*Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*
**33** (2017), 331–337
www.emis.de/journals
ISSN 1786-0091

# QUASI-CONFORMALLY FLAT AND PROJECTIVELY FLAT TRANS-SASAKIAN MANIFOLDS

VIBHA SRIVASTAVA AND P. N. PANDEY

ABSTRACT. The object of the present paper is to study quasi-conformally flat and projectively flat contact manifolds. We have also studied Quasi-conformally flat and projectively flat trans-Sasakian manifolds. We obtained condition for trans-Sasakian manifold to be quasi-conformally flat and projectively flat. The value of scalar curvature has been obtained in quasi-conformally flat and projectively flat trans-Sasakian manifolds.

## 1. INTRODUCTION

Oubina [11] introduced a manifold which generalizes both $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds such manifold was called a trans-Sasakian manifold of type $(\alpha, \beta)$. Sasakian, Kenmotsu, and cosympletic manifold are particular cases of trans-Sasakian manifolds. Trans-Sasakian manifolds of type $(0,0), (\alpha, 0)$ and $(0, \beta)$ are called cosympletic [2], $\alpha$-Sasakian ([3], [14]) and $\beta$-Kenmotsu ([3], [8]) respectively. Concept of nearly trans-Sasakian manifold was introduced by C. Gherghe [7]. Marrero [9] constructed a three dimensional trans-Sasakian manifold. Prasad and Srivastava [12] obtained certain results on trans-Sasakian manifolds. Jeong- Sik kim et al. [7] studied a generalized Ricci-recurrent trans-Sasakian manifolds. A quasi-conformal curvature tensor $\check{C}$ was defined by Yano and Sawaki [17] as follows:

$$\begin{aligned} (1.1)\quad & \check{C}(X,Y)Z \\ &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\qquad - \frac{r}{(2n+1)}\left(\frac{a}{2n} + 2b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

where $a$ and $b$ are constants and $R, S, Q$ and $r$ are the Riemannian curvature-tensor, the Ricci - tensor, the Ricci operator and the scalar curvature of the manifold respectively. A $(2n+1)-$dimensional Riemannian manifold $(M, g)$

2010 *Mathematics Subject Classification.* 53C50,53C15.
*Key words and phrases.* Quasi-conformal curvature tensor, Projective curvature tensor, scalar curvature tensor, Trans-Sasakian manifolds.

is quasi-Conformally flat if $\check{C} = 0$. Various properties of the quasi-conformal curvature tensor on contact metric manifolds have been studied by several geometers [1, 4, 5, 13] etc. If $a = 1$ and $b = -\frac{1}{2n-1}$, then $\check{C}$ becomes the conformal curvature tensor $C$ which is given by

$$(1.2)\quad C(X,Y)Z$$
$$= R(X,Y)Z - \frac{1}{2n-1}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\} + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}.$$

The Weyl projective curvature tensor $P$ of type $(1,3)$ on a $(2n+1)-$dimensional Riemannian manifold $(M,g)$ is defined as

$$(1.3)\qquad P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y],$$

for any $X, Y, Z \in TM$. The manifold $(M,g)$ is said to be projectively flat if $P$ vanishes identically on $M$. In this paper we study quasi-conformally flat and projectively flat trans-Sasakian manifolds.

## 2. Preliminaries

Let $M$ be a $(2n+1)$- dimensional almost contact metric manifold [1] with almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a compatible Riemannian metric on $M$ such that

$$(2.1)\qquad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0,$$

$$(2.2)\qquad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

$$(2.3)\qquad g(\phi X, Y) = -g(X, \phi Y), \quad g(X,\xi) = \eta(X),$$

for all $X, Y \in TM$.

An almost contact metric manifold is said to be contact manifold if

$$(2.4)\qquad d\eta(X,Y) = \Phi(X,Y) = g(X, \phi Y),$$

$\Phi(X,Y)$ is being called fundamental $2-$form of $M$.

An almost contact metric manifold $M$ is called trans-Sasakian manifold if

$$(2.5)\qquad (\nabla_X\phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta g\{(\phi X, Y)\xi - \eta(Y)\phi X\},$$

where $\nabla$ is Levi-Civita connection of Riemannian metric $g$ and $\alpha$ and $\beta$ are smooth functions on $M$. From equation (2.5) and equations (2.1), (2.2) and (2.3), we get

$$(2.6)\qquad (\nabla_X\phi)\xi = -\alpha\phi X + \beta[X - \eta(X)\xi],$$

$$(2.7)\qquad (\nabla_X\eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

In a $(2n+1)$- dimensional trans-Sasakian manifold, we have ([6], [3])

$$R(X,Y)\xi = (\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y] + (Y\alpha)\phi X-(X\alpha)\phi Y + (Y\beta)\phi^2 X-(X\beta)\phi^2 Y, \tag{2.8}$$

