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A CLASS OF BERWALDIAN FINSLER METRICS

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Abstract. In this paper, we study projectively flat Finsler metrics defined by the Euclidean metric and related 1-forms. For this class of Finsler metrics, we find the necessary and sufficient condition to be Berwaldian. Then, we obtain the differential equations that characterize these metrics with vanishing Douglas curvature.

1. Introduction

The Berwald metrics are very important in Finsler geometry. They were first investigated by L. Berwald. The geodesics of a Finsler metric $F(x, y)$ on a smooth manifold M are determined by the systems of second order differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0, \tag{1}$$

where $G^i = G^i(x, y)$ are scalar functions on TM_0 and called by spray coefficients. They define a global vector field $G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ on TM_0, which is called spray. By definition, F is called a Berwald metric if $G^i = G^i(x, y)$ are quadratic in $y \in T_xM$ at every point x, i.e.

$$G^i = \frac{1}{2}\Gamma^i_{kh}(x)y^hy^k. \tag{2}$$

In [7], Peyghan-Tayebi considered a class of Finsler metrics called generalized Berwald metrics which contains the class of Berwald metrics as a special case. They find some interesting curvature properties of generalized Berwald metrics. Very recently, Tayebi-Barzegari study generalized Berwald manifold with (α, β)-metrics and showed that a Finsler manifold with (α, β)-Finsler function of sign property is a generalized Berwald manifold if and only if there exists a covariant derivative such that it is compatible with α and β and equivalently if and only if the dual vector field $\beta^\sharp$ is of constant Riemannian length [9].

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A Finsler metric is said to be locally projectively equivalent to a Riemannian metric g if at every point x, there is a local coordinate neighborhood in which the geodesics of F coincide with that of g as point sets. In this case, the spray coefficients G^i are in the following form

$$G^i = \frac{1}{2}\Gamma^i_{kh}(x)y^h y^k + P(x,y)y^i. \tag{3}$$

Finsler metrics with this property are called Douglas metrics. Obviously, the Douglas metrics are more generalized than Berwald metrics.

In [8], Shen constructed a group of projectively flat metrics with $\mathbf{K} = 0$ as the following

$$F(x,y) = \Big\{1 + < a,x > + \frac{(1-|x|^2) < a,y >}{\sqrt{|y|^2 - (|x|^2|y|^2 - < x,y >^2)} + < x,y >}\Big\} \times \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - < x,y >^2)} + < x,y >)^2}{(1-|x|^2)^2(\sqrt{|y|^2 - (|x|^2|y|^2 - < x,y >^2})}.$$

Let us put

$$r = |y|,\ u = |x|^2,\ s = \frac{< x,y >}{|y|},\ v = < a,x >,\ t = \frac{< a,y >}{|y|},\ |a| < 1. \tag{4}$$

Then, the above metric can be written as

$$F = r\Big\{1 + v + \frac{(1-u)t}{\sqrt{1-u+s^2}+s}\Big\}\frac{(\sqrt{1-u+s^2}+s)^2}{(1-u)^2\sqrt{1-u+s^2}}.$$

In [11], Tayebi-Shahbazi Nia find found a group of projectively flat Finsler metrics composed by u, v, s and t with double square roots. Then it is natural to ask if there exist more projectively flag Finsler metrics defined by the Euclidean metric $|y|$ and the 1−forms $< x,y >, < a,y >$?

These motivate us to study the following Finsler metric

$$F = r\phi(u,s,v,t), \tag{5}$$

where $x \in \mathbb{R}^n$, $y \in T_x\mathbb{R}^n$, $a = a_i y^i$ is a constant 1−form, $<,>$ is the standard inner product of $\mathbb{R}^n$ and ϕ is a C^∞ function [2, 3]. When $a = 0$, then the metric F in (5) becomes a spherically symmetric. When $a \neq 0$, F in (5) is neither spherically symmetric nor general-(α, β) metric [4]. In this paper, we prove the following:

Theorem 1.1. *Let $F = r\phi(u,s,v,t)$ be a Finsler metric on an open subset $U \subset \mathbb{R}^n$ with dimension $n \geq 3$ in (5). Then F is a Berwald metric if and only if the following PDE's hold*

$$P - sP_s - tP_t = 0, \quad P_{ss} = P_{tt} = P_{st} = 0, \quad Q_s - sQ_{ss} - tQ_{st} = 0,$$
$$Q_t - tQ_{tt} - sQ_{st} = 0, \quad Q_{sss} = Q_{sst} = Q_{stt} = Q_{ttt} = 0, \quad R_s - sR_{ss} - tR_{st} = 0,$$
$$R_t - tR_{tt} - sR_{st} = 0, \quad R_{sss} = R_{sst} = R_{stt} = R_{ttt} = 0.$$

2. Preliminaries

A *Finsler metric* on a manifold M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on TM_0; (ii) $F(x, \lambda y) = \lambda F(x, y)$ $\lambda > 0$; and (iii) For any tangent vector $y \in T_xM$, the vertical Hessian of $F^2/2$ given by

$$g_{ij}(x,y) = \Big[\frac{1}{2}F^2\Big]_{y^iy^j}$$

is positive definite.

Every Finsler metric F induces a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ is defined by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big\}y^jy^k,$$

where the matrix (g^{ij}) means the inverse of matrix (g_{ij}).

From [4], we have the Hessian matrix $g_{ij}(x,y) := \frac{1}{2}[F^2]_{y^iy^j}$ of F in (5) as follows

$$\begin{aligned} g_{ij} &= C_0\delta_{ij} + C_1a_ia_j + C_2\frac{y_i}{r}\frac{y_j}{r} + C_3\Big(a_j\frac{y_i}{r} + a_i\frac{y_j}{r}) + C_4(x_j\frac{y_i}{r} + x_i\frac{y_j}{r}\Big)\\ &\quad + C_5\Big(a_jx_i + a_ix_j\Big) + C_6x_ix_j, \end{aligned}$$

where

$$\begin{aligned} C_0 &= \phi^2 - s\phi\phi_s - t\phi\phi_t,\\ C_1 &= \phi_t^2 + \phi\phi_{tt},\\ C_2 &= s^2(\phi_s^2 + \phi\phi_{ss}) + t^2(\phi_t^2 + \phi\phi_{tt}) + 2ts(\phi_s\phi_t + \phi\phi_{st}) - s\phi\phi_s - t\phi\phi_t,\\ C_3 &= \phi\phi_t - s(\phi_s\phi_t + \phi\phi_{st}) - t(\phi_s^2 + \phi\phi_{ss}),\\ C_4 &= \phi\phi_s - s(\phi_s^2 + \phi\phi_{ss}) - t(\phi_s\phi_t + \phi\phi_{st}),\\ C_5 &= \phi_s\phi_t + \phi\phi_{st},\\ C_6 &= \phi_s^2 + \phi\phi_{ss}. \end{aligned}$$

In order to compute the geodesic spray coefficients of F in (5), let us denote

$$g_{ij} = C_0\Big(F_{ij} + \gamma\frac{y_i}{r}\frac{y_j}{r}\Big),$$

where

$$\begin{aligned} &\gamma = -\frac{C_3}{C_0}, \quad F_{ij} = E_{ij} + \theta N_iN_j, \qquad \theta = \frac{C_5}{C_0}, \qquad N_i = a_i + x_i,\\ &E_{ij} = D_{ij} + \xi M_iM_j, \qquad \xi = \frac{C_3}{C_0}, \qquad M_i = a_i + \frac{y_i}{r},\\ &D_{ij} = B_{ij} + \epsilon a_ia_j, \qquad \epsilon = \frac{C_1 - C_3 - C_5}{C_0},\\ &B_{ij} = A_{ij} + \lambda L_iL_j, \qquad \lambda = \frac{C_2}{C_0}, \qquad L_i = \frac{C_4}{C_2}x_i + \frac{y_i}{r}, \end{aligned}$$

$$A_{ij} = \delta_{ij} + \mu x_i x_j, \qquad \mu = \frac{C_2 C_6 - C_4^2 - C_2 C_5}{C_0 C_2}.$$

Therefore, the inverse of the metric tensor is given by

$$g^{ij} = C_0^{-1}\Big\{\delta^{ij} - \zeta x^i x^j - \tau L^i L^j - \nu (B^{ij})^2 a_i a_j - \sigma M^i M^j - \kappa N^i N^j - \alpha (F^{ij})^2 y_i y_j\Big\},$$

where

$$\begin{aligned}
\zeta &= \frac{\mu}{1+\mu u}, \quad \tau = \frac{\lambda}{1+\lambda L^2}, \qquad L^i = \omega x^i + \frac{y^i}{r},\\
\nu &= \frac{\epsilon}{1+\epsilon a^2}, \qquad B^{ij} a_j = b_1 x^i + b_2 \frac{y^i}{r} + a^i,\\
\sigma &= \frac{\xi}{1+\xi M^2}, \qquad M^i = d_1 x^i + d_2 \frac{y^i}{r} + d_3 a^i,\\
\kappa &= \frac{\theta}{1+\theta N^2}, \qquad N^i = e_1 x^i + e_2 \frac{y^i}{r} + e_3 a^i,\\
\alpha &= \frac{\gamma}{1+\gamma y^2}, \qquad F^{ij} y_j = f_1 x^i + f_2 \frac{y^i}{r} + f_3 a^i.
\end{aligned}$$

Since $\omega, L^2, b_1, b_2, a^2, d_1, d_2, d_3, M^2, e_1, e_2, e_3, N^2, f_1, f_2, f_3$ and y^2 are too long, they are listed in Appendix.

