted by the symbol$(QE)_n$.

Again, U. C. De and G. C. Ghosh defined generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold if its Ricci-tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) \tag{3}$$

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where a, b, c, are non-zero scalars and A, B are two 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X) \tag{4}$$

U, V being unit vectors which are orthogonal, i.e.

$$g(U,V) = 0. \tag{5}$$

The vector fields U and V are called the generators of the manifold. This type of manifold are denoted by $G(QE)_n$.

The k-nullity distribution [15] of a Riemannian manifold M is defined by

$$N(k) : p \to N_p(k) = \{Z \in T_pM \setminus R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}. \tag{6}$$

for all $X, Y \in TM$ and k is a smooth function. M. M. Tripathi and Jeong Jik Kim [14] introduced the notion of $N(k)$-quasi Einstein manifold which defined as follows: If the generator U belongs to the k-nullity distribution $N(k)$, then a quasi Einstein manifold (M^n, g) is called $N(k)$-quasi Einstein manifold.

In [13], H. G. Nagaraja introduced the concept of $N(k)$-mixed quasi Einstein manifold and mixed quasi constant curvature.A non flat Riemannian manifold (M^n, g) is called a $N(k)$-mixed quasi Einstein manifold if its Ricci tensor of type $(0,2)$ is non zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)B(Y) + cB(X)A(Y), \tag{7}$$

where a, b, c, are smooth functions and A, B are non zero 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X)\ \forall X, \tag{8}$$

U, V being the orthogonal unit vector fields called generators of the manifold belong to $N(k)$. Such a manifold is denoted by the symbol $N(k) - (MQE)_n$.

Again a Riemannian manifold (M^n, g) is called of mixed quasi constant curvature if it is conformally flat and curvature tensor $\acute{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned} \acute{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ q[g(X,W)A(Y)B(Z) - g(X,Z)A(Y)B(W) \\ &+ g(X,W)A(Z)B(Y) - g(X,Z)A(W)B(Y)] \\ &+ s[g(Y,Z)A(W)B(X) - g(Y,W)A(Z)B(X) \\ &+ g(Y,Z)A(X)B(W) - g(Y,W)A(X)B(Z)]. \end{aligned} \tag{9}$$

2. Preliminaries

We know in a n-dimensional $(n > 2)$ Riemannian manifold the covariant quasi conformal curvature tensor is defined as ([1], [3], [7], [11])

$$\begin{aligned} \tilde{C}(X,Y,Z,W) &= \acute{a}\acute{R}(X,Y,Z,W) + \acute{b}[S(Y,Z)g(X,W) \\ &- S(X,Z)g(Y,W) + g(Y,Z)g(QX,W) - g(X,W)g(QY,W)] \\ &- \frac{r}{n}[\frac{\acute{a}}{n-1} + 2\acute{b}][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \end{aligned} \tag{10}$$

where

$$g(C(X,Y)Z,W) = \tilde{C}(X,Y,Z,W). \tag{11}$$

$$g(R(X,Y)Z,W) = \tilde{R}(X,Y,Z,W). \tag{12}$$

The projective curvature tensor is denoted by $\tilde{P}(X,Y,Z,W)$ and in a V_n $(n > 2)$ it is defined as

$$\begin{aligned} (13)\quad & \tilde{P}(X,Y,Z,W) \\ & = \tilde{R}(X,Y,Z,W) - \frac{1}{n-1}[S(Y,Z)g(X,W) - S(Y,W)g(X,W)]. \end{aligned}$$

From (7) and (8), we get

$$S(X,X) = a|X|^2 + (b+c)|g(X,U)g(X,V)|, \ \forall X. \tag{14}$$

Let θ_1 be the angle between U and any vector X; θ_2 be the angle between V and any vector X. Then

$$\cos\theta_1 = \frac{g(X,U)}{\sqrt{g(U,U)}\sqrt{g(X,X)}} = \frac{g(X,U)}{\sqrt{g(X,X)}}(\text{ as } g(U,U) = 1)$$

and $\cos\theta_2 = \frac{g(X,V)}{\sqrt{g(X,X)}}$. If $b > 0$ and $c > 0$ we have from (14)

$$(a+b+c)|X|^2 \geq a|X|^2 + (b+c)|g(X,U)g(X,V)| = S(X,X) \tag{15}$$

Now, contracting (7) over X and Y, we get

$$r = na \tag{16}$$

where r is the scalar curvature.

Again from (7) we have

$$S(U,U) = a \tag{17}$$

$$S(V,V) = a \tag{18}$$

If X is a unit vector field, then $S(X,X)$ is the Ricci-curvature in the direction of X.

Q be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S, where

$$g(QX,Y) = S(X,Y) \ \forall X,Y \in TM. \tag{19}$$

Let l^2 denote the squares of the lengths of the Ricci-tensor S. Then

$$l^2 = \sum_{i=1}^{n} S(Qe_i, e_i) \tag{20}$$

where $\{e_i\}$, $i = 1,2,\ldots,n$ is an orthonormal basis of the tangent space at a point of $N(k) - (MQE)_n$.

