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SOME OPTIMAL INEQUALITIES FOR SCREEN CONFORMAL HALF-LIGHTLIKE SUBMANIFOLDS

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Abstract. In this paper, some relations involving the main intrinsic and extrinsic invariants for a half-lightlike submanifold of a Lorentzian manifold are given. Some results for screen conformal half-lightlike submanifolds and their leaves are obtained with the help of these relations.

1. Introduction

To establish relationships between intrinsic and extrinsic invariants of a submanifold is one of the most fundamental problems in submanifolds theory. Some principal inequalities for submanifolds of a real space form were initially proved by B.-Y. Chen [5, 6, 7, 8]. Then, the study of this topic has attracted a lot of attention during the last two decades. Related inequalities have been established for different kinds of Riemannian submanifolds in ambient manifolds endowed with different kinds of structures by various geometers (see [2, 22, 24, 25, 26, 27, 29, 30] etc.).

Beside these facts, there exist also some useful relations involving curvatures for submanifolds of a semi-Riemannian manifold. Recently, B.-Y. Chen [9] showed very significant applications of these type inequalities to nondegenerate submanifolds and the authors [21] proved some general inequalities for r-lightlike submanifolds of a semi-Riemannaian manifold. Also, the authors and S. Keleş [17, 18] give some relations involving the curvatures on lightlike hypersurfaces of a Lorentzian manifold.

The main purpose of this paper is to show some relations involving the intrinsic and extrinsic invariants for half-lightlike submanifolds of a Lorentzian manifold. The paper is arranged as follows. In section 2, some basic facts about half-lightlike submanifolds are mentioned. In section 3, curvatures on half-lightlike submanifolds are investigated. In section 4, some inequalities for screen conformal half-lightlike submanifolds are established. In section 5, some

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results related to screen conformal half-lightlike submanifolds and their leaves are obtained.

2. Half-lightlike submanifolds

Let $(\widetilde{M},\widetilde{g})$ be an $(n+3)$-dimensional Lorentzian manifold with a non-degenerate metric $\widetilde{g}$ of constant index 1 and (M,g) be an $(n+1)$-dimensional lightlike submanifold of $(\widetilde{M},\widetilde{g})$, where g is the induced degenerate metric from $\widetilde{g}$. Then there exists a smooth distribution $\operatorname{Rad} T_pM$, called *radical space* of the tangent space T_pM at $p\in M$, defined by

$$\operatorname{Rad} T_pM = \{\xi \in T_pM : g_p(\xi, X) = 0 \text{ for all } X \in T_pM\}. \tag{2.1}$$

The complementary non-degenerate vector bundle $S(TM)$ of $\operatorname{Rad} TM$ in TM is called *screen bundle* of M. Thus, we have

$$TM = \operatorname{Rad} TM \oplus_{orth} S(TM), \tag{2.2}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. The submanifold $(M,g,S(TM))$ is called a *half-lightlike submanifold* if rank of the radical space is one [12, 14].

Let $(M,g,S(TM))$ be an $(n+1)$-dimensional half-lightlike submanifold of $(\widetilde{M},\widetilde{g})$. Then there exist a one dimensional non-degenerate sub-bundle $\mathcal{D}$ spanned by u and a one-dimensional degenerate sub-bundle $\operatorname{ltr}(TM)$ spanned by N such that

$$\begin{gathered}\widetilde{g}(\xi,u)=0, \quad \widetilde{g}(N,\xi)=\mp 1,\\ \widetilde{g}(N,u)=\widetilde{g}(N,N)=0, \quad \widetilde{g}(u,u)\neq 0.\end{gathered} \tag{2.3}$$