$$R(\xi,Y)X = (\alpha^2-\beta^2)[g(X,Y)\xi-\eta(X)Y] + 2\alpha\beta[g(\phi X,Y)\xi-\eta(X)\phi Y] + (X\alpha)\phi Y+g(\phi X,Y)(\mathrm{grad}\alpha) + (X\beta)[Y-\eta(Y)\xi]-g(\phi X,\phi Y)(\mathrm{grad}\beta), \tag{2.9}$$

$$\eta(R(\xi,Y)X) = g(R(\xi,Y)X,\xi) = (\alpha^2-\beta^2-\xi\beta)[g(X,Y)-\eta(X)\eta(Y)] \tag{2.10}$$

and

$$2\alpha\beta+\xi\alpha=0. \tag{2.11}$$

In a trans-Sasakian manifold, we also have [6]

$$S(X,\xi) = (2n(\alpha^2-\beta^2)-\xi\beta)\eta(X)-(2n-1)X\beta-(\phi X)\alpha \tag{2.12}$$

and

$$Q\xi = (2n(\alpha^2-\beta^2)-\xi\beta)\xi-(2n-1)\mathrm{grad}\beta+\phi(\mathrm{grad}\alpha). \tag{2.13}$$

An almost contact metric manifold $M$ is said to be $\eta-$Einstein if its Ricci-tensor $S$ is of the form

$$S(X,Y) = ag(X,Y)+b\eta(X)\eta(Y), \tag{2.14}$$

where $a$ and $b$ are smooth functions on $M$. An $\eta-$Einstein manifold becomes Einstein manifold if $b=0$, i.e

$$S(X,Y) = ag(X,Y). \tag{2.15}$$

If $\{e_1, e_2, ..., e_{2n}, e_{2n+1}=\xi\}$ be a local ortho-normal basis of tangent space in a $(2n+1)-$ dimensional almost contact manifold $M$, then we have

$$\sum_{i=1}^{2n+1} g(e_i,e_i) = (2n+1). \tag{2.16}$$

$$\sum_{i=1}^{2n+1} g(e_i,Y)S(X,e_i) = \sum_{i=1}^{2n+1} R(e_i,Y,X,e_i) = S(X,Y). \tag{2.17}$$

## 3. QUASI-CONFORMALLY FLAT AND PROJECTIVELY FLAT MANIFOLDS

Let $M$ be a $(2n+1)$- dimensional quasi-conformally flat manifold, then from equation (1.1) we have

$$R(X,Y)Z = \frac{b}{a}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX] + \frac{r}{(2n+1)\,a}(\frac{a}{2n} + 2b)[g(Y,Z)X - g(X,Z)Y]. \tag{3.1}$$

Let $\{e_{1,}e_{2,}...e_{2n}, e_{2n+1} = \xi\}$ be a local ortho-normal basis of tangent space. Putting $Y = Z = e_i$, in equation (3.1), we get

$$S\,(X,W) = \frac{r}{2n+1} g\,(X,W) \qquad if\ a + (2n-1)\,b \neq 0. \tag{3.2}$$

Hence from equation (3.2), we can state the following theorem:

**Theorem 3.1.** *A quasi-conformally flat manifold is an Einstein manifold if $a + (2n-1)\,b \neq 0$. If $a + (2n-1)\,b = 0$, then $\check{C}(X,Y)Z = aC\,(X,Y)\,Z$, or $\check{C}(X,Y)Z = -\,(2n-1)\,bC\,(X,Y)\,Z$.*

This leads to:

**Corollary 3.2.** *A quasi-conformally flat manifold is conformally flat if $a + (2n-1)\,b = 0$ and $a \neq 0$ (equivalently if $a + (2n-1)\,b = 0$ and $b \neq 0$).*

**Corollary 3.3.** *If $a+(2n-1)\,b = 0$, then quasi-conformally curvature becomes constant multiple of conformal curvature tensor.*

If the manifold is projectively flat then from equation(1.3), we have

$$R(X,Y)Z = \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{3.3}$$

Putting $Y = Z = e_i$ in equation (3.3), we get

$$S\,(X,W) = \frac{1}{2n}\,[rg\,(X,W) - S\,(X,W)]\,, \tag{3.4}$$

$$S\,(X,W) = \frac{r}{2n+1} g\,(X,W)\,. \tag{3.5}$$

Hence a projectively flat manifold is an Einstein manifold.