Now from (7) we get

$$S(Qe_i, e_i) = ag(Qe_i, e_i)bA(Qe_iB(e_i) + cB(Qe_i)A(e_i) \tag{21}$$

i.e. $l^2 = na^2 + b^2 + c^2$.

3. Quasi conformally flat $N(k)$- mixed quasi Einstein manifold

Let a $N(k)$- mixed quasi Einstein manifold is quasi conformally flat. Considering $\tilde{C}(X,Y)Z = 0$. For all vector fields X, Y, Z it follows form (10) that

$$(22)\quad R(X,Y)Z = \frac{r}{n\acute{a}}[\frac{\acute{a}}{n-1} + 2\acute{b}][g(Y,Z)X - g(X,Z)Y] - \frac{\acute{b}}{\acute{a}}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY],$$

or

$$(23)\quad g(R(X,Y)Z,W) = \frac{r}{n\acute{a}}[\frac{\acute{a}}{n-1} + 2\acute{b}][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] - \frac{\acute{b}}{\acute{a}}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)g(QX,W) - g(X,Z)g(QY,W)].$$

Using (7), (8), (12) in (22) we get

$$(24)\quad \acute{R}(X,Y,Z,W) = \{\frac{r}{n\acute{a}}[\frac{\acute{a}}{n-1} + 2\acute{b}] - \frac{2a\acute{b}}{\acute{a}}\}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] - \frac{b\acute{b}}{\acute{a}}[g(X,W)A(Y)B(Z) - g(Y,W)A(X)B(Z) + g(Y,Z)A(X)B(W) - g(X,Z)A(Y)B(W)] - \frac{c\acute{c}}{\acute{a}}[g(X,W)B(Y)A(Z) - g(Y,W)B(X)A(Z) + g(Y,Z)B(X)A(W) - g(X,Z)B(Y)A(W)].$$

Thus from (24) we get

Theorem 3.1. *A quasi conformally flat $N(k) - (MQE)_n$ is a manifold of mixed quasi constant curvature.*

Corollary 3.1. *A conharmonically flat $N(k) - (MQE)_n$ is a manifold of mixed quasi constant curvature.*

Corollary 3.2. *A projectively flat $N(k) - (MQE)_n$ is not a manifold of mixed quasi constant curvature.*

4. Ricci Semi-symmetric $N(k) - (MQE)_n (n > 3)$

Chaki and Maity proved that $(QE)_n (n > 3)$ is Ricci Semi-symmetric if and only if $A(R(X,Y)Z = 0$. Let us suppose that $N(k) - (MQE)_n$ $(n > 3)$ is Ricci-Semi symmetric. Then

$$(25)\quad A(R(X,Y)Z = 0.$$

From (25) we get

$$A(Q(X)) = 0 \tag{26}$$

where Q be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S. Then

$$g(QX, Y) = S(X, Y). \tag{27}$$

Then from (7) we get

$$A(Q(X)) = aA(X) + cB(X). \tag{28}$$

From (26) and (28) it follows that

$$aA(X) + cB(X) = 0. \tag{29}$$

Thus we can state the following

Theorem 4.1. *If a $N(k) - (MQE)_n$ is Ricci Semi symmetric than $aA(X) + cB(X) = 0$.*

5. $N(k) - (MQE)_n (n > 3)$ with divergence free quasi conformal curvature tensor

We know quasi conformal curvature tensor is said to be conservative if divergence of $\acute{C}$ vanishes , i.e. $\operatorname{div} \acute{C} = 0$. In this section we obtain a sufficient condition for a $N(k) - (MQE)_n$ be a quasi conformally conservative. In a $N(k) - (MQE)_n$ if a, b and c are constant, then contracting (7) we obtain

$$r = na, \text{ i.e. } dr = 0 \tag{30}$$

where r is the scalar curvature. Using (30) in (10) we get

$$\begin{aligned} (31)\quad (\nabla_W \acute{C})(X, Y, Z) &= a_1 (\nabla_W R)(X, Y)Z \\ &+ b_1[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y + g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y]. \end{aligned}$$

We know that

$$(\operatorname{div} R)(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z). \tag{32}$$

Now from (7) we get

$$\begin{aligned} (33)\quad (\nabla_X S)(Y, Z) = b[(\nabla_X A)(Y)B(Z) + (\nabla_X B)(Z)A(Y)] \\ + c[(\nabla_X B)(Y)A(Z) + (\nabla_X A)(X)B(Z)] \end{aligned}$$

where b and c are constant. Hence contracting (31) and using (33) we obtain

$$\begin{aligned}(\operatorname{div}\acute{C})(X,Y,Z) = {} & 2b(a_1+b_1)[(\nabla_X A)(Y)B(Z)+(\nabla_X B)(Z)A(Y) \\ & -(\nabla_Y A)(X)B(Z)-(\nabla_Y B)(Z)A(X)] \\ & +2c(a_1+b_1)[(\nabla_X B)(Y)A(Z)+(\nabla_X A)(Z)B(Y) \\ & -(\nabla_Y B)(X)A(Z)-(\nabla_Y A)(Z)B(X)] \\ & +bb_1[(\nabla_U A)(X)+B(X)\operatorname{div}U+(\nabla_U B)(X)+A(X)\operatorname{div}U]g(Y,Z) \\ & -cb_1[(\nabla_U A)(Y)+B(Y)\operatorname{div}U+(\nabla_U B)(Y)+A(Y)\operatorname{div}U]g(X,Z).\end{aligned} \tag{34}$$