Here, $\mathcal{D}$ is called *screen transversal bundle* and $\operatorname{ltr}(TM)$ is called *lightlike transversal bundle* [13, 16]. From the equations (2.2) and (2.3), we have the following decomposition:

$$T\widetilde{M} = S(TM) \oplus_{\mathrm{orth}} \mathcal{D} \oplus_{\mathrm{orth}} (\operatorname{Rad} TM \oplus \operatorname{ltr}(TM)), \tag{2.4}$$

where $\oplus$ denotes the direct sum but it is not orthogonal.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$. The Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^t_X N, \tag{2.6}$$

$$\widetilde{\nabla}_X u = -A_u X + \nabla^t_X u \tag{2.7}$$

for all $X,Y\in\Gamma(TM)$, where $\nabla_X Y, A_N X, A_u X \in \Gamma(TM)$ and $h(X,Y), \nabla^t_X N$, $\nabla^t_X u \in \Gamma(\operatorname{tr}(TM))$. Here, h and A_N are called *second fundamental form* and *shape operator* of M, respectively. From the equations (2.4), (2.5), (2.6) and (2.7), we have

$$h(X,Y) = B(X,Y)N + D(X,Y)u, \tag{2.8}$$

$$\nabla^t_X N = \rho_1(X)N + \rho_2(X)u \tag{2.9}$$

and

$$\nabla^t_X u = \epsilon_1(X)N + \epsilon_2(X)u. \tag{2.10}$$

Thus, we write (2.5), (2.6) and (2.7) again as follows:

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)u, \tag{2.11}$$

$$\widetilde{\nabla}_X N = -A_N X + \rho_1(X)N + \rho_2(X)u, \tag{2.12}$$

$$\widetilde{\nabla}_X u = -A_u X + \epsilon_1(X)N + \epsilon_2(X)u. \tag{2.13}$$

From (2.3) and (2.11), we see that B is symmetric, it is independent of choosing screen distribution and it vanishes on $\mathrm{Rad}\,TM$.

Let P be the projection morphism of $\Gamma(TM)$ to $\Gamma(S(TM))$. We also write from (2.2) that

$$\nabla_X PY = \nabla^*_X PY + h^*(X,Y), \tag{2.14}$$

$$\nabla_X \xi = -A^*_\xi(X) - \rho_1(X)\xi, \tag{2.15}$$

where $\nabla^*_X PY, A^*_\xi(X) \in \Gamma(S(TM))$ and $h^*(X,Y) \in \Gamma(\mathrm{Rad}\,TM)$. Here, A^*_ξ is called *local shape operator* and h^* is called *local second fundamental form* given by

$$h^*(X,Y) = C(X,PY)\xi. \tag{2.16}$$

Using (2.3), (2.11), (2.12), (2.13), (2.14) and (2.15), we have the followings:

$$B(X,Y) = g(A^*_\xi X, Y), \tag{2.17}$$

$$C(X,PY) = g(A_N X, PY), \tag{2.18}$$

$$D(X,PY) = g(A_u X, PY), \tag{2.19}$$

$$D(X,Y) = g(A_u X, PY) - \epsilon_1(X)\eta(Y). \tag{2.20}$$

The manifold $(M, g, S(TM))$ is called a *totally geodesic half-lightlike submanifold* if

$$B(X,Y) = D(X,Y) = 0, \ \forall X, Y \in \Gamma(TM). \tag{2.21}$$

Furthermore, $(M, g, S(TM))$ is called *irrotational* [23] if h vanishes on $\mathrm{Rad}\,TM$ and it is called *totally geodesic* if h vanishes on $\Gamma(TM)$, identically [13]. If there exists a smooth transversal vector field H such that

$$h(X,Y) = g(X,Y)H \tag{2.22}$$

for all $X, Y \in \Gamma(TM)$, then the submanifold is called *totally umbilical* [15]. Furthermore, $(M, g, S(TM))$ is called *minimal* if it is irrotational and

$$\mathrm{trace}_{S(TM)} h = 0, \tag{2.23}$$

where $\mathrm{trace}_{S(TM)}$ denotes the trace restricted to $S(TM)$ with respect to the degenerate metric g [4].

Let $(M, g, S(TM))$ be an $(n+1)$-dimensional half-lightlike submanifold and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. Let us consider

$$\mu_1 = \frac{1}{n}\sum_{j=1}^{n} B(e_j, e_j) \text{ and } \mu_2 = \frac{1}{n}\sum_{j=1}^{n} D(e_j, e_j). \tag{2.24}$$

Then it is clear from (2.23) and (2.24) that M is minimal if and only if $\mu_1 = \mu_2 = 0$.