## 4. Quasi-conformally flat trans-Sasakian manifolds

If a trans-Sasakian manifold is quasi-conformally flat then from equation (1.1) we have

$$
\begin{aligned}
(4.1)\qquad R(X,Y,Z,W) &= \frac{b}{a}[S(X,Z)g(Y,W) \\
&-S(Y,Z)g(X,W)+g(X,Z)S(Y,W) \\
&-g(Y,Z)S(X,W)]+\frac{r}{(2n+1)\,a}(\frac{a}{2n}+2b)[g(Y,Z)g(X,W) \\
&-g(X,Z)g(Y,W)].
\end{aligned}
$$

On putting $Z=\xi$ in equation (4.1) and using equations (2.3) and (2.12), we get

$$
\begin{aligned}
(4.2)\qquad g\left(R(X,Y)\,\xi,W\right) = \frac{b}{a}[&\{\left(2n\left(\alpha^2-\beta^2\right)-\xi\beta\right)\eta\left(X\right) \\
&-\left(2n-1\right)X\beta-\left(\phi X\right)\alpha\}g\left(Y,W\right) \\
&-\{\left(2n\left(\alpha^2-\beta^2\right)-\xi\beta\right)\eta\left(Y\right) \\
&-\left(2n-1\right)Y\beta-\left(\phi Y\right)\alpha\}g\left(X,W\right) \\
&+\eta\left(X\right)S\left(Y,W\right)-\eta\left(Y\right)S\left(X,W\right)] \\
&+\frac{r}{(2n+1)\,a}(\frac{a}{2n}+2b)\{\eta\left(Y\right)g\left(X,W\right) \\
&-\eta\left(X\right)g\left(Y,W\right)\}
\end{aligned}
$$

Again putting $X=\xi$ in equation (4.2) and using equations $(2.1)\,,(2.3)\,,(2.10)\,,(2.11)$ and (2.12), we get

$$
\begin{aligned}
(4.3)\quad S\left(Y,W\right) = [&\frac{r}{(2n+1)\,a}(\frac{a}{2n}+2b)-\frac{a}{b}(2n\left(\alpha^2-\beta^2\right)-\xi\beta)]g\left(Y,W\right) \\
&+[2\left(2n\left(\alpha^2-\beta^2\right)-\xi\beta\right)+\frac{a}{b}\left(\alpha^2-\beta^2-\xi\beta\right) \\
&-\frac{r}{(2n+1)\,b}(\frac{a}{2n}+2b)]\eta\left(Y\right)\eta\left(W\right) \\
&-\{(2n-1)\,Y\beta-\left(\phi Y\right)\alpha\}\eta\left(W\right) \\
&-\{((2n-1)\,W\beta-\left(\phi W\right)\alpha)\}\eta\left(Y\right).
\end{aligned}
$$

From equations (3.2) and (4.3), we get

$$
(4.4)\qquad r=2n\left(2n+1\right)\left(\alpha^2-\beta^2-\xi\beta\right),
$$

and

$$
(4.5)\qquad \left(2n-1\right)\operatorname{grad}\beta-\phi\left(\operatorname{grad}\alpha\right)=\left(2n-1\right)\left(\xi\beta\right)\xi.
$$

This leads to:

**Theorem 4.1.** *A trans-Sasakian manifold can not be quasi-conformally flat unless* $(2n-1)\operatorname{grad}\beta-\phi\left(\operatorname{grad}\alpha\right)-\left(2n-1\right)\left(\xi\beta\right)\xi$ *is zero.*

From the equations (4.4) and (4.5), we have the following:

**Corollary 4.2.** *If a trans-Sasakian manifold is quasi-conformally flat then scalar curvature tensor* $r = 2n\,(2n+1)\,(\alpha^2 - \beta^2 - \xi\beta)$*, where* $\alpha$ *and* $\beta$ *are related by* $(2n-1)\,\mathrm{grad}\,\beta - \phi\,(grad\alpha) = (2n-1)\,(\xi\beta)\,\xi$.

## 5. Projectively flat trans-Sasakian manifolds

If a trans-Sasakian manifold is projectively flat then from equation (1.3), we have

$$(5.1)\quad R(X,Y,Z,W) = g\,(R(X,Y)\,Z,W) = \frac{1}{2n}[S(Y,Z)g\,(X,W) - S(X,Z)g\,(Y,W)].$$

On putting $W = \xi$ in equation (5.1) and using equations (2.3) and (2.12), we get

$$(5.2)\qquad \eta\,((R(X,Y)\,Z) = \frac{1}{2n}[S(Y,Z)\eta\,(X) - S(X,Z)\eta\,(Y)].$$

Again putting $X = \xi$ in equation (5.2) and using equations $(2.1)\,,(2.3)\,,(2.10)\,,(2.11)$ and (2.12), we get

$$S\,(Y,Z) = 2n\left(\alpha^2 - \beta^2 - \xi\beta\right)g\,(Y,Z) + (2n-1)\,(\xi\beta)\,\eta\,(Y)\,\eta\,(Z) + ((2n-1)\,(Z\beta) + (\phi Z)\,\alpha)\,\eta\,(Y)\,.$$