On the other hand, by the definition of the geodesic spray coefficients, we have

$$G^i := \frac{1}{4} g^{il}\left\{\left(F^2\right)_{x^k y^l} y^k - \left(F^2\right)_{x^l}\right\} = \frac{F_{x^k} y^k}{2F} y^i + \frac{F}{2} g^{il}\Big(F_{x^k y^l} y^k - F_{x^l}\Big).$$

Since $F_{x^k} = 2r\phi_u x_k + \phi_s y_k + r\phi_v a_k$, one can write the first part as

$$\frac{F_{x^k} y^k}{2F} y^i = \frac{r}{2\phi}\Big(2s\phi_u + \phi_s + t\phi_v\Big) y_i. \tag{6}$$

At the same time, it can be computed that

$$\begin{aligned}
F_{x^k y^l} &= \Big(2r\phi_u x_k + \phi_s y_k + r\phi_v a_k\Big)_{y^l}\\
&= \frac{2}{r}\Big(\phi_u - s\phi_{us} - t\phi_{ut}\Big) x_k y_l + \frac{1}{r}\Big(\phi_v - s\phi_{vs} - t\phi_{vt}\Big) a_k y_l\\
&\quad - \frac{1}{r^2}\Big(s\phi_{ss} + t\phi_{st}\Big) y_k y_l + \frac{1}{r}\Big(\phi_{st} a_l y_k + \phi_{ss} x_l y_k\Big)\\
&\quad + 2\Big(\phi_{us} x_l x_k + \phi_{ut} a_l x_k\Big) + \phi_{vt} a_k a_l + \phi_s \delta_{lk} + \phi_{vs} a_k x_l.
\end{aligned}$$

Hence

$$\begin{aligned}
F_{x^k y^l} y^k - F_{x^l} &= r\Big(2s\phi_{us} + \phi_{ss} + t\phi_{sv} - 2\phi_u\Big)\Big(x_l - s\frac{y_l}{r}\Big)\\
&\quad + r\Big(2s\phi_{ut} + \phi_{st} + t\phi_{vt} - \phi_v\Big)\Big(a_l - t\frac{y_l}{r}\Big).
\end{aligned} \tag{7}$$

Combining (6) and (7), the geodesic spray coefficients become

$$G^i = \frac{r}{2\phi}\Big(2s\phi_u + \phi_s + t\phi_v\Big)y_i + \frac{r^2\phi}{2}g^{il}\Bigg\{\Big(2s\phi_{us} + \phi_{ss} + t\phi_{sv} - 2\phi_u\Big)\Big(x_l - s\frac{y_l}{r}\Big) + \Big(2s\phi_{ut} + \phi_{st} + t\phi_{vt} - \phi_v\Big)\Big(a_l - t\frac{y_l}{r}\Big)\Bigg\}.$$

So we only need to compute

$$g^{il}\left(x_l - s\frac{y_l}{r}\right) = C_0^{-1}\left[Cx^i + D\frac{y^i}{r} + Ea^i\right], \tag{8}$$

and

$$g^{il}\left(a_l - t\frac{y_l}{r}\right) = C_0^{-1}\left[Gx^i + H\frac{y^i}{r} + Ia^i\right], \tag{9}$$

where C, D, E, G, H and I are again too long, they are listed in Appendix. By putting (8) and (9) into G^i and simplifying the result it, one will finally come to the formula

$$G^i = rPy^i + r^2Qx^i + r^2Ra^i, \tag{10}$$

where P, Q and R being long, they are again listed in Appendix. When $a = 0$, then the geodesics spray coefficients in (10) become the geodesics spray coefficients of spherically symmetric Finsler metrics [6].

2.1. **Berwald curvature.** The Berwald curvature of a Finsler metric is a tensor defined in local coordinates as follows

$$B := {B^i}_{jkl}dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i},$$

where

$${B^i}_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

For a Finsler metric in (5), we already know its geodesic spray coefficients can be written as $G^i = rPy^i + r^2Qx^i + r^2Ra^i$.

Proposition 2.1. *Let $F = r\phi(u, s, v, t)$ be a Finsler metric on an open subset $U \subset \mathbb{R}^n$ with dimension $n \geq 3$ in (5). Then Berwald curvature of F is given by*

$$\begin{aligned}
{B^i}_{jkl} = &+\frac{1}{r}\{\delta^i_j\left[P_{ss}x^kx^l + P_{tt}a^ka^l + P_{st}\left(x^ka^l + x^la^k\right)\right] \\
&+(P - sP_s - tP_t)\delta^i_j\delta_{kl}\}(j \to k \to l \to j) \\
&-\frac{1}{r^2}[(sP_{ss} + tP_{st})\delta^i_jx^ky^l + (sP_{ss} + tP_{st})\delta^i_jy^kx^l \\
&+(sP_{ss} + tP_{st})y^i\delta_{jk}x^l + (tP_{tt} + sP_{st})\delta^i_ja^ky^l + (tP_{tt} + sP_{st})\delta^i_jy^ka^l
\end{aligned}$$

$$
\begin{aligned}
&+(tP_{tt}+sP_{ts})y^i\delta_{jk}a^l](j\to k\to l\to j)+\frac{1}{r^3}[(s^2P_{ss}+t^2P_{tt}+sP_s\\
&+tP_t+stP_{st}+stP_{ts}-P)\left(\delta^i_j y^k y^l+y^i\delta_{jk}y^l\right)](j\to k\to l\to j)\\
&+\frac{1}{r^5}(3P-s^3P_{sss}-t^3P_{ttt}-s^2tP_{tss}-st^2P_{stt}-s^2tP_{sst}-st^2P_{tts}\\
&-s^2tP_{sts}-st^2P_{tst}-6s^2P_{ss}-6t^2P_{tt}-6stP_{st}-6stP_{ts}-3sP_s\\
&-3tP_t)y^iy^jy^ky^l+\frac{y^i}{r^2}\left(P_{sss}x^jx^kx^l+P_{ttt}a^ja^ka^l\right)\\
&+\frac{1}{r^4}[\left(s^2P_{sss}+t^2P_{stt}+stP_{sst}+stP_{sts}+3sP_{ss}+3tP_{st}\right)y^iy^jy^kx^l\\
&+(s^2P_{sst}+t^2P_{ttt}+stP_{stt}+stP_{tst}+3tP_{tt}\\
&+3sP_{st})y^iy^jy^ka^l](j\to k\to l\to j)\\
&-\frac{1}{r^3}[(P_{ss}+sP_{sss}+tP_{sst})y^iy^jx^kx^l\\
&+(P_{st}+sP_{sst}+tP_{stt})y^iy^ja^kx^l+(P_{st}+sP_{sts}+tP_{stt})y^iy^jx^ka^l\\
&+(P_{tt}+tP_{ttt}+sP_{stt})y^iy^ja^ka^l](j\to k\to l\to j)\\
&+\frac{1}{r^2}\left(P_{stt}y^ix^ja^ka^l+P_{sst}y^ia^jx^kx^l\right)(j\to k\to l\to j)\\
&+\frac{1}{r}[(Q_s-sQ_{ss}-tQ_{st})x^i\delta_{jk}x^l\\
&+(Q_t-tQ_{tt}-sQ_{ts})x^i\delta_{jk}a^l](j\to k\to l\to j)\\
&+\frac{1}{r^3}[(s^2Q_{sss}+t^2Q_{stt}+stQ_{sst}+stQ_{sts}+sQ_{ss}+tQ_{st}\\
&-Q_s)x^ix^jy^ky^l+(t^2Q_{ttt}+s^2Q_{sst}+stQ_{stt}+stQ_{tst}+tQ_{tt}+sQ_{st}\\
&-Q_t)x^iy^jy^ka^l](j\to k\to l\to j)+\frac{1}{r^2}(s^2Q_{ss}+t^2Q_{tt}\\
&+stQ_{ts}+stQ_{st}-sQ_s-tQ_t)x^iy^j\delta_{kl}(j\to k\to l\to j)\\
&-\frac{1}{r^2}\left(tQ_{sst}+sQ_{sss}\right)x^ix^jx^ly^k(j\to k\to l\to j)\\
&+\frac{x^i}{r}\left(Q_{sss}x^jx^kx^l+Q_{ttt}a^ja^ka^l\right)\\
&+\frac{1}{r^4}(3sQ_s+3tQ_t-3s^2Q_{ss}-3t^2Q_{tt}-3stQ_{ts}-3stQ_{st}-s^3Q_{sss}\\
&-t^3Q_{ttt}-s^2tQ_{sst}-st^2Q_{tts}-s^2tQ_{sts}-st^2Q_{tst}\\
&-s^2tQ_{tss}-st^2Q_{stt})x^iy^jy^ky^l\\
&+\frac{1}{r}\left(Q_{sst}x^ix^jx^ka^l+Q_{stt}x^ix^ja^ka^l\right)(j\to k\to l\to j)\\
&-\frac{1}{r^2}[\left(sQ_{sst}+tQ_{stt}\right)x^ix^jy^ka^l+\left(sQ_{sts}+tQ_{stt}\right)x^ix^ja^ky^l\\
&+\left(sQ_{stt}+tQ_{ttt}\right)x^iy^ja^ka^l](j\to k\to l\to j)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{r}[(R_t - tR_{tt} - sR_{ts})a^i\delta_{jk}a^l\\
&+(R_s - sR_{ss} - tR_{st})a^i\delta_{jk}x^l](j\to k\to l\to j)\\
&+\frac{1}{r^3}[(t^2R_{ttt} + s^2R_{sst} + stR_{stt} + stR_{tst} + tR_{tt} + sR_{ts} - R_t)a^ia^jy^ky^l\\
&+(s^2R_{sss} + t^2R_{stt} + stR_{sst} + stR_{sts} + sR_{ss}\\
&+tR_{st} - R_s)a^iy^jy^kx^l](j\to k\to l\to j) + \frac{1}{r^2}(s^2R_{ss} + t^2R_{tt}\\
&+stR_{st} + stR_{ts} - sR_s - tR_t)a^iy^j\delta_{kl}(j\to k\to l\to j)\\
&-\frac{1}{r^2}(sR_{stt} + tR_{ttt})a^ia^ja^ky^l(j\to k\to l\to j)\\
&+\frac{1}{r}\left(R_{ttt}a^ia^ja^ka^l + R_{sss}a^ix^jx^kx^l\right)\\
&+\frac{1}{r^4}(3sR_s + 3tR_t - 3s^2R_{ss} - 3t^2R_{tt} - 3stR_{st} - 3stR_{ts} - s^3R_{sss}\\
&-t^3R_{ttt} - s^2tR_{sst} - st^2R_{stt} - s^2tR_{sts} - st^2R_{tst} - s^2tR_{tss}\\
&-st^2R_{tts})a^iy^jy^ky^l\\
&+\frac{1}{r}\left(R_{stt}a^ia^ja^kx^l + R_{sst}a^ia^jx^kx^l\right)(j\to k\to l\to j)\\
&-\frac{1}{r^2}[(tR_{stt} + sR_{sts})a^ia^jy^kx^l + (tR_{stt} + sR_{sst})a^ia^jx^ky^l\\
&+(tR_{sst} + sR_{sss})a^iy^jx^kx^l](j\to k\to l\to j). \qquad (11)
\end{aligned}
$$

When $a = 0$, then the Berwald curvature in (11) becomes the Berwald curvature of spherically symmetric Finsler metrics [6].