Let φ be a non-zero function on a neigborhood $\mathcal{U}$ of M. Then M is called *screen locally conformal* if A_N and A^*_ξ related by

$$A_N = \varphi A^*_\xi, \tag{2.25}$$

i.e.,

$$C(X, PY) = \varphi B(X, Y) \tag{2.26}$$

for all $X, Y \in \Gamma(TM)$ [10].

3. Curvatures on half-lightlike submanifolds

Let $(M, g, S(TM))$ be an $(n+1)$-dimensional half-lightlike submanifold of an $(n+3)$-dimensional Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. Let us denote the Riemannian curvature tensors of $\widetilde{M}$ and M by $\widetilde{R}$ and R, respectively. Then the following relations between these tensors hold:

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)PZ, PW) &= g(R(X,Y)PZ, PW) \\ &\quad + B(X, PZ)C(Y, PW) - B(Y, PZ)C(X, PW) \\ &\qquad + [D(X, PZ)D(Y, PW) - D(Y, PZ)D(X, PW)],\end{aligned} \tag{3.1}$$

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)PZ, \xi) &= (\nabla_X B)(Y, PZ) - (\nabla_Y B)(X, PZ) \\ &\quad + \rho_1(X)B(Y, PZ) - \rho_1(Y)B(X, PZ) \\ &\qquad + \varepsilon_1(X)D(Y, PZ) - \varepsilon_1(Y)D(X, PZ),\end{aligned} \tag{3.2}$$

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)PZ, N) &= g(R(X,Y)PZ, N) \\ &\qquad + [\rho_2(Y)D(X, PZ) - \rho_2(X)D(Y, PZ)],\end{aligned} \tag{3.3}$$

$$\widetilde{g}(\widetilde{R}(X,Y)\xi, N) = g(R(X,Y)\xi, N) + \rho_2(X)\epsilon_1(Y) - \rho_2(Y)\epsilon_1(X) \tag{3.4}$$

for all $X, Y, Z, U \in \Gamma(TM)$ [16].

Now let us choose a 2-dimensional non-degenerate plane section

$$\Pi = \mathrm{Span}\{X, Y\}$$

in T_pM, $p \in M$. Then the sectional curvature at p is expressed by [3]

$$K(\Pi) = \frac{g(R(X,Y)Y, X)}{g(X,X)g(Y,Y) - g(X,Y)^2}. \tag{3.5}$$

From the equations (3.1) and (3.5), we get the following lemma:

Lemma 3.1. *Let $(M, g, S(TM))$ be a half-lightlike submanifold of a Lorentzian manifold. Then we have*

$$K(\Pi) = \widetilde{K}(\Pi) + B(Y,Y)C(X,X) - B(X,Y)C(Y,X) + D(X,X)D(Y,Y) - D(X,Y)^2 \tag{3.6}$$

for any non-degenerate plane section $\Pi = \mathrm{Span}\{X, Y\}$ in T_pM, $p \in M$.

From the equation (2.18) and Lemma 3.1, it is clear that the sectional curvature map defined on any half-lightlike submanifold doesn't need to be symmetric and thus it hasn't a geometric meaning which is different from the Riemannian context.

Now, we recall the following result of K. L. Duggal and A. Bejancu [13]:

Theorem 3.2. *Let $(M, g, S(TM))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then the following assertions are equivalent:*

i) *$S(TM)$ is integrable.*
ii) *h^* is symmetric on $\Gamma(S(TM))$.*
iii) *A_N is self-adjoint on $\Gamma(S(TM))$ with respect to g.*

As a consequence of Theorem 3.2, we can state the following:

Corollary 3.3. *Let $(M, g, S(TM))$ be a half-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. The sectional curvature map is symmetric if and only if $S(TM)$ is integrable.*

Remark 3.4. Taking into consideration Corollary 3.3, we see that the sectional curvature of any half-lightlike submanifold includes important geometric meanings as in the Riemannian context when its screen distribution is integrable.

The screen Ricci tensor, denoted by $\mathrm{Ric}_{S(TM)}$, is defined by

$$\mathrm{Ric}_{S(TM)}(X, Y) = \mathrm{trace}|_{S(TM)}\{Z \to R(X, Z)Y\} \tag{3.7}$$

for any $X, Y, Z \in \Gamma(S(TM))$ [17, 22]. The screen Ricci curvature at a unit vector field X in $\Gamma(S(TM))$ is given by

$$\mathrm{Ric}_{S(TM)}(X) = \sum_{j=1}^{n} g(R(X, e_j)e_j, X), \tag{3.8}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\Gamma(S(TM))$.

