ON THE CONVERGENCE OF CESÀRO MEANS OF NEGATIVE ORDER OF WALSH-FOURIER SERIES

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ABSTRACT. In this paper we investigate the convergence of Cesàro means of negative order of Walsh-Fourier series of functions of generalized bounded oscillation.

Let \( r_0(x) \) be a function defined on \( R := (-\infty, \infty) \) by

\[
  r_0(x) = \begin{cases} 
  1, & \text{if } x \in \left[0, \frac{1}{2}\right) \\
  -1, & \text{if } x \in \left[\frac{1}{2}, 1\right) 
  \end{cases},
\]

\[ r_0(x+1) = r_0(x). \]

The Rademacher system is defined by

\[
  r_n(x) = r_0(2^n x), \quad n \geq 1 \text{ and } x \in [0, 1). 
\]

Let \( w_0, w_1, \ldots \) represent the Walsh functions, i.e., \( w_0(x) = 1 \) and if \( k = 2^{n_1} + \ldots + 2^{n_s} \) is a positive integer with \( n_1 > n_2 > \ldots > n_s \) then \( w_k(x) = r_{n_1}(x) \times \cdots \times r_{n_s}(x) \).

The idea of using products of Rademacher’s functions to define the Walsh system originated from Paley [16].

The Walsh-Dirichlet kernel is defined by

\[
  D_n(x) = \sum_{k=0}^{n-1} w_k(x). 
\]

Recall that

\[
  D_{2^n}(x) = \begin{cases} 
  2^n, & \text{if } x \in \left[0, \frac{1}{2^n}\right) \\
  0, & \text{if } x \in \left[\frac{1}{2^n}, 1\right) 
  \end{cases}. 
\]

Suppose that \( f \) is a Lebesgue integrable function on \([0,1]\) and 1-periodic. Then its Walsh-Fourier series is defined by

\[
  \sum_{k=0}^{\infty} \hat{f}(k) w_k(x),
\]

\( 2020 \) Mathematics Subject Classification. 42C10.

Key words and phrases. Walsh-Fourier series, Cesàro means, generalized bounded variation.
where
\[ \hat{f}(k) = \int_0^1 f(t) w_k(t) \, dt \]
is called the \( k \)-th Walsh-Fourier coefficient of the function \( f \). Denote by \( S_n(f, x) \) the \( n \)-th partial sum of the Walsh-Fourier series of the function \( f \), namely
\[ S_n(f, x) = \sum_{k=0}^{n-1} \hat{f}(k) w_k(x). \]

The Cesàro \((C, \alpha)\)-means of the Walsh-Fourier series are defined as
\[ \sigma_n^\alpha(f, x) = \frac{1}{A_\alpha^n} \sum_{k=0}^{n} A_\alpha^{n-k} \hat{f}(k) w_k(x), \]
where
\[ A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha + 1) \cdots (\alpha + n)}{n!}, \quad \alpha \neq -1, -2, \ldots. \]

Let \( C([0,1]) \) denote the space of continuous functions \( f \) with period 1. If \( f \in C([0,1]) \), then the function
\[ w(\delta, f) = \sup \{|f(x') - f(x'')| : |x' - x''| \leq \delta, \ x', x'' \in [0,1]| \}
\]is called the modulus of continuity of the function \( f \). The modulus of continuity of an arbitrary function \( f \in C([0,1]) \) has the following properties:

1) \( \omega(0) = 0 \),
2) \( \omega(\delta) \) is nondecreasing,
3) \( \omega(\delta) \) is continuous on \([0,1]\),
4) \( \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2) \) for \( 0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 1. \)

An arbitrary function \( \omega(\delta) \) which is defined on \([0,1]\) and has properties 1) – 4) is called a modulus of continuity. If the modulus of continuity \( \omega(\delta) \) is given, then \( H_\omega \) denotes the class of functions \( f \in C([0,1]) \) for which
\[ \omega(\delta, f) = O(\omega(\delta)) \quad \text{as} \quad \delta \to 0. \]

\( C_w([0,1]) \) is the collection of functions \( f : [0,1] \to R \) that are uniformly continuous from the dyadic topology of \([0,1]\) to the usual topology of \( R \), or for short: uniformly \( W \)-continuous.

Let \( f \) be defined on \([0,1]\). We shall represent the dyadic modulus of continuity by
\[ \hat{\omega}(\delta, f) = \sup_{0 \leq h \leq \delta} \sup_x |f(x \oplus h) - f(x)|, \]
where \( \oplus \) denotes dyadic addition (see [12] or [18]).

The problems of summability of Cesàro means of the Walsh-Fourier series were studied in [4], [7], [10], [9], [8], [16], [18], [17].

Tevzadze [19] has studied the uniform convergence of Cesàro means of negative order of the Walsh-Fourier series. In particular, in terms of modulus of
continuity and variation of function \( f \in C_w ([0, 1]) \) he has proved the criterion for the uniform summability by the Cesàro method of negative order of Fourier series with respect to the Walsh system.

In [9] Goginava investigated the problem of estimating the deviation of \( f \in L_p \) from its Cesàro means of negative order in the \( L_p \)-metric, \( p \in [1, \infty) \). Analogous results for Walsh-Kaczmarz system were proved by Nagy [15] and Gát, Nagy [6].

In his monograph [23, part 1, chapter 4] Zhizhiashvili investigated the behavior of Cesàro means of negative order of trigonometric Fourier series in detail.

The notion of a function-bounded variation was introduced by Jordan [13]. Generalizing this notion Wiener [21] considered the class of function \( V_p \). Young [22] introduced the notion of the function of bounded \( \Phi \)-variation. Waterman [20] studied the class of function of bounded \( \Lambda \)-variation, and Chanturia [3] defined the notion of the modulus of variation of a function. In 1990, Kita and Yoneda [14] introduced the notion of the generalized Wiener’s class \( BV (p (n) \uparrow p) \).

Generalizing the class \( BV (p (n) \uparrow p) \), Akhobadze [1, 2] considered the classes of function \( BV (p (n) \uparrow p, \phi) \) and \( B \Lambda (p (n) \uparrow p, \phi) \).

Definition 1. [11] Let \( 1 \leq p (n) \uparrow p \) as \( n \to \infty \) where \( 1 \leq p \leq \infty \). We say that a function belongs to the \( BO (p (n) \uparrow p) \) class if

\[
O (f; p(n) \uparrow p) := \sup_n \left\{ 2^{n-1} \sup_{l=0}^{2^n-1} \sup_{t,u \in [2^{-n},(l+1)2^{-n})} |f(t) - f(u)|^{p(n)} \right\}^{1/p(n)} < \infty.
\]

When \( p(n) = p \) for all \( n \), \( BO (p (n) \uparrow p) \) coincides with the class of \( p \)-bounded fluctuation \( BF_p \) [18].

Estimates of the Fourier coefficients of functions of bounded fluctuation with respect to the Vilenkin system were studied by Gát and Toledo [5].

In [11] Goginava proved that the following statements are true.

**Theorem 1.** Let \( f \) be a function in the class \( BO (p (n) \uparrow \infty) \) and

\[
\dot{\omega} \left( \frac{1}{2^n}, f \right) = o \left( \frac{1}{p(n+1) \log_2 p(n+1)} \right) \text{ as } n \to \infty.
\]

Then the Walsh-Fourier series of the function \( f \) converges uniformly in \([0, 1]\).

**Theorem 2.** Let \( p(2n) \leq cp(n), n \in P \) and \( p(n) \log_2 p(n) = o(n) \) as \( n \to \infty \). If \( \omega