Proof. Let F be a Finsler metric in (5). From (4) and (10), we have

$$
\begin{aligned}
\frac{\partial G^i}{\partial y^i} &= r_{y^j}Py^i + r(P_s s_{y^j} + P_t t_{y^j})y^i + rP\delta^i_j + 2y^jQx^i\\
&\quad +r^2(Q_s s_{y^j} + Q_t t_{y^j})x^i + 2y^jRa^i + r^2(R_s s_{y^j} + R_t t_{y^j})a^i, \qquad (12)
\end{aligned}
$$

where we have used $\frac{\partial u}{\partial y^i} = 0$ and $\frac{\partial r^2}{\partial y^j} = 2y^j$. By (12), we obtain

$$
\begin{aligned}
\frac{\partial^2 G^i}{\partial y^j\partial y^k} &= [(P_s s_{y^k} + P_t t_{y^k})y^ir_{y^j} + P\delta^i_k r_{y^j} + r(P_s s_{y^j} + P_t t_{y^j})\delta^i_k\\
&\quad +2(Q_s s_{y^j} + Q_t t_{y^j})y^kx^i + 2(R_s s_{y^j} + R_t t_{y^j})y^ka^i](j\leftrightarrow k)\\
&\quad +r(P_{ss}s_{y^k} + P_{st}t_{y^k})y^is_{y^j} + P_sry^is_{y^jy^k} + Py^ir_{y^jy^k}\\
&\quad +r(P_{ts}s_{y^k} + P_{tt}t_{y^k})y^it_{y^j} + P_try^it_{y^jy^k} + 2Qx^i\delta_{jk}\\
&\quad +r^2(Q_{ss}s_{y^k} + Q_{st}t_{y^k})x^is_{y^j} + r^2Q_sx^is_{y^jy^k} + r^2Q_tx^it_{y^jy^k}\\
&\quad +r^2(Q_{ts}s_{y^k} + Q_{tt}t_{y^k})x^it_{y^j} + 2Ra^i\delta_{jk} + r^2R_sa^is_{y^jy^k}\\
&\quad +r^2(R_{ss}s_{y^k} + R_{st}t_{y^k})a^is_{y^j} + r^2(R_{ts}s_{y^k} + R_{tt}t_{y^k})a^it_{y^j}
\end{aligned}
$$

$$+r^2R_ta^it_{y^jy^k},$$

where $j \leftrightarrow k$ denotes symmetrization. By definition, we get

$$\begin{aligned}
B^i{}_{jkl} =& \left(P_{tt}y^it_{y^j}t_{y^k}r_{y^l} + P_ty^it_{y^jy^k}r_{y^l} + P_ty^it_{y^j}r_{y^ky^l}\right)(j \to k \to l \to j)\\
&+[P_s\left(s_{y^j}r_{y^k} + s_{y^k}r_{y^j}\right)\delta^i_l + Pr_{y^jy^k}\delta^i_l\\
&+P_{ss}rs_{y^j}s_{y^k}\delta^i_l](j \to k \to l \to j)\\
&+(P_{ss}y^is_{y^j}s_{y^k}r_{y^l} + P_sy^is_{y^jy^k}r_{y^l} + P_sy^is_{y^j}r_{y^ky^l})(j \to k \to l \to j)\\
&+\left[P_t\left(t_{y^j}r_{y^k} + t_{y^k}r_{y^j}\right)\delta^i_l + P_{tt}rt_{y^j}t_{y^k}\delta^i_l\right](j \to k \to l \to j)\\
&+\left(P_srs_{y^jy^k}\delta^i_l + P_{ss}ry^is_{y^j}s_{y^ky^l}\right)(j \to k \to l \to j)\\
&+\left(P_trt_{y^jy^k}\delta^i_l + P_{tt}ry^it_{y^j}t_{y^ky^l}\right)(j \to k \to l \to j)\\
&+[P_{sst}ry^is_{y^j}s_{y^k}t_{y^l} + P_{stt}ry^is_{y^j}t_{y^k}t_{y^l}\\
&+P_{st}ry^i\left(s_{y^jy^k}t_{y^l} + s_{y^j}t_{y^ky^l}\right)](j \to k \to l \to j)\\
&+P_{st}\{[y^i\left(s_{y^j}t_{y^k} + s_{y^k}t_{y^j}\right)r_{y^l}\\
&+r\left(s_{y^j}t_{y^k} + s_{y^k}t_{y^j}\right)\delta^i_l]\}(j \to k \to l \to j)\\
&+2x^i\left(Q_ss_{y^j}\delta_{kl} + Q_{ss}y^js_{y^k}s_{y^l} + Q_sy^js_{y^ky^l}\right)(j \to k \to l \to j)\\
&+2x^i\left(Q_tt_{y^j}\delta_{kl} + Q_{tt}y^jt_{y^k}t_{y^l} + Q_ty^jt_{y^ky^l}\right)(j \to k \to l \to j)\\
&+[Q_{sst}r^2x^is_{y^j}s_{y^k}t_{y^l} + Q_{stt}r^2x^is_{y^j}t_{y^k}t_{y^l}\\
&+Q_{st}r^2x^i\left(s_{y^jy^k}t_{y^l} + s_{y^j}t_{y^ky^l}\right)](j \to k \to l \to j)\\
&+[2Q_{st}x^iy^j\left(s_{y^k}t_{y^l} + s_{y^l}t_{y^k}\right)\\
&+r^2x^i\left(Q_{ss}s_{y^j}s_{y^ky^l} + Q_{tt}t_{y^j}t_{y^ky^l}\right)](j \to k \to l \to j)\\
&+2a^i\left(R_ss_{y^j}\delta_{kl} + R_{ss}y^js_{y^k}s_{y^l} + R_sy^js_{y^ky^l}\right)(j \to k \to l \to j)\\
&+2a^i\left(R_tt_{y^j}\delta_{kl} + R_{tt}y^jt_{y^k}t_{y^l} + R_ty^jt_{y^ky^l}\right)(j \to k \to l \to j)\\
&+[R_{sst}r^2a^is_{y^j}s_{y^k}t_{y^l} + R_{stt}r^2a^is_{y^j}t_{y^k}t_{y^l}\\
&+R_{st}r^2a^i\left(s_{y^jy^k}t_{y^l} + s_{y^j}t_{y^ky^l}\right)](j \to k \to l \to j)\\
&+[2R_{st}a^iy^j\left(s_{y^k}t_{y^l} + s_{y^l}t_{y^k}\right)\\
&+r^2a^i\left(R_{ss}s_{y^j}s_{y^ky^l} + R_{tt}t_{y^j}t_{y^ky^l}\right)](j \to k \to l \to j)\\
&+P_{sss}ry^is_{y^j}s_{y^k}s_{y^l} + Py^ir_{y^jy^ky^l} + P_sry^is_{y^jy^ky^l} + P_{ttt}ry^it_{y^j}t_{y^k}t_{y^l}\\
&+P_try^it_{y^jy^ky^l} + Q_{sss}r^2x^is_{y^j}s_{y^k}s_{y^l} + Q_sr^2x^is_{y^jy^ky^l}\\
&+Q_{ttt}r^2x^it_{y^j}t_{y^k}t_{y^l} + Q_tr^2x^it_{y^jy^ky^l} + R_{sss}r^2a^is_{y^j}s_{y^k}s_{y^l}\\
&+R_sr^2a^is_{y^jy^ky^l} + R_{ttt}r^2a^it_{y^j}t_{y^k}t_{y^l} + R_tr^2a^it_{y^jy^ky^l}, \qquad (13)
\end{aligned}$$

where $j \to k \to l \to j$ denotes cyclic permutation.
Observe that

$$r_{y^j} = \frac{y^j}{r}, \qquad (14)$$

$$r_{y^jy^k} = \frac{r^2\delta_{jk} - y^jy^k}{r^3}, \tag{15}$$

$$r_{y^jy^ky^l} = \frac{3y^jy^ky^l - r^2\delta_{jk}y^l(j \to k \to l \to j)}{r^5}. \tag{16}$$

where we have used (4). Direct computations yield

$$s_{y^j} = \frac{rx^j - sy^j}{r^2},\ s_{y^jy^k} = \frac{3sy^jy^k - rx^jy^k - rx^ky^j - sr^2\delta_{jk}}{r^4}, \tag{17}$$

$$\begin{aligned} s_{y^jy^ky^l} = & \frac{1}{r^5}(3x^jy^ky^l + 3sr\delta_{jk}y^l - r^2x^j\delta_{kl})(j \to k \to l \to j) \\ & -\frac{15}{r^6}sy^jy^ky^l, \end{aligned} \tag{18}$$

$$t_{y^j} = \frac{ra^j - ty^j}{r^2}, \tag{19}$$

$$t_{y^jy^k} = \frac{3ty^jy^k - ra^jy^k - ra^ky^j - tr^2\delta_{jk}}{r^4}, \tag{20}$$

$$t_{y^jy^ky^l} = \frac{1}{r^5}(3a^jy^ky^l + 3tr\delta_{jk}y^l - r^2a^j\delta_{kl})(j \to k \to l \to j) \tag{21}$$

$$-\frac{15}{r^6}ty^jy^ky^l.$$

From (13)-(21) we conclude the proof. □

Proof of theorem 1.1. As we know, a Finsler metric F is called Berwald metric if Berwald curvature is zero. From (11) , a Finsler metric F in (5) is a Berwald metric if and only if P, Q and R in its geodesic spray coefficients must satisfy

$$\begin{cases} P - sP_s - tP_t = 0, \\ P_{ss} = P_{tt} = P_{st} = 0, \\ Q_s - sQ_{ss} - tQ_{st} = 0, \\ Q_t - tQ_{tt} - sQ_{st} = 0, \\ Q_{sss} = Q_{sst} = Q_{stt} = Q_{ttt} = 0, \\ R_s - sR_{ss} - tR_{st} = 0, \\ R_t - tR_{tt} - sR_{st} = 0, \\ R_{sss} = R_{sst} = R_{stt} = R_{ttt} = 0. \end{cases}$$

From these equations, one can first solve P, Q and R, then completely determine the metric function F. □

2.2. **Douglas curvature.** In [1], Douglas introduced the local function $D_j{}^i{}_{kl}$ as follows

$$D_j{}^i{}_{kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right), \tag{22}$$

These functions are called Douglas curvature and a Finsler metric F is said to be a Douglas metric if $D_j{}^i{}_{kl} = 0$ [10, 12].

Proposition 2.2. *Let $F = r\phi(u, s, v, t)$ be a Finsler metric on an open subset $U \subset \mathbb{R}^n$ with dimension $n \geq 3$ in (5). Then F has vanishing Douglas curvature if and only if the following hold*

$$\begin{cases} Q_s - sQ_{ss} - tQ_{st} = 0, \\ Q_t - tQ_{tt} - sQ_{st} = 0, \\ Q_{sss} = Q_{sst} = Q_{stt} = Q_{ttt} = 0, \\ R_s - sR_{ss} - tR_{st} = 0, \\ R_t - tR_{tt} - sR_{st} = 0, \\ R_{sss} = R_{sst} = R_{stt} = R_{ttt} = 0. \end{cases} \tag{23}$$

Proof. Let F be a Finsler metric in (5). From (4) and (10), we have

$$\begin{aligned} \frac{\partial G^j}{\partial y^j} &= r_{y^j} P y^j + r(P_s s_{y^j} + P_t t_{y^j}) y^j + nrP + 2Q < x, y > \\ &\quad + r^2(Q_s s_{y^j} + Q_t t_{y^j}) x^j + 2R < a, y > + r^2(R_s s_{y^j} + R_t t_{y^j}) a^j \\ &= r\big[(n+1)P + 2sQ + 2tR + (u - s^2)Q_s + (v - st)Q_t + (v - st)R_s \\ &\quad + (a^2 - t^2)R_t\big]. \end{aligned}$$

It follows that

$$G^i - \frac{1}{n+1}\frac{\partial G^j}{\partial y^j} y^i = rZy^i + r^2 Q x^i + r^2 R a^i,$$

where

$$\begin{aligned} Z = -\frac{1}{n+1}&[2sQ + 2tR + (u - s^2)Q_s + (v - st)Q_t + (v - st)R_s \\ &+ (a^2 - t^2)R_t]. \end{aligned} \tag{24}$$

Substituting G^i into (22) we get

$$D_j{}^i{}_{kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(rZy^i + r^2 Q x^i + r^2 R a^i\right).$$

In this case, to get Douglas curvature, one can replace Z and P in (11). When $a=0$, then the Douglas curvature becomes the Douglas curvature of spherically symmetric Finsler metrics [5].