The screen scalar curvature at a point $p \in M$, denoted by $r_{S(TM)}(p)$, is defined by

$$r_{S(TM)}(p) = \frac{1}{2}\sum_{i,j=1}^{n} g(R(e_i, e_j)e_j, e_i) = \frac{1}{2}\sum_{i,j=1}^{n} K_{ij}. \tag{3.9}$$

Using Lemma 3.1 and the equation (3.9), we have the following lemma:

Lemma 3.5. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional half-lightlike submanifold $(\widetilde{M}, \widetilde{g})$ with integrable screen distribution $S(TM)$. Suppose $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\Gamma(S(TM))$. Then we have*

$$2r_{S(TM)}(p) = 2\widetilde{r}_{S(TM)}(p) + \sum_{i,j=1}^{n} [B(e_i, e_i)C(e_j, e_j) - B(e_i, e_j)C(e_j, e_i)] + \sum_{i,j=1}^{n} [D(e_i, e_i)D(e_j, e_j) - D(e_i, e_j)^2], \tag{3.10}$$

where $\widetilde{r}_{S(TM)}(p)$ is the scalar curvature of screen distribution of $\widetilde{M}$ (See the equation (2.3) in [19]*) defined by*

$$\widetilde{r}_{S(TM)}(p) = \frac{1}{2} \sum_{i,j=1}^{n} \widetilde{g}(\widetilde{R}(e_i, e_j)e_j, e_i). \tag{3.11}$$

Putting (2.24) and (2.26) in Lemma 3.5, we obtain the following corollary:

Corollary 3.6. *If $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$, then we have*

$$2r_{S(TM)}(p) = 2\widetilde{r}_{S(TM)}(p) + \varphi n^2 \mu_1^2 + n^2 \mu_2^2 - \sum_{i,j=1}^{n} [\varphi B(e_i, e_j)^2 + D(e_i, e_j)^2]. \tag{3.12}$$

Now, we shall mention the other curvatures, namely the null sectional curvature, the Ricci type tensor and the scalar curvature for a half-lightlike submanifold.

Let ξ be a null vector of T_pM. A plane Π is called a *null plane* if it contains ξ and e_i such that $\widetilde{g}(\xi, e_i) = 0$ and $\widetilde{g}(e_i, e_i) \neq 0$. *The null sectional curvature* of Π is defined by

$$K^{\text{null}}(\Pi) \equiv K^{\text{null}}(e_i) = \frac{g(R_p(e_i, \xi)\xi, e_i)}{g_p(e_i, e_i)}. \tag{3.13}$$

We note that the null sectional curvature measures differences in length of two spacelike geodesic constructed from the degenerate plane section Π and it is independent of the choice of the spacelike vector e_i but it depends quadratically on the null vector ξ. For more details, we refer to [1] and [3].

Proposition 3.7. [16] *Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a Lorentzian space form $\widetilde{M}(c)$ of constant curvature c. The null sectional curvature of M is given by*

$$K^{null}(e_i) = D(\xi, \xi)D(e_i, e_i) - D(e_i, \xi)D(\xi, e_i). \tag{3.14}$$

From Proposition 3.7, it is clear that the null sectional curvature vanishes on irrotational screen locally conformal half-lightlike submanifolds of a semi-Riemannian space form.

The *Ricci type tensor*, denoted by $R^{(0,2)}$, defined by

$$R^{(0,2)}(X,Y) = \sum_{j=1}^{n} g(R(e_j,X)Y,e_j) + \widetilde{g}(R(\xi,X)Y,N) \tag{3.15}$$

for any $X, Y \in \Gamma(TM)$.

From the equations (3.1)-(3.4), it can be shown that the Ricci type tensor doesn't need to be symmetric as the sectional curvature map. This tensor is called *Ricci tensor* if it is symmetric.

Taking the trace in the equation (3.15), it can be obtained a scalar at $p \in M$ such that

$$\tau(p) = \sum_{i,j=1}^{n} K_{ij} + \sum_{i=1}^{n} \left(K^{\text{null}}(e_i) + K_N(e_i) \right), \tag{3.16}$$

where $K_N(e_i) = \widetilde{g}(R(\xi,e_i)e_i,N)$ for $i \in \{1,\ldots,n\}$.