Now if we consider the generator U of the manifold is a recurrent vector field [6] with associated 1-form A, not being the 1-form of recurrence, gives $\nabla_X U = D(X)U$, where D is the 1-form of recurrence, we get

$$g(\nabla_X U, Y) = g(D(X)U, Y), \text{ i.e. } (\nabla_X A)(Y) = D(X)A(Y) \tag{35}$$

so we get

$$\begin{aligned}(\operatorname{div}\acute{C})(X,Y,Z) = {} & 2b(a_1+b_1)[D(X)A(Y)B(Z)+D(X)B(Z)A(Y) \\ & -D(Y)A(X)B(Z)-D(Y)B(Z)A(X)] \\ & +2c(a_1+b_1)[D(X)B(Y)A(Z)+D(X)A(Z)B(Y) \\ & -D(Y)B(X)A(Z)-D(Y)A(Z)B(X)] \\ & +bb_1[D(U)A(X)+D(U)B(X)]g(Y,Z) \\ & -cb_1[D(U)A(Y)+D(U)B(Y)]g(X,Z).\end{aligned} \tag{36}$$

Since $(\nabla_X A)(U) = 0$, it follows from (35) we get $D(X) = 0$. Hence from (36) we get $(\operatorname{div}\acute{C})(X,Y,Z) = 0$. Thus we get

Theorem 5.1. *If in a $N(k) - (MQE)_n (n > 3)$ the associated scalars are constants and generator U of the manifold is a recurrent vector field with the associated 1-form A not being the 1-form of recurrence, then the manifold is quasi-conformally conservative.*

6. Sufficient condition for a compact, orientable $N(k) - (MQE)_n$(n≥3) without boundary to be isometric to a sphere

In this section we consider a compact, orientable $N(k) - (MQE)_n$ without boundary having constant associated scalars a, b, c. Then from (16) and (19), it follows that the scalar curvature is constant and so also is the length of the Ricci-tensor.

We further suppose that $N(k) - (MQE)_n$ under consideration admits a non-isometric conformal motion generated by a vector field X. Since l^2 is constant, it follows that

$$\pounds_X l^2 = 0. \tag{37}$$

where $\pounds_X$ denotes Lie differentiation with respect to X. Now, it is known ([2], [4], [5], [9], [12]) that if a compact Riemannian manifold M of dimension $n > 2$ with constant scalar curvature admits an infinitesimal non-isometric conformal transformation X such that $\pounds_X l^2 = 0$ then M is isometric to a sphere. But a sphere is Einstein so that b and c vanish which is a contradiction. This leads to the following theorem.

Theorem 6.1. *A compact orientable $N(k)$-mixed quasi Einstein manifold $N(k) - (MQE)_n (n \geq 3)$ without boundary does not admit a non-isometric conformal vector field.*

7. Killing vector field in a compact orientable $N(k) - (MQE)_n (n \geq 3)$ without boundary

In this section, we consider a compact, orientable $N(k) - (MQE)_n$ $(n \geq 3)$ without boundary with a, b, c as associated scalars and U and V as the generators.

It is known [4] that in such a manifold M, the following relation holds

$$\int_M [S(X,X) - |\nabla X|^2 - (divX)^2] dv \leq 0 \ \forall X. \tag{38}$$

If X is a Killing vector field, then $\operatorname{div} X = 0$ [4]. Hence (38) takes the form

$$\int_M [S(X,X) - |\nabla X|^2] dv = 0. \tag{39}$$

Let $b > 0$, $c > 0$ then by (15)

$$(a+b+c)|X|^2 \geq S(X,X) \tag{40}$$

Therefore,

$$(a+b+c)|X|^2 - |\nabla X|^2 \geq S(X,X) - |\nabla X|^2 \tag{41}$$

Consequently,

$$\int_M [(a+b+c)|X|^2 - |\nabla X|^2] dv \geq \int_M [S(X,X) - |\nabla X|^2] dv \tag{42}$$

and by (39)

$$\int_M [(a+b+c)|X|^2 - |\nabla X|^2] dv \geq 0 \tag{43}$$

If $a + b + c$ <0, then

$$\int_M [(a+b+c)|X|^2 - |\nabla X|^2] dv = 0. \tag{44}$$

Therefore, $X = 0$. This leads to the following theorem.

Theorem 7.1. *If in a compact, orientable $N(k) - (MQE)_n (n \geq 3)$ without boundary the associated scalars are such that $b > 0, c > 0$ and $a + b + c < 0$ then there exists no non-zero killing vector field in this manifold.*

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