Therefore, a Finsler metric F in (5) is a Douglas metric if, and only if,

$$Z - sZ_s - tZ_t = 0, \tag{25}$$

$$Z_{ss} = Z_{tt} = Z_{st} = 0, \tag{26}$$

$$Q_s - sQ_{ss} - tQ_{st} = 0, \tag{27}$$

$$Q_t - tQ_{tt} - sQ_{st} = 0, \tag{28}$$

$$Q_{sss} = Q_{sst} = Q_{stt} = Q_{ttt} = 0, \tag{29}$$

$$R_s - sR_{ss} - tR_{st} = 0, \tag{30}$$

$$R_t - tR_{tt} - sR_{st} = 0, \tag{31}$$

$$R_{sss} = R_{sst} = R_{stt} = R_{ttt} = 0. \tag{32}$$

Plugging (24) into (25), we have

$$\begin{aligned} -(n+1)(Z - sZ_s - tZ_t) = {} & (Q_s - sQ_{ss} - tQ_{st})\left(u - s^2\right) \\ & + (Q_t - tQ_{tt} - sQ_{st})(v - st) \\ & + (R_s - sR_{ss} - tR_{st})(v - st) \\ & + (R_t - tR_{tt} - sR_{st})\left(a^2 - t^2\right). \end{aligned}$$

Thus (27), (28), (30) and (31) imply (25). Finally, (26) is easy to obtain from (27)-(32). Then (25)-(32) can be reduced to (23). $\square$

3. Appendix

$$\begin{aligned}
\omega &:= \frac{C_4}{C_2} - \frac{C_4}{C_2}\zeta u - \zeta s, \\
L^2 &:= s\omega + \frac{C_4}{C_2}(u\omega + s) + 1, \\
b_1 &:= -v\tau\,\omega^2 - t\tau\,\omega - v\zeta, \\
b_2 &:= -v\tau\,\omega - t\tau, \\
B^{ij}a_j a_i &:= tb_2 + vb_1 + a^2, \\
d_1 &:= -s\nu\,{b_1}^2 - s\tau\,\omega^2 - t\nu\,b_1b_2 - v\nu\,{b_1}^2 - v\tau\,\omega^2 - a^2\nu\,b_1 - t\nu\,b_1 - t\tau\,\omega \\
&\quad -\nu\,b_1b_2 - s\xi - v\xi - \tau\,\omega, \\
d_2 &:= -s\nu\,b_1b_2 - t\nu\,{b_2}^2 - v\nu\,b_1b_2 - a^2\nu\,b_2 - s\tau\,\omega - t\nu\,b_2 - v\tau\,\omega - \nu\,{b_2}^2 - t\tau \\
&\quad -\tau + 1, \\
d_3 &:= -s\nu\,b_1 - t\nu\,b_2 - v\nu\,b_1 - a^2\nu - t\nu - \nu\,b_2 + 1, \\
M^2 &:= (s+v)\,d_1 + (t+1)\,d_2 + \left(a^2 + t\right)d_3, \\
e_1 &:= -a^2\sigma\,d_1d_3 - s\nu\,b_1b_2 - s\sigma\,d_1d_2 - t\nu\,b_1b_2 - t\sigma\,d_1d_2 - u\nu\,{b_1}^2 - u\sigma\,{d_1}^2
\end{aligned}$$

$$
\begin{aligned}
&-u\tau\,\omega^2 - v\nu\,{b_1}^2 - v\sigma\,{d_1}^2 - v\sigma\,d_1d_3 - v\tau\,\omega^2 - a^2\nu\,b_1 - s\tau\,\omega - t\tau\,\omega\\
&-v\nu\,b_1 - u\xi - v\xi + 1,\\
e_2 &:= -a^2\sigma\,d_2d_3 - s\nu\,{b_2}^2 - s\sigma\,{d_2}^2 - t\nu\,{b_2}^2 - t\sigma\,{d_2}^2 - u\nu\,b_1b_2 - u\sigma\,d_1d_2\\
&\quad -v\nu\,b_1b_2 - v\sigma\,d_1d_2 - v\sigma\,d_2d_3 - a^2\nu\,b_2 - u\tau\,\omega - v\nu\,b_2 - v\tau\,\omega - s\tau - t\tau,\\
e_3 &:= -a^2\sigma\,{d_3}^2 - s\sigma\,d_2d_3 - t\sigma\,d_2d_3 - u\sigma\,d_1d_3 - v\sigma\,d_1d_3 - v\sigma\,{d_3}^2 - s\nu\,b_2\\
&\quad -t\nu\,b_2 - u\nu\,b_1 - v\nu\,b_1 - a^2\nu - \nu v + 1,\\
N^2 &:= (u+v)\,e_1 + (s+t)\,e_2 + \left(a^2+v\right)e_3,\\
f_1 &:= -s\kappa\,{e_1}^2 - s\nu\,{b_1}^2 - s\sigma\,{d_1}^2 - s\tau\,\omega^2 - t\kappa\,e_1e_3 - t\sigma\,d_1d_3 - t\nu\,b_1 - \kappa\,e_1e_2\\
&\quad -\nu\,b_1b_2 - \sigma\,d_1d_2 - s\xi - \tau\,\omega,\\
f_2 &:= -s\kappa\,e_1e_2 - s\nu\,b_1b_2 - s\sigma\,d_1d_2 - t\kappa\,e_2e_3 - t\sigma\,d_2d_3 - s\tau\,\omega - t\nu\,b_2 - \kappa\,{e_2}^2\\
&\quad -\nu\,{b_2}^2 - \sigma\,{d_2}^2 - \tau + 1,\\
f_3 &:= -s\kappa\,e_1e_3 - s\sigma\,d_1d_3 - t\kappa\,{e_3}^2 - t\sigma\,{d_3}^2 - s\nu\,b_1 - \kappa\,e_2e_3 - \sigma\,d_2d_3 - t\nu\\
&\quad -\nu\,b_2,\\
y^2 &:= sf_1 + tf_3 + f_2,\\
C &:= s^2\alpha\,{f_1}^2 + s^2\kappa\,{e_1}^2 + s^2\nu\,{b_1}^2 + s^2\sigma\,{d_1}^2 + s^2\tau\,\omega^2 + st\alpha\,f_1f_3 + st\kappa\,e_1e_3\\
&\quad +st\sigma\,d_1d_3 + st\nu\,b_1 - u\alpha\,{f_1}^2 - u\kappa\,{e_1}^2 - u\nu\,{b_1}^2 - u\sigma\,{d_1}^2 - u\tau\,\omega^2\\
&\quad -v\alpha\,f_1f_3 - v\kappa\,e_1e_3 - v\sigma\,d_1d_3 + s^2\xi - v\nu\,b_1 - u\xi + 1,\\
D &:= s^2\kappa\,e_1e_2 + s^2\nu\,b_1b_2 + s^2\sigma\,d_1d_2 + st\alpha\,f_2f_3 + st\kappa\,e_2e_3 + st\sigma\,d_2d_3 + s^2\tau\,\omega\\
&\quad +st\nu\,b_2 - u\alpha\,f_1f_2 - u\kappa\,e_1e_2 - u\nu\,b_1b_2 - u\sigma\,d_1d_2 - v\alpha\,f_2f_3 - v\kappa\,e_2e_3\\
&\quad -v\sigma\,d_2d_3 - u\tau\,\omega - v\nu\,b_2 + \alpha\,f_1f_2 - s,\\
E &:= s^2\alpha\,f_1f_3 + s^2\kappa\,e_1e_3 + s^2\sigma\,d_1d_3 + st\alpha\,{f_3}^2 + st\kappa\,{e_3}^2 + st\sigma\,{d_3}^2 + s^2\nu\,b_1\\
&\quad -u\alpha\,f_1f_3 - u\kappa\,e_1e_3 - u\sigma\,d_1d_3 - v\alpha\,{f_3}^2 - v\kappa\,{e_3}^2 - v\sigma\,{d_3}^2 + st\nu - u\nu\,b_1\\
&\quad -v\nu,\\
G &:= st\alpha\,{f_1}^2 + st\kappa\,{e_1}^2 + st\nu\,{b_1}^2 + st\sigma\,{d_1}^2 + st\tau\,\omega^2 + t^2\alpha\,f_1f_3 + t^2\kappa\,e_1e_3\\
&\quad +t^2\sigma\,d_1d_3 - a^2\alpha\,f_1f_3 - a^2\kappa\,e_1e_3 - a^2\sigma\,d_1d_3 + t^2\nu\,b_1 - v\alpha\,{f_1}^2 - v\kappa\,{e_1}^2\\
&\quad -v\nu\,{b_1}^2 - v\sigma\,{d_1}^2 - v\tau\,\omega^2 - a^2\nu\,b_1 + st\xi - v\xi,\\
H &:= st\alpha\,f_1f_2 + st\kappa\,e_1e_2 + st\nu\,b_1b_2 + st\sigma\,d_1d_2 + t^2\alpha\,f_2f_3 + t^2\kappa\,e_2e_3\\
&\quad +t^2\sigma\,d_2d_3 - a^2\alpha\,f_2f_3 - a^2\kappa\,e_2e_3 - a^2\sigma\,d_2d_3 + st\tau\,\omega + t^2\nu\,b_2 - v\alpha\,f_1f_2\\
&\quad -v\kappa\,e_1e_2 - v\nu\,b_1b_2 - v\sigma\,d_1d_2 - a^2\nu\,b_2 - v\tau\,\omega - t,\\
I &:= st\alpha\,f_1f_3 + st\kappa\,e_1e_3 + st\sigma\,d_1d_3 + t^2\alpha\,{f_3}^2 + t^2\kappa\,{e_3}^2 + t^2\sigma\,{d_3}^2 - a^2\alpha\,{f_3}^2\\
&\quad -a^2\kappa\,{e_3}^2 - a^2\sigma\,{d_3}^2 + st\nu\,b_1 - v\alpha\,f_1f_3 - v\kappa\,e_1e_3 - v\sigma\,d_1d_3 + t^2\nu - v\nu\,b_1\\
&\quad -a^2\nu + 1,\\
P &:= A + \frac{BD+FH}{C_0},
\end{aligned}
$$