Remark 3.8. The scalar $\tau(p)$ is called scalar curvature at $p \in M$ if the tensor $R^{(0,2)}$ is symmetric. Otherwise, $\tau(p)$ can't be called scalar curvature since it is not possible to calculate it from a tensor quantity $R^{(0,2)}$ (See [11]).

4. Some optimal inequalities for half-lightlike submanifolds

We begin this section with the following algebraic lemma:

Lemma 4.1. [28] *If $a_1,\ldots,a_n$ are n-real numbers $(n>1)$, then*

$$\frac{1}{n}(\sum_{i=1}^{n} a_i)^2 \leq \sum_{i=1}^{n} a_i^2, \tag{4.1}$$

with equality if and only if $a_1 = \ldots = a_n$.

Now, we are going to establish an inequality involving the intrinsic and extrinsic invariants for screen conformal half-lightlike submanifolds of a Lorentzian manifold.

Theorem 4.2. *Let $(M,g,S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold of an $(n+3)$-dimensional Lorentzian manifold $\widetilde{M}$ with $\varphi > 0$. Then*

$$2r_{S(TM)}(p) \leq 2\widetilde{r}_{S(TM)}(p) + n(n-1)(\varphi\mu_1^2 + \mu_2^2). \tag{4.2}$$

The equality case of (4.2) holds for all $p \in M$ if and only if $S(TM)$ is totally umbilical in M.

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. Then we have from Corollary 3.6 that

$$2r_{S(TM)}(p) = 2\widetilde{r}_{S(TM)}(p) + \varphi n^2\mu_1^2 - \varphi \sum_{i\neq j=1}^{n} B(e_i, e_j)^2 - \varphi \sum_{i=1}^{n} B(e_i, e_i)^2 + n^2\mu_2^2 - \sum_{i\neq j=1}^{n} D(e_i, e_j)^2 - \sum_{i=1}^{n} D(e_i, e_i)^2. \tag{4.3}$$

If we use Lemma 4.1 in (4.3), we obtain (4.2).

The equality case of (4.2) is true if and only if

$$B_{11} = \cdots = B_{nn},\ B_{ij} = 0,\ \text{for } i \neq j \in \{1, \dots, n\},$$
$$D_{11} = \cdots = D_{nn},\ D_{ij} = 0,\ \text{for } i \neq j \in \{1, \dots, n\},$$

which imply that $S(TM)$ is totally umbilical in M. □

Considering Theorem 4.2, we get the following corollaries:

Corollary 4.3. *Let $(M, g, S(TM))$ be an $(n + 1)$-dimensional irrotational screen conformal half-lightlike submanifold of an $(n+3)$-dimensional Lorentzian manifold $\widetilde{M}$ with $\varphi > 0$. Then we have*

$$2r_{S(TM)}(p) \leq 2\widetilde{r}_{S(TM)}(p) + n(n-1)(\varphi\mu_1^2 + \mu_2^2). \tag{4.4}$$

The equality case of (4.4) holds for all $p \in M$ if and only if M is totally umbilical.

Corollary 4.4. *Let $(M, g, S(TM))$ be an $(n + 1)$-dimensional screen conformal half-lightlike submanifold of an $(n+3)$-dimensional Lorentzian space form $R_1^{n+3}(c)$ of constant curvature c. If $\varphi > 0$, then we have*

$$2r_{S(TM)}(p) \leq n(n-1)(c + \varphi\mu_1^2 + \mu_2^2). \tag{4.5}$$

The equality case of (4.5) holds for all $p \in M$ if and only if M is totally umbilical.

Corollary 4.5. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold of an $(n+3)$-dimensional semi-Euclidean space E_1^{n+3}. If $\varphi > 0$, then we gave*

$$2r_{S(TM)}(p) \leq n(n-1)(\varphi\mu_1^2 + \mu_2^2). \tag{4.6}$$

The equality case of (4.6) holds for all $p \in M$ if and only if M is totally umbilical.