From equation (3.5) and (5.3), we get

$$(5.3)\qquad r = 2n\,(2n+1)\left(\alpha^2 - \beta^2 - \xi\beta\right),$$

and

$$(5.4)\qquad (2n-1)\,(d\beta - \xi\,(\beta)\,\eta) + d\alpha o\phi = 0.$$

This leads to:

**Theorem 5.1.** *A trans-Sasakian manifold can not be projectively flat unless* $(2n-1)\,(d\beta - (\xi\beta)\,\eta) + d\alpha o\phi$ *is zero.*

From the equations(5.4) and (5.5), we have the following:

**Corollary 5.2.** *If a trans-Sasakian manifold is projectively flat than scalar curvature tensor* $r = 2n\,(2n+1)\,(\alpha^2 - \beta^2 - \xi\beta)$*, where* $\alpha$ *and* $\beta$ *are related* $(2n-1)\,(d\beta - \xi\,(\beta)\,\eta) + d\alpha 0\phi = 0$.

## Acknowledgement

The first author is thankful to University Grants Commission, New Delhi, India for financial support in the form of UCG-Dr. D. S. Kothari Post Doctoral Fellowship.

## References


[1] K. Amur and Y. B. Maralabhavi. On quasi-conformally flat spaces. *Tensor N. S.,* $31\,(2) : 194 - 198, 1977$.

[2] D. E. Blair. Contact manifolds in Riemannian geometry. Lecture Notes in Math. 509 Springer Verlag, 1976.

[3] D. E. Blair. Riemannian geometry of Contact and Symplectic manifolds. volume 203 of Progress in Mathematics. Birkhauser Boston, Inc., Boston, MA, 2002.

[4] U. C. De , J. B. Jun and A. K. Gazi. Sasakian manifolds with quasi- Conformal curvature, *Bull Korean Math. Soc.,* $45\,(2) : 213 - 319, 2008$.

[5] U.C. De and Y. Matsuyama. Quasi-conformally flat manifolds satisfying certain condition on the Ricci tensor. *SUT J. Math.,* $42\,(2) : 295 - 303, 2006$.

[6] C. Gherghe. Harmonicity on nearly trans-Sasakian manifolds. *Demonstratio Math.,* $33\,(2002)\,, 151 - 157$.

[7] Jeong-Sik Kim, R. Prasad and M. M. Tripathi. On generalized Ricci recurrent trans-Sasakian manifolds. *J. Korean Math. Soc.,* $39\,(6) : 953 - 961, 2002$.

[8] K. Kenmotsu. A class of almost contact Riemannian manifolds. *Tohoku Math J.,* $24\,(1) : 93 - 103, 1972$.

[9] J. C. Marrero. The local structure of trans-Sasakian manifold. *Ann. Mat. Pura. Appl.*, $162\,(1) : 77 - 86, 1992$.

[10] K. Nomizu. On hypersurfaces satisfying certain condition on the curvature. *Tohoku Math. J .*, $20 : 46 - 59, 1968$.

[11] J. A. Oubina. New Classes of almost contact metric structure. *Pub. Math. Debrecen,* $32 : 187 - 193, 1985$.

[12] R. Prasad and V. Srivastava. Some results on trans-Sasakain manifolds. *Math. Vesnik. J. of Mathematics,* $65\,(3) : 346 - 362, 2013$ .

[13] A. A. Shaikh and K. K. Baishya. On weakly quasi-conformally symmetric manifolds. Soochow *J . Math,* $31\,(4) : 581 - 582, 2005$.

[14] S. Sasaki. On differentiable manifolds with certain structure which are closed related to an almost contact structure. *Tohoku Math. J.,* $12\,(3) : 459 - 476, 1960$.

[15] Z. I. Szabo. Structure theorems on Riemannian space satisfying $R(X,Y)R = 0$. I. The local version. *J . Diff. Geom.,* $17\,(4) : 531 - 582$, 1982.

[16] Z.I. Szabo. Structure Classification and contraction of complete hypersurfaces satisfying $R(X,Y)R = 0$. *Acta Sci. Math.*, $47 : 321 - 348, 1984$.

[17] K. Yano and S. Sawaki. Riemannian manifolds admitting a conformal transformation group. *J. Differential Geom.,* $2\,(2) : 161 - 184, 1968$.


*Received November 10, 2016.*


Vibha Srivastava,
Department of Mathematics,
University of Allahabad, Allahabad-211002
*E-mail address*: `vibha.one22@rediffmail.com`

P. N. Pandey,
Department of Mathematics,
University of Allahabad, Allahabad-211002
*E-mail address*: `pnpiaps@gmail.com`