$$Q := \frac{BC + FG}{C_0},$$
$$R := \frac{BE + IF}{C_0}.$$

By using Maple program, we can reformulate the equations above as the followings:

$$\begin{aligned}
\omega = &-\big(-s^3\phi^2\phi_{ss}\phi_{st} - s^3\phi\,{\phi_s}^2\phi_{st} - s^3{\phi_s}^3\phi_t - st^2\phi^2{\phi_{st}}^2 - st^2\phi_s{\phi_t}^3 + s^2\phi^2\phi_s\phi_{st}\\
&+s^2\phi\,\phi_s^2\phi_t - t^2\phi^2\phi_t\phi_{st} - t^2\phi\,\phi_s{\phi_t}^2 + s\phi^3\phi_{ss} + s\phi^2{\phi_s}^2 + t\phi^3\phi_{st} - 2\,s^2t\phi^2{\phi_{st}}^2\\
&-2\,s^2t{\phi_s}^2{\phi_t}^2 + 2\,t\phi^2\phi_s\phi_t - \phi^3\phi_s - 2\,st\phi^2\phi_t\phi_{ss} - 4s^2t\phi\,\phi_s\phi_t\phi_{st} - 2\,st^2\phi\,\phi_s\phi_t\phi_{st}\\
&+st^2\phi\,{\phi_s}^2\phi_{tt} - st^2\phi\,\phi_s\phi_t\phi_{tt} + st^2\phi\,{\phi_t}^2\phi_{ss} - st^2\phi\,{\phi_t}^2\phi_{st} + st\phi^2\phi_s\phi_{st} + st\phi^2\phi_t\phi_{st}\\
&-st\phi\,{\phi_s}^2\phi_t + st\phi\,\phi_s{\phi_t}^2 - s^3\phi\,\phi_s\phi_t\phi_{ss} + st^2\phi^2\phi_{ss}\phi_{tt} - st^2\phi^2\phi_{st}\phi_{tt}\big)\\
&\big(-4\,stu\phi\,\phi_s\phi_t\phi_{st} - s^3\phi^2\phi_s\phi_{ss} - s^3\phi\,{\phi_s}^3 - s^2u{\phi_s}^3\phi_t - t^3\phi^2\phi_t\phi_{tt} - t^3\phi\,{\phi_t}^3\\
&-t^2u\phi^2{\phi_{st}}^2 - t^2u\phi_s{\phi_t}^3 + s^2\phi^3\phi_{ss} + su\phi\,{\phi_s}^3 + t^2\phi^3\phi_{tt} - s\phi^3\phi_s - t\phi^3\phi_t - u\phi^2{\phi_s}^2\\
&+2\,st\phi^3\phi_{st} + 2\,t^2\phi^2{\phi_t}^2 + 2\,s^2\phi^2{\phi_s}^2 - 2\,s^2t\phi^2\phi_s\phi_{st} - 3\,s^2t\phi\,{\phi_s}^2\phi_t - 2\,st^2\phi^2\phi_t\phi_{st}\\
&-3\,st^2\phi\,\phi_s{\phi_t}^2 - 2\,stu\phi^2{\phi_{st}}^2 - 2\,stu{\phi_s}^2{\phi_t}^2 + 4\,st\phi^2\phi_s\phi_t + 2\,tu\phi^2\phi_s\phi_{st} + tu\phi\,{\phi_s}^2\phi_t\\
&+tu\phi\,\phi_s{\phi_t}^2 - 2\,t^2u\phi\,\phi_s\phi_t\phi_{st} - s^2t\phi^2\phi_t\phi_{ss} - s^2u\phi^2\phi_{ss}\phi_{st} - s^2u\phi\,{\phi_s}^2\phi_{st}\\
&-s^2u\phi\,\phi_s\phi_t\phi_{ss} - st^2\phi^2\phi_s\phi_{tt} + t^2u\phi^2\phi_{ss}\phi_{tt} - t^2u\phi^2\phi_{st}\phi_{tt} + t^2u\phi\,{\phi_s}^2\phi_{tt}\\
&-t^2u\phi\,\phi_s\phi_t\phi_{tt} + t^2u\phi\,{\phi_t}^2\phi_{ss} - t^2u\phi\,{\phi_t}^2\phi_{st} + su\phi^2\phi_s\phi_{ss} + su\phi^2\phi_s\phi_{st} + su\phi\,{\phi_s}^2\phi_t\\
&-tu\phi^2\phi_t\phi_{ss} + tu\phi^2\phi_t\phi_{st}\big)\,,\\
L^2 = &-\big(-s^2t^2u\phi\,{\phi_s}^2{\phi_t}^2\phi_{ss} + 2\,s^2t^2u\phi\,{\phi_s}^3\phi_t\phi_{st} + 2\,s^2t^2u\phi\,{\phi_s}^3\phi_t\phi_{tt} + 14\,s^2t^2u\phi\,{\phi_s}^2{\phi_t}^2\phi_{st}\\
&+2\,s^2t^2u\phi\,\phi_s{\phi_t}^3\phi_{ss} - 2\,st^3u\phi^3\phi_{ss}\phi_{st}\phi_{tt} - 2\,st^3u\phi^2{\phi_s}^2\phi_{st}\phi_{tt} + 6\,st^3u\phi^2\phi_s\phi_t{\phi_{st}}^2\\
&-2\,st^3u\phi^2{\phi_t}^2\phi_{ss}\phi_{st} - 2\,st^3u\phi\,{\phi_s}^3\phi_t\phi_{tt} + 4\,st^3u\phi\,{\phi_s}^2{\phi_t}^2\phi_{st} + 4\,st^3u\phi\,{\phi_s}^2{\phi_t}^2\phi_{tt}\\
&-2\,st^3u\phi\,\phi_s{\phi_t}^3\phi_{ss} + 8\,st^3u\phi\,\phi_s{\phi_t}^3\phi_{st} + 2\,t^4u\phi^2\phi_s\phi_t\phi_{st}\phi_{tt} - 2\,s^2tu\phi^3\phi_s\phi_{ss}\phi_{st}\\
&-2\,s^2tu\phi^3\phi_t\phi_{ss}\phi_{st} + 2\,s^2tu\phi^2{\phi_s}^2\phi_t\phi_{ss} - 10\,s^2tu\phi^2{\phi_s}^2\phi_t\phi_{st} - 2\,s^2tu\phi^2\phi_s{\phi_t}^2\phi_{ss}\\
&-2st^2u\phi^3\phi_s\phi_{st}\phi_{tt} + 4\,st^2u\phi^3\phi_t\phi_{ss}\phi_{st} - 4\,st^2u\phi^2{\phi_s}^2\phi_t\phi_{st} - 2\,st^2u\phi^2{\phi_s}^2\phi_t\phi_{tt}\\
&+4\,st^2u\phi^2\phi_s{\phi_t}^2\phi_{ss} - 10\,st^2u\phi^2\phi_s{\phi_t}^2\phi_{st} - 4\,stu\phi^3\phi_s\phi_t\phi_{ss} + 2\,stu\phi^3\phi_s\phi_t\phi_{st}\\
&-8\,s^5t\phi^2\phi_s\phi_t\phi_{ss}\phi_{st} - 2\,s^4t^2\phi^2\phi_s\phi_t\phi_{ss}\phi_{st} - 2\,s^4t^2\phi^2\phi_s\phi_t\phi_{ss}\phi_{tt} + 2\,s^3t^3\phi^2\phi_s\phi_t\phi_{ss}\phi_{tt}\\
&-8\,s^3t^3\phi^2\phi_s\phi_t\phi_{st}\phi_{tt} - 2\,s^2t^4\phi^2\phi_s\phi_t\phi_{st}\phi_{tt} + 8\,s^3tu\phi\,{\phi_s}^3\phi_t\phi_{st} + 4\,s^3tu\phi\,{\phi_s}^2{\phi_t}^2\phi_{ss}\\
&+2\,s^2t^2u\phi^3\phi_{ss}\phi_{st}\phi_{tt} - 2\,s^2t^2u\phi^2{\phi_s}^2\phi_{ss}\phi_{tt} + 2\,s^2t^2u\phi^2{\phi_s}^2\phi_{st}\phi_{tt} + 12\,s^2t^2u\phi^2\phi_s\phi_t{\phi_{st}}^2\\
&+2\,s^2t^2u\phi^2{\phi_t}^2\phi_{ss}\phi_{st} - 2\,st^3\phi^4\phi_{st}\phi_{tt} - 4\,st^3\phi^3{\phi_t}^2\phi_{st} - 6\,st^3\phi^2\phi_s{\phi_t}^3 + 2\,t^3u\phi^2{\phi_t}^3\phi_{ss}\\
&-2\,t^3u\phi^2{\phi_t}^3\phi_{st} - 2\,t^3u\phi\,\phi_s{\phi_t}^4 - 2\,s^2t\phi^4\phi_s\phi_{st} - 2\,s^2t\phi^3{\phi_s}^2\phi_t - 2\,s^2u\phi^3{\phi_s}^2\phi_{ss}\\
&+2\,st^2\phi^4\phi_t\phi_{st} + 3\,st^2\phi^3\phi_s{\phi_t}^2 + 2\,su\phi^4\phi_s\phi_{ss} + 2\,tu\phi^4\phi_s\phi_{st} + 2\,tu\phi^3{\phi_s}^2\phi_t\\
&-2\,s^6\phi^2{\phi_s}^2\phi_{ss}\phi_{st} - 2\,s^6\phi\,{\phi_s}^3\phi_t\phi_{ss} - 4\,s^5t\phi^3\phi_{ss}{\phi_{st}}^2 - 4\,s^5t\phi^2{\phi_s}^2{\phi_{st}}^2\\
&-4\,s^3t^3\phi^3{\phi_{st}}^2\phi_{tt} - 4\,s^3t^3\phi^2{\phi_t}^2{\phi_{st}}^2 + 2\,s^5\phi^3\phi_s\phi_{ss}\phi_{st} + 2\,s^5\phi^2{\phi_s}^2\phi_t\phi_{ss} + 4\,s^4t\phi^3\phi_s{\phi_{st}}^2\\
&-2\,s^4t\phi^3\phi_t{\phi_{ss}}^2 + 2\,s^4t\phi^2{\phi_s}^3\phi_{st} + 6\,s^4t\phi\,{\phi_s}^3{\phi_t}^2 + 5\,s^3t^2\phi^3\phi_s{\phi_{st}}^2 + 4\,s^3t^2\phi^3\phi_t{\phi_{st}}^2
\end{aligned}$$