Now, we recall the following theorem of D. H. Jin in [20]:

Theorem 4.6. *Let $(M, g, S(TM))$ be a half-lightlike submanifold of semi-Riemannian manifold $\widetilde{M}$. If the coscreen distribution $\mathcal{D}$ is a conformal Killing on $\widetilde{M}$, then there exists a smooth function δ such that*

$$D(X, Y) = \delta g(X, Y) \tag{4.7}$$

for all $X, Y \in TM$.

From Theorem 4.2 and Theorem 4.6, we obtain the following corollary:

Corollary 4.7. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold of an $(n+3)$-dimensional Lorentzian manifold $\widetilde{M}$. If the equality case of (4.2) holds for all $p \in M$, then $\mathcal{D}$ is a homothetic Killing on $\widetilde{M}$ with $\delta = 1$.*

If we put (3.3), (3.12) and (3.14) in (3.16), we have the following lemma:

Lemma 4.8. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold of an $(n+3)$-dimensional Lorentzian space form $R_1^{n+3}(c)$. Suppose $\{e_1 \dots, e_n\}$ is an orthonormal basis of $\Gamma(S(TM))$. Then we have*

$$\begin{aligned} \tau(p) = \left(c + \varphi\mu_1^2 + \mu_2^2\right) n^2 - \sum_{i=1}^{n} D(e_i, \xi) \left[D(e_i, \xi) + \rho_2(e_i)\right] \\ + n\mu_2 \left(D(\xi, \xi) + \rho_2(\xi)\right) - \sum_{i,j=1}^{n} \left[\varphi B(e_i, e_j)^2 + D(e_i, e_j)^2\right]. \end{aligned} \tag{4.8}$$

Using Lemma 4.8, we get the followings:

Theorem 4.9. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold of an $(n+3)$-dimensional Lorentzian space form $R_1^{n+3}(c)$. Then we have*

$$\tau(p) \leq \left(c + \varphi\mu_1^2 + \mu_2^2\right) n^2 + n\mu_2 \left(D(\xi, \xi) + \rho_2(\xi)\right) - \sum_{i=1}^{n} \left[D(e_i, \xi)\rho_2(e_i)\right]. \tag{4.9}$$

If the equality case of (4.9) holds for all $p \in M$, then $S(TM)$ is totally geodesic.

Corollary 4.10. *If $(M, g, S(TM))$ is an $(n+1)$-dimensional irrotational screen conformal half-lightlike submanifold of an $(n+3)$-dimensional semi-Euclidean space E_1^{n+3}. Then we have*

$$\tau(p) \leq \left(\varphi\mu_1^2 + \mu_2^2\right) n^2 + n\mu_2\rho_2(\xi). \tag{4.10}$$

The equality case of (4.10) holds for all $p \in M$ if and only if M is totally geodesic.

If we consider Lemma 4.1 in Lemma 4.8 and use similar arguments as in the proof of Theorem 4.2, we obtain the following theorem:

Theorem 4.11. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold of $R_1^{n+3}(c)$. Then we have*

$$\begin{aligned} \tau(p) \leq n^2 c + n(n-1)(\varphi\mu_1^2 + \mu_2^2) + n\mu_2 \left(D(\xi, \xi) + \rho_2(\xi)\right) \\ - \sum_{i=1}^{n} \left[D(e_i, \xi)\rho_2(e_i)\right]. \end{aligned} \tag{4.11}$$

The equality case of (4.11) *holds for all* $p \in M$ *if and only if* M *is totally umbilical.*

Corollary 4.12. *Let* $(M, g, S(TM))$ *be an* $(n+1)$*-dimensional irrotational screen conformal half-lightlike submanifold of* $R_1^{n+3}(c)$*. Then*

$$\tau(p) \leq n^2 c + n(n-1)\left(\varphi\mu_1^2 + \mu_2^2\right) + n\mu_2\rho_2(\xi). \tag{4.12}$$

The equality case of (4.12) *holds for all* $p \in M$ *if and only if* M *is totally umbilical.*