$$+6\, s^3t^2\phi\,\phi_s^2\phi_t^3 + 4\, s^3tu\phi_s^4\phi_t^2 - 2\, s^2t^3\phi^2\phi_t^3\phi_{ss} + 2\, s^2t^3\phi^2\phi_t^3\phi_{st} + 3\, s^2t^3\phi\,\phi_s^2\phi_t^3$$
$$+2\, s^2t^3\phi\,\phi_s\phi_t^4 + 4\, s^2t^2u\phi^3\phi_{st}^3 + 6\, s^2t^2u\phi_s^3\phi_t^3 + 2\, st^4\phi^2\phi_t^3\phi_{st} + 3\, st^4\phi\,\phi_s\phi_t^4$$
$$+2\, st^3u\phi^3\phi_{st}^3 + 4\, st^3u\phi_s^2\phi_t^4 + 2\, s^3t\phi^4\phi_{ss}\phi_{st} + 2\, s^3t\phi^2\phi_s^3\phi_t - 2\, s^3t\phi^2\phi_s^2\phi_t^2$$
$$-2\, s^3u\phi^2\phi_s^3\phi_{st} - 2\, s^3u\phi\,\phi_s^4\phi_t - 2\, s^2t^2\phi^2\phi_s^2\phi_t^2 - 3\, t^4\phi^2\phi_t^4 + 8\, s^3tu\phi^2\phi_s\phi_t\phi_{ss}\phi_{st}$$
$$+2\, s^2t^2u\phi^2\phi_s\phi_t\phi_{ss}\phi_{st} + 2\, s^2t^2u\phi^2\phi_s\phi_t\phi_{ss}\phi_{tt} - 2\, st^3u\phi^2\phi_s\phi_t\phi_{ss}\phi_{tt}$$
$$+8\, st^3u\phi^2\phi_s\phi_t\phi_{st}\phi_{tt} - 8\, s^5t\phi\,\phi_s^3\phi_t\phi_{st} - 4\, s^5t\phi\,\phi_s^2\phi_t^2\phi_{ss} - 2\, s^4t^2\phi^3\phi_{ss}\phi_{st}\phi_{tt}$$
$$+2\, s^4t^2\phi^2\phi_s^2\phi_{ss}\phi_{tt} - 2\, s^4t^2\phi^2\phi_s^2\phi_{st}\phi_{tt} - 12\, s^4t^2\phi^2\phi_s\phi_t\phi_{st}^2 - 2\, s^4t^2\phi^2\phi_t^2\phi_{ss}\phi_{st}$$
$$-2\, s^4t^2\phi\,\phi_s^3\phi_t\phi_{st} - 2\, s^4t^2\phi\,\phi_s^3\phi_t\phi_{tt} - 14\, s^4t^2\phi\,\phi_s^2\phi_t^2\phi_{st} - 2\, s^4t^2\phi\,\phi_s\phi_t^3\phi_{ss}$$
$$+2\, s^3t^3\phi^3\phi_{ss}\phi_{st}\phi_{tt} + 2\, s^3t^3\phi^2\phi_s^2\phi_{st}\phi_{tt} - 6\, s^3t^3\phi^2\phi_s\phi_t\phi_{st}^2 + 2\, s^3t^3\phi^2\phi_t^2\phi_{ss}\phi_{st}$$
$$+2\, s^3t^3\phi\,\phi_s^3\phi_t\phi_{tt} - 4\, s^3t^3\phi\,\phi_s^2\phi_t^2\phi_{st} - 4\, s^3t^3\phi\,\phi_s^2\phi_t^2\phi_{tt} + 2\, s^3t^3\phi\,\phi_s\phi_t^3\phi_{ss}$$
$$-8\, s^3t^3\phi\,\phi_s\phi_t^3\phi_{st} + 2\, s^2t^4\phi^2\phi_t^2\phi_{ss}\phi_{tt} - 2\, s^2t^4\phi^2\phi_t^2\phi_{st}\phi_{tt} - 2\, s^2t^4\phi\,\phi_s\phi_t^3\phi_{st}$$
$$-2\, s^2t^4\phi\,\phi_s\phi_t^3\phi_{tt} + 2\, s^4t\phi^3\phi_s\phi_{ss}\phi_{st} + 2\, s^4t\phi^3\phi_t\phi_{ss}\phi_{st} + s^2u\phi^2\phi_s^3\phi_t$$
$$+t^2u\phi^3\phi_s^2\phi_{tt} - t^2u\phi^3\phi_t^2\phi_{ss} + t^2u\phi^3\phi_t^2\phi_{st} - t^2u\phi^2\phi_s^2\phi_t^2 + t^2u\phi^2\phi_s\phi_t^3$$
$$-s^2t^2\phi^3\phi_t^2\phi_{st} - s^2t^2\phi^2\phi_s\phi_t^3 + s^2u\phi^3\phi_s^2\phi_{st} - t^4u\phi^3\phi_{ss}\phi_{tt}^2 + t^4u\phi^3\phi_{st}^2\phi_{tt}$$
$$+t^4u\phi^3\phi_{st}\phi_{tt}^2 - t^4u\phi^2\phi_s^2\phi_{tt}^2 + t^4u\phi^2\phi_s\phi_t\phi_{tt}^2 + t^4u\phi^2\phi_t^2\phi_{st}^2 - t^4u\phi\,\phi_s^2\phi_t^2\phi_{tt}$$
$$-t^4u\phi\,\phi_t^4\phi_{ss} + t^4u\phi\,\phi_t^4\phi_{st} - s^2t^2\phi^3\phi_s^2\phi_{tt} + s^2t^2\phi^3\phi_t^2\phi_{ss} + s^2t^3\phi^3\phi_t\phi_{st}^2$$
$$-s^2t^2u\phi^3\phi_{ss}^2\phi_{tt} + s^2t^2u\phi^3\phi_{ss}\phi_{st}^2 + s^2t^2u\phi^2\phi_s^2\phi_{st}^2 - s^2t^2u\phi^2\phi_t^2\phi_{ss}^2$$
$$-s^2t^2u\phi\,\phi_s^4\phi_{tt} + st^4\phi^3\phi_s\phi_{tt}^2 + s^2t^4\phi\,\phi_s^2\phi_t^2\phi_{tt} + s^2t^4\phi\,\phi_t^4\phi_{ss} - s^2t^4\phi\,\phi_t^4\phi_{st}$$
$$+s^4u\phi^3\phi_{ss}^2\phi_{st} + s^4u\phi^2\phi_s\phi_t\phi_{ss}^2 + s^4u\phi\,\phi_s^4\phi_{st} + s^3t^2\phi\,\phi_s^3\phi_t^2 - s^4t^2\phi^2\phi_s^2\phi_{st}^2$$
$$+s^4t^2\phi^2\phi_t^2\phi_{ss}^2 + s^4t^2\phi\,\phi_s^4\phi_{tt} + s^4t^2\phi\,\phi_s^2\phi_t^2\phi_{ss} + s^2t^4\phi^3\phi_{ss}\phi_{tt}^2 - s^2t^4\phi^3\phi_{st}^2\phi_{tt}$$
$$-s^2t^4\phi^3\phi_{st}\phi_{tt}^2 + s^2t^4\phi^2\phi_s^2\phi_{tt}^2 - s^2t^4\phi^2\phi_s\phi_t\phi_{tt}^2 - s^2t^4\phi^2\phi_t^2\phi_{st}^2 - s^6\phi^2\phi_s\phi_t\phi_{ss}^2$$
$$+s^4t^2\phi^3\phi_{ss}^2\phi_{tt} - s^4t^2\phi^3\phi_{ss}\phi_{st}^2 - t^4\phi^4\phi_{tt}^2 - s^2u\phi^4\phi_{ss}^2 - s^2u\phi^2\phi_s^4 - t^2u\phi^4\phi_{st}^2$$
$$+t^5\phi^3\phi_t\phi_{tt}^2 + t^5\phi\,\phi_t^5 + t^4u\phi_s\phi_t^5 + s^4\phi^4\phi_{ss}^2 - s^4\phi^3\phi_s^2\phi_{st} + s^4\phi^2\phi_s^4 - s^4\phi^2\phi_s^3\phi_t$$
$$+s^4u\phi_s^5\phi_t - s^2t^4\phi_s\phi_t^5 - s^6\phi^3\phi_{ss}^2\phi_{st} - s^6\phi\,\phi_s^4\phi_{st} - s^6\phi_s^5\phi_t - 6\, st^2u\phi\,\phi_s^2\phi_t^3$$
$$-2\, t^3u\phi^3\phi_s\phi_{st}\phi_{tt} + 2\, t^3u\phi^3\phi_t\phi_{ss}\phi_{tt} - 2\, t^3u\phi^3\phi_t\phi_{st}\phi_{tt} - 2\, t^3u\phi^2\phi_s\phi_t^2\phi_{st}$$
$$-2\, t^3u\phi^2\phi_s\phi_t^2\phi_{tt} - 2\, stu\phi^4\phi_{ss}\phi_{st} - 2\, stu\phi^2\phi_s^3\phi_t + 2\, stu\phi^2\phi_s^2\phi_t^2$$
$$-2\, t^2u\phi^3\phi_s\phi_t\phi_{st} - 2\, s^4t\phi^2\phi_s^2\phi_t\phi_{ss} + 10\, s^4t\phi^2\phi_s^2\phi_t\phi_{st} + 2\, s^4t\phi^2\phi_s\phi_t^2\phi_{ss}$$
$$+2\, s^4u\phi^2\phi_s^2\phi_{ss}\phi_{st} + 2\, s^4u\phi\,\phi_s^3\phi_t\phi_{ss} + 2\, s^3t^2\phi^3\phi_s\phi_{st}\phi_{tt} - 4\, s^3t^2\phi^3\phi_t\phi_{ss}\phi_{st}$$
$$+6\, s^3t^2\phi^2\phi_s^2\phi_t\phi_{st} + 2\, s^3t^2\phi^2\phi_s^2\phi_t\phi_{tt} - 4\, s^3t^2\phi^2\phi_s\phi_t^2\phi_{ss} + 10\, s^3t^2\phi^2\phi_s\phi_t^2\phi_{st}$$
$$+4\, s^3tu\phi^3\phi_{ss}\phi_{st}^2 + 4\, s^3tu\phi^2\phi_s^2\phi_{st}^2 + 4\, s^2t^3\phi^3\phi_s\phi_{st}\phi_{tt} - 2\, s^2t^3\phi^3\phi_t\phi_{ss}\phi_{tt}$$
$$+2\, s^2t^3\phi^3\phi_t\phi_{st}\phi_{tt} + 2\, s^2t^3\phi^2\phi_s^2\phi_t\phi_{tt} + 6\, s^2t^3\phi^2\phi_s\phi_t^2\phi_{st} + 2\, s^2t^3\phi^2\phi_s\phi_t^2\phi_{tt}$$
$$+2\, st^4\phi^3\phi_t\phi_{st}\phi_{tt} + 4\, st^4\phi^2\phi_s\phi_t^2\phi_{tt} + 4\, st^3u\phi^3\phi_{st}^2\phi_{tt} + 4\, st^3u\phi^2\phi_t^2\phi_{st}^2$$
$$-2\, t^4u\phi^2\phi_t^2\phi_{ss}\phi_{tt} + 2\, t^4u\phi^2\phi_t^2\phi_{st}\phi_{tt} + 2\, t^4u\phi\,\phi_s\phi_t^3\phi_{st} + 2\, t^4u\phi\,\phi_s\phi_t^3\phi_{tt}$$
$$+4\, s^3t\phi^3\phi_s\phi_t\phi_{ss} - 2\, s^3t\phi^3\phi_s\phi_t\phi_{st} - 2\, s^3u\phi^3\phi_s\phi_{ss}\phi_{st} - 2\, s^3u\phi^2\phi_s^2\phi_t\phi_{ss}$$
$$-2\, s^2t^2\phi^3\phi_s\phi_t\phi_{st} - 4\, s^2tu\phi^3\phi_s\phi_{st}^2 + 2\, s^2tu\phi^3\phi_t\phi_{ss}^2 - 2\, s^2tu\phi^2\phi_s^3\phi_{st}$$

$$
\begin{aligned}
&-6\,s^2tu\phi\,{\phi_s}^3{\phi_t}^2-4\,st^3\phi^3\phi_s\phi_t\phi_{tt}-4\,st^2u\phi^3\phi_s{\phi_{st}}^2-4\,st^2u\phi^3\phi_t{\phi_{st}}^2-4\,s^5t{\phi_s}^4{\phi_t}^2\\
&-4\,s^4t^2\phi^3{\phi_{st}}^3-6\,s^4t^2{\phi_s}^3{\phi_t}^3-2\,s^3t^3\phi^3{\phi_{st}}^3-4\,s^3t^3{\phi_s}^2{\phi_t}^4+2\,s^5\phi^2{\phi_s}^3\phi_{st}\\
&+2\,s^5\phi\,{\phi_s}^4\phi_t+2\,t^5\phi^2{\phi_t}^3\phi_{tt}+2\,s^4\phi^3{\phi_s}^2\phi_{ss}-4\,t^4\phi^3{\phi_t}^2\phi_{tt}-2\,s^3\phi^4\phi_s\phi_{ss}\\
&+2\,t^3\phi^4\phi_t\phi_{tt}+2\,su\phi^3{\phi_s}^3+s^2\phi^4{\phi_s}^2-t^2\phi^4{\phi_t}^2-u\phi^4{\phi_s}^2-2\,s^3\phi^3{\phi_s}^3+3\,t^3\phi^3{\phi_t}^3\big)\\
&\setminus\big(\big(s^2{\phi_s}^2+s^2\phi\,\phi_{ss}+t^2{\phi_t}^2+t^2\phi\,\phi_{tt}+2\,ts\phi_s\phi_t+2\,ts\phi\,\phi_{st}-s\phi\,\phi_s-t\phi\,\phi_t\big)\\
&\big(-4\,stu\phi\,\phi_s\phi_t\phi_{st}-s^3\phi^2\phi_s\phi_{ss}-s^3\phi\,{\phi_s}^3-s^2u{\phi_s}^3\phi_t-t^3\phi^2\phi_t\phi_{tt}-t^3\phi\,{\phi_t}^3\\
&-t^2u\phi^2{\phi_{st}}^2-t^2u\phi_s{\phi_t}^3+s^2\phi^3\phi_{ss}+su\phi\,{\phi_s}^3+t^2\phi^3\phi_{tt}-s\phi^3\phi_s-t\phi^3\phi_t-u\phi^2{\phi_s}^2\\
&+2\,st\phi^3\phi_{st}+2\,t^2\phi^2{\phi_t}^2+2\,s^2\phi^2{\phi_s}^2-2\,s^2t\phi^2\phi_s\phi_{st}-3\,s^2t\phi\,{\phi_s}^2\phi_t-2\,st^2\phi^2\phi_t\phi_{st}\\
&-3\,st^2\phi\,\phi_s{\phi_t}^2-2\,stu\phi^2{\phi_{st}}^2-2\,stu{\phi_s}^2{\phi_t}^2+4\,st\phi^2\phi_s\phi_t+2\,tu\phi^2\phi_s\phi_{st}+tu\phi\,{\phi_s}^2\phi_t\\
&+tu\phi\,\phi_s{\phi_t}^2-2\,t^2u\phi\,\phi_s\phi_t\phi_{st}-s^2t\phi^2\phi_t\phi_{ss}-s^2u\phi^2\phi_{ss}\phi_{st}-s^2u\phi\,{\phi_s}^2\phi_{st}\\
&-s^2u\phi\,\phi_s\phi_t\phi_{ss}-st^2\phi^2\phi_s\phi_{tt}+t^2u\phi^2\phi_{ss}\phi_{tt}-t^2u\phi^2\phi_{st}\phi_{tt}+t^2u\phi\,{\phi_s}^2\phi_{tt}\\
&-t^2u\phi\,\phi_s\phi_t\phi_{tt}+t^2u\phi\,{\phi_t}^2\phi_{ss}-t^2u\phi\,{\phi_t}^2\phi_{st}+su\phi^2\phi_s\phi_{ss}+su\phi^2\phi_s\phi_{st}+su\phi\,{\phi_s}^2\phi_t\\
&-tu\phi^2\phi_t\phi_{ss}+tu\phi^2\phi_t\phi_{st}\big)\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
b_1=\;&\big(4\,stv\phi\,\phi_s\phi_t\phi_{st}+s^2t\phi^2\phi_s\phi_{st}+s^2t\phi\,{\phi_s}^2\phi_t+st^2\phi^2\phi_t\phi_{st}+st^2\phi\,\phi_s{\phi_t}^2-s^3t\phi^2\phi_{ss}\phi_{st}\\
&-s^3t\phi\,{\phi_s}^2\phi_{st}-s^3t\phi\,\phi_s\phi_t\phi_{ss}-2\,st^2\phi^2\phi_t\phi_{ss}+2\,stv\phi^2{\phi_{st}}^2+2\,stv{\phi_s}^2{\phi_t}^2-2\,sv\phi^2\phi_s\phi_{st}\\
&-2\,sv\phi\,{\phi_s}^2\phi_t-2\,tv\phi^2\phi_s\phi_{st}+2\,tv\phi^2\phi_t\phi_{ss}-2\,tv\phi^2\phi_t\phi_{st}-2\,tv\phi\,\phi_s{\phi_t}^2-2\,s^2t^2\phi^2{\phi_{st}}^2\\
&-2\,s^2t^2{\phi_s}^2{\phi_t}^2+2\,t^2\phi^2\phi_s\phi_t-st^3\phi_s{\phi_t}^3+s^2v{\phi_s}^3\phi_t-t^3\phi^2\phi_t\phi_{st}-t^3\phi\,\phi_s{\phi_t}^2+t^2v\phi^2{\phi_{st}}^2\\
&+t^2v\phi_s{\phi_t}^3+st\phi^3\phi_{ss}+st\phi^2{\phi_s}^2+t^2\phi^3\phi_{st}-t\phi^3\phi_s-v\phi^3\phi_{ss}+v\phi^3\phi_{st}+v\phi^2\phi_s\phi_t\\
&-s^3t{\phi_s}^3\phi_t-st^3\phi^2{\phi_{st}}^2-4\,s^2t^2\phi\,\phi_s\phi_t\phi_{st}-2\,st^3\phi\,\phi_s\phi_t\phi_{st}+2\,t^2v\phi\,\phi_s\phi_t\phi_{st}-t^2v\phi\,{\phi_t}^2\phi_{ss}\\
&+t^2v\phi\,{\phi_t}^2\phi_{st}+st^3\phi^2\phi_{ss}\phi_{tt}-st^3\phi^2\phi_{st}\phi_{tt}+st^3\phi\,{\phi_s}^2\phi_{tt}-st^3\phi\,\phi_s\phi_t\phi_{tt}+st^3\phi\,{\phi_t}^2\phi_{ss}\\
&-st^3\phi\,{\phi_t}^2\phi_{st}+s^2v\phi^2\phi_{ss}\phi_{st}+s^2v\phi\,{\phi_s}^2\phi_{st}+s^2v\phi\,\phi_s\phi_t\phi_{ss}+st^2\phi^2\phi_s\phi_{st}-st^2\phi\,{\phi_s}^2\phi_t\\
&-t^2v\phi^2\phi_{ss}\phi_{tt}+t^2v\phi^2\phi_{st}\phi_{tt}-t^2v\phi\,{\phi_s}^2\phi_{tt}+t^2v\phi\,\phi_s\phi_t\phi_{tt}\big)\setminus\big(-4\,stu\phi\,\phi_s\phi_t\phi_{st}\\
&-s^2u{\phi_s}^3\phi_t-t^3\phi^2\phi_t\phi_{tt}-t^3\phi\,{\phi_t}^3-t^2u\phi^2{\phi_{st}}^2-t^2u\phi_s{\phi_t}^3-s^2\phi^3\phi_{ss}+t^2\phi^3\phi_{tt}-s\phi^3\phi_s\\
&-3\,t\phi^3\phi_t+3\,t^2\phi^2{\phi_t}^2-2\,s^2t\phi^2\phi_s\phi_{st}-st^2\phi\,\phi_s{\phi_t}^2-2\,stu\phi^2{\phi_{st}}^2-2\,stu{\phi_s}^2{\phi_t}^2\\
&+2\,st\phi^2\phi_s\phi_t+2\,tu\phi^2\phi_s\phi_{st}+2\,tu\phi\,\phi_s{\phi_t}^2-2\,t^2u\phi\,\phi_s\phi_t\phi_{st}+2\,s^2t\phi^2\phi_t\phi_{ss}-s^2u\phi^2\phi_{ss}\phi_{st}\\
&-s^2u\phi\,{\phi_s}^2\phi_{st}-s^2u\phi\,\phi_s\phi_t\phi_{ss}-st^2\phi^2\phi_s\phi_{tt}+t^2u\phi^2\phi_{ss}\phi_{tt}-t^2u\phi^2\phi_{st}\phi_{tt}+t^2u\phi\,{\phi_s}^2\phi_{tt}\\
&-t^2u\phi\,\phi_s\phi_t\phi_{tt}+t^2u\phi\,{\phi_t}^2\phi_{ss}-t^2u\phi\,{\phi_t}^2\phi_{st}+2\,su\phi^2\phi_s\phi_{st}+2\,su\phi\,{\phi_s}^2\phi_t-2\,tu\phi^2\phi_t\phi_{ss}\\
&+2\,tu\phi^2\phi_t\phi_{st}+\phi^4+s^2t^2\phi_s{\phi_t}^3-s^3\phi^2\phi_s\phi_{st}-s^3\phi\,{\phi_s}^2\phi_t+u\phi^3\phi_{ss}-u\phi^3\phi_{st}-u\phi^2\phi_s\phi_t\\
&+s^2t^2\phi^2{\phi_{st}}^2+2\,s^3t\phi^2{\phi_{st}}^2+2\,s^3t{\phi_s}^2{\phi_t}^2+s^4\phi^2\phi_{ss}\phi_{st}+s^4\phi\,{\phi_s}^2\phi_{st}+s^4{\phi_s}^3\phi_t\\
&+2\,s^2t^2\phi\,\phi_s\phi_t\phi_{st}+4\,s^3t\phi\,\phi_s\phi_t\phi_{st}+s^4\phi\,\phi_s\phi_t\phi_{ss}-s^2t^2\phi^2\phi_{ss}\phi_{tt}+s^2t^2\phi^2\phi_{st}\phi_{tt}\\
&-s^2t^2\phi\,{\phi_s}^2\phi_{tt}+s^2t^2\phi\,\phi_s\phi_t\phi_{tt}-s^2t^2\phi\,{\phi_t}^2\phi_{ss}+s^2t^2\phi\,{\phi_t}^2\phi_{st}-s^2t\phi^2\phi_t\phi_{st}-s^2t\phi\,\phi_s{\phi_t}^2\big)\,,\\
b_2=-&\,\big(-2\,s^2t^2\phi^2\phi_s\phi_{st}-3\,s^2t^2\phi\,{\phi_s}^2\phi_t+2\,s^2tv\phi^2{\phi_{st}}^2+2\,s^2tv{\phi_s}^2{\phi_t}^2-2\,st^3\phi^2\phi_t\phi_{st}
\end{aligned}
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\begin{aligned}
&-3\,st^3\phi\,\phi_s{\phi_t}^2-2\,st^2u\phi^2{\phi_{st}}^2-2\,st^2u{\phi_s}^2{\phi_t}^2+4\,st^2\phi^2\phi_s\phi_t+2\,t^2u\phi^2\phi_s\phi_{st}\\
&-2\,tv\phi^2\phi_s\phi_t+2\,s^2t\phi^2{\phi_s}^2+2\,st^2\phi^3\phi_{st}-st\phi^3\phi_s-sv\phi^3\phi_{ss}-sv\phi^2{\phi_s}^2-t^2\phi^3\phi_t\\
&-tu\phi^2{\phi_s}^2-tv\phi^3\phi_{st}+v\phi^3\phi_s-t^4\phi^2\phi_t\phi_{tt}-t^4\phi\,{\phi_t}^3-t^3u\phi^2{\phi_{st}}^2-t^3u\phi_s{\phi_t}^3\\
&+s^2t\phi^3\phi_{ss}+t^3\phi^3\phi_{tt}-s^3t\phi\,{\phi_s}^3+s^3v{\phi_s}^3\phi_t-s^2tu\phi\,\phi_s\phi_t\phi_{ss}+st^2v\phi\,\phi_s\phi_t\phi_{tt}\\
&+4\,s^2tv\phi\,\phi_s\phi_t\phi_{st}-4\,st^2u\phi\,\phi_s\phi_t\phi_{st}+2\,st^2v\phi\,\phi_s\phi_t\phi_{st}+2\,t^3\phi^2{\phi_t}^2-2\,t^3u\phi\,\phi_s\phi_t\phi_{st}\\
&+2\,stv\phi^2\phi_t\phi_{ss}+stu\phi^2\phi_s\phi_{ss}+stu\phi^2\phi_s\phi_{st}+stu\phi\,{\phi_s}^3+stu\phi\,{\phi_s}^2\phi_t-stv\phi^2\phi_s\phi_{st}\\
&-stv\phi^2\phi_t\phi_{st}+stv\phi\,{\phi_s}^2\phi_t-stv\phi\,\phi_s{\phi_t}^2-t^2u\phi^2\phi_t\phi_{ss}+t^2u\phi^2\phi_t\phi_{st}+t^2u\phi\,{\phi_s}^2\phi_t\\
&+t^2u\phi\,\phi_s{\phi_t}^2+t^2v\phi^2\phi_t\phi_{st}+t^2v\phi\,\phi_s{\phi_t}^2-s^2t^2\phi^2\phi_t\phi_{ss}-s^2tu\phi^2\phi_{ss}\phi_{st}\\
&-s^2tu\phi\,{\phi_s}^2\phi_{st}-s^2tu{\phi_s}^3\phi_t-st^3\phi^2\phi_s\phi_{tt}-st^2v\phi^2\phi_{ss}\phi_{tt}+st^2v\phi^2{\phi_{st}}^2\\
&+st^2v\phi^2\phi_{st}\phi_{tt}-st^2v\phi\,{\phi_s}^2\phi_{tt}-st^2v\phi\,{\phi_t}^2\phi_{ss}+st^2v\phi\,{\phi_t}^2\phi_{st}+st^2v\phi_s{\phi_t}^3\\
&+t^3u\phi^2\phi_{ss}\phi_{tt}-t^3u\phi^2\phi_{st}\phi_{tt}+t^3u\phi\,{\phi_s}^2\phi_{tt}-t^3u\phi\,\phi_s\phi_t\phi_{tt}+t^3u\phi\,{\phi_t}^2\phi_{ss}\\
&-t^3u\phi\,{\phi_t}^2\phi_{st}-s^2v\phi^2\phi_s\phi_{st}-s^2v\phi\,{\phi_s}^2\phi_t-s^3t\phi^2\phi_s\phi_{ss}+s^3v\phi^2\phi_{ss}\phi_{st}\\
&+s^3v\phi\,{\phi_s}^2\phi_{st}+s^3v\phi\,\phi_s\phi_t\phi_{ss}\big)\setminus\big(-4\,stu\phi\,\phi_s\phi_t\phi_{st}-s^2u{\phi_s}^3\phi_t-t^3\phi^2\phi_t\phi_{tt}-t^3\phi\,{\phi_t}^3\\
&-t^2u\phi^2{\phi_{st}}^2-t^2u\phi_s{\phi_t}^3-s^2\phi^3\phi_{ss}+t^2\phi^3\phi_{tt}-s\phi^3\phi_s-3\,t\phi^3\phi_t+3\,t^2\phi^2{\phi_t}^2\\
&-2\,s^2t\phi^2\phi_s\phi_{st}-st^2\phi\,\phi_s{\phi_t}^2-2\,stu\phi^2{\phi_{st}}^2-2\,stu{\phi_s}^2{\phi_t}^2+2\,st\phi^2\phi_s\phi_t\\
&+2\,tu\phi^2\phi_s\phi_{st}+2\,tu\phi\,\phi_s{\phi_t}^2-2\,t^2u\phi\,\phi_s\phi_t\phi_{st}+2\,s^2t\phi^2\phi_t\phi_{ss}-s^2u\phi^2\phi_{ss}\phi_{st}\\
&-s^2u\phi\,{\phi_s}^2\phi_{st}-s^2u\phi\,\phi_s\phi_t\phi_{ss}-st^2\phi^2\phi_s\phi_{tt}+t^2u\phi^2\phi_{ss}\phi_{tt}-t^2u\phi^2\phi_{st}\phi_{tt}+t^2u\phi\,{\phi_s}^2\phi_{tt}\\
&-t^2u\phi\,\phi_s\phi_t\phi_{tt}+t^2u\phi\,{\phi_t}^2\phi_{ss}-t^2u\phi\,{\phi_t}^2\phi_{st}+2\,su\phi^2\phi_s\phi_{st}+2\,su\phi\,{\phi_s}^2\phi_t-2\,tu\phi^2\phi_t\phi_{ss}\\
&+2\,tu\phi^2\phi_t\phi_{st}+\phi^4+s^2t^2\phi_s{\phi_t}^3-s^3\phi^2\phi_s\phi_{st}-s^3\phi\,{\phi_s}^2\phi_t+u\phi^3\phi_{ss}-u\phi^3\phi_{st}\\
&-u\phi^2\phi_s\phi_t+s^2t^2\phi^2{\phi_{st}}^2+2\,s^3t\phi^2{\phi_{st}}^2+2\,s^3t{\phi_s}^2{\phi_t}^2+s^4\phi^2\phi_{ss}\phi_{st}+s^4\phi\,{\phi_s}^2\phi_{st}\\
&+s^4{\phi_s}^3\phi_t+2\,s^2t^2\phi\,\phi_s\phi_t\phi_{st}+4\,s^3t\phi\,\phi_s\phi_t\phi_{st}+s^4\phi\,\phi_s\phi_t\phi_{ss}-s^2t^2\phi^2\phi_{ss}\phi_{tt}\\
&+s^2t^2\phi^2\phi_{st}\phi_{tt}-s^2t^2\phi\,{\phi_s}^2\phi_{tt}+s^2t^2\phi\,\phi_s\phi_t\phi_{tt}-s^2t^2\phi\,{\phi_t}^2\phi_{ss}+s^2t^2\phi\,{\phi_t}^2\phi_{st}\\
&-s^2t\phi^2\phi_t\phi_{st}-s^2t\phi\,\phi_s{\phi_t}^2\big)\,.
\end{aligned}
$$