5. Some results on leaves of half-lightlike submanifolds

Let $(M, g, S(TM))$ be an $(n+1)$-dimensional half-lightlike submanifold of an $(n+3)$-dimensional Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. Suppose $S(TM)$ is integrable and (M', g') is an n-dimensional leaf of $S(TM)$ immersed in $\widetilde{M}$ as a codimension 2 with the non-degenerate metric g'. Denote the induced connection of M' by ∇'. Using the equations (2.11) and (2.16), we can write

$$\widetilde{\nabla}_X Y = \nabla'_X Y + B(X,Y)N + C(X,Y)\xi + D(X,Y)u \tag{5.1}$$

for all $X, Y \in \Gamma(S(TM))$. It follows that the second fundamental form of M', denoted by h', is given by

$$h'(X,Y) = B(X,Y)N + C(X,Y)\xi + D(X,Y)u \tag{5.2}$$

for all $X, Y \in \Gamma(S(TM))$. Hence, we have

$$\|h'(X,Y)\|^2 = 2B(X,Y)C(X,Y) + D(X,Y)^2. \tag{5.3}$$

The mean curvature vector $H'(p)$ at $p \in M'$ is a vector field satisfying

$$H'(p) = \frac{1}{n}\left(\mathrm{trace}|_{S(TM)}(h')\right) \tag{5.4}$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}\left[B(e_i,e_i)N + C(e_i,e_i)\xi + D(e_i,e_i)u\right]\right) = \mu_1 N + \mu_2 u + \sum_{i=1}^{n} C(e_i,e_i)\xi.$$

If M is screen conformal, from the equations (2.26) and (5.4), we get

$$\|H'(p)\|^2 = 2\varphi\mu_1^2 + \mu_2^2. \tag{5.5}$$

Now, we recall the following Theorem and Corollary of K. Duggal and B. Sahin (See Theorem 4.4.6 and Corollary 4.4.8 in [16]).

Theorem 5.1. *Let* $(M, g, S(TM))$ *be a screen conformal half-lightlike submanifold of a semi-Riemannian manifold* $\widetilde{M}$ *with a leaf* M' *of* $S(TM)$*. Then*

a) M *is totally geodesic,*
b) M *is totally umbilical,*
c) M *is minimal*

if and only if M' *is so immersed as a submanifold of* $\widetilde{M}$ *and* ε_1 *vanishes on* M*.*

Corollary 5.2. *Let $(M, g, S(TM))$ be an irrotational screen conformal half-lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ with a leaf M' of $S(TM)$. Then*

a) *M is totally geodesic,*
b) *M is totally umbilical,*
c) *M is minimal*

if and only if M' is so immersed as a submanifold of $\widetilde{M}$.

Corollary 5.3. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold and M' be an n-dimensional leaf of $S(TM)$ immersed in $R_1^{n+3}(c)$. Then we have*

$$2r_{S(TM)}(p) \leq n(n-1)\left(c + \|H'(p)\|^2 - \varphi\mu_1^2\right). \tag{5.6}$$

The equality case of (5.6) holds for all $p \in M'$ if and only if both M' and M are totally umbilical.

Proof. If we put (5.5) in (4.5), we easily obtain (5.6). The rest part of proof is clear from Theorem 5.1. □

Corollary 5.4. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold and M' be an n-dimensional leaf of $S(TM)$ immersed in E_1^{n+3}. Then we have*

$$2r_{S(TM)}(p) \leq n(n-1)\left(\|H'(p)\|^2 - \varphi\mu_1^2\right). \tag{5.7}$$

The equality case of (5.7) holds for all $p \in M'$ if and only if both M' and M are totally umbilical.

Corollary 5.5. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold and M' be an n-dimensional leaf of $S(TM)$ immersed in $R_1^{n+3}(c)$. Then we have*

$$\begin{aligned} \tau(p) \leq n^2 c + n(n-1)\left(\|H'(p)\|^2 - \varphi\mu_1^2\right) + n\mu_2\left(D(\xi,\xi) + \rho_2(\xi)\right) \\ - \sum_{i=1}^{n}\left[D(e_i,\xi)\rho_2(e_i)\right]. \end{aligned} \tag{5.8}$$

The equality case of (5.8) *holds for all $p \in M'$ if and only if both M' and M are totally umbilical.*

Proof. If we put (5.5) in (4.11), the proof of this theorem is straightforward. □