References

[1] J. Douglas. The general geometry of paths. Ann. of Math. (2), 29(1-4): 143–168, 1927/28.

[2] E. Guo, H. Liu, and X. Mo. On spherically symmetric Finsler metrics with isotropic Berwald curvature. Int. J. Geom. Methods Mod. Phys., 10(10):1350054, 13, 2013.

[3] L. Huang and X. Mo. Projectively flat Finsler metrics with orthogonal invariance. Ann. Polon. Math., 107(3):259–270, 2013.

[4] W. Liu and B. Li. Projectively flat Finsler metrics defined by the Euclidean metric and related 1-forms. Differential Geom. Appl., 46:14–24, 2016.

[5] X. Mo, N. M. Solórzano, and K. Tenenblat. On spherically symmetric Finsler metrics with vanishing Douglas curvature. Differential Geom. Appl., 31(6):746–758, 2013.

[6] X. Mo and L. Zhou. The curvature of spherically symmetric Finsler metrics in R^n. arXiv:1202.4543, 2012.

[7] E. Peyghan and A. Tayebi. Generalized Berwald metrics. Turkish J. Math., 36(3):475–484, 2012.

[8] Z. Shen. Projectively flat Finsler metrics of constant flag curvature. Trans. Amer. Math. Soc., 355(4):1713–1728, 2003.

[9] A. Tayebi and M. Barzegari. Generalized Berwald spaces with (α, β)-metrics. Indag. Math. (N.S.), 27(3):670–683, 2016.

[10] A. Tayebi and H. Sadeghi. On generalized Douglas-Weyl (α, β)-metrics. Acta Math. Sin. (Engl. Ser.), 31(10):1611–1620, 2015.

[11] A. Tayebi and M. Shahbazi Nia. A new class of projectively flat Finsler metrics with constant flag curvature $\mathbf{K} = 1$. Differential Geom. Appl., 41:123–133, 2015.

[12] A. Tayebi and T. Tabatabaeifar. Douglas-Randers manifolds with vanishing stretch tensor. Publ. Math. Debrecen, 86(3-4):423–432, 2015.

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