Corollary 5.6. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional irrotational screen conformal half-lightlike submanifold and M' be an n-dimensional leaf of $S(TM)$ immersed in E_1^{n+3}. Then we have*

$$\tau(p) \leq n(n-1)\left(\|H'(p)\|^2 - \varphi\mu_1^2\right) + n\mu_2\rho_2(\xi). \tag{5.9}$$

The equality case of (5.9) *holds for all $p \in M'$ if and only if both M' and M are totally umbilical.*

Corollary 5.7. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional screen conformal half-lightlike submanifold and M' be an n-dimensional leaf of $S(TM)$ immersed in $R_1^{n+3}(c)$. Then we have*

$$\tau(p) \leq n^2 c + n^2 \left(\|H'(p)\|^2 - \varphi\mu_1^2\right) + n\mu_2 \left(D(\xi,\xi) + \rho_2(\xi)\right) - \sum_{i=1}^{n} \left[D(e_i,\xi)\rho_2(e_i)\right]. \tag{5.10}$$

If the equality case of (5.10) *holds for all $p \in M'$, then both M' and M are totally geodesic.*

Proof. If we put (5.5) in (4.9), the proof of this theorem is straightforward. □

Corollary 5.8. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional irrotational screen conformal half-lightlike submanifold and M' be an n-dimensional leaf of $S(TM)$ immersed in E_1^{n+3}. Then we have*

$$\tau(p) \leq n^2 \left(\|H'(p)\|^2 - \varphi\mu_1^2\right) + n\mu_2\rho_2(\xi). \tag{5.11}$$

The equality case of (5.11) *holds for all $p \in M'$ if and only if both M' and M are totally geodesic.*

Example 5.9. Consider in R_1^7 with signature $(-,+,+,+,+,+,+)$ a submanifold M given by the equations

$$x_4 = (x_1^2 - x_2^2)^{\frac{1}{2}},\ x_3 = (1 - x_5^2)^{\frac{1}{2}},\ x_6 = (1 - x_7^2)^{\frac{1}{2}},\ x_2, x_5, x_7 > 0. \tag{5.12}$$

Then we have

$$\begin{aligned} \operatorname{Rad} TM &= \operatorname{Span}\{\xi = x_1\partial x_1 + x_2\partial x_2 + x_4\partial x_4\}, \\ S(TM) &= \operatorname{Span}\{Z_1 = x_4\partial x_1 + x_1\partial x_4,\ Z_2 = -x_5\partial x_3 + x_3\partial x_5, \\ &\qquad Z_3 = -x_6\partial x_7 + x_7\partial x_6,\ Z_4 = x_6\partial x_7 + x_7\partial x_6\}, \end{aligned}$$

and

$$\mathcal{D} = \operatorname{Span}\{u = x_3\partial x_3 + x_5\partial x_5\}.$$

Hence, M is a half lightlike submanifold of R_1^7 with $S(TM) = \operatorname{Span}\{U_1, U_2, U_3, U_4\}$. Also, the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = \frac{1}{2}\{-x_1\partial x_1 + x_2\partial x_2 + x_4\partial x_4\}.$$

Thus, we have

$$\begin{aligned} \widetilde{\nabla}_{Z_1}\xi = Z_1,\ \widetilde{\nabla}_{Z_2}\xi = \widetilde{\nabla}_{Z_3}\xi = \widetilde{\nabla}_{Z_4}\xi = 0,\quad \widetilde{\nabla}_{\xi}\xi = \xi, \\ \widetilde{\nabla}_{Z_1}N = \frac{1}{2x_1}Z_1,\ \widetilde{\nabla}_{Z_2}N = \widetilde{\nabla}_{Z_3}N = \widetilde{\nabla}_{Z_4}N = 0,\ \widetilde{\nabla}_{\xi}N = -N \end{aligned}$$

and $(M, g, S(TM)$ is screen conformal with $\varphi = \frac{1}{2x_1}$. From (2.5), (2.6) and (2.7), we get

$$B(Z_1, Z_1) = -x_2^2,\ D(Z_2, Z_2) = -1 \tag{5.13}$$

and the other components of B and D vanishes.

Consider M' to be a 4-dimensional leaf of $S(TM)$. Then it follows that

$$TM' = \mathrm{Span}\{Z_1, Z_2, Z_3, Z_4\}.$$

By a straightforward computation, it can be obtained that the M' satisfies corollaries of this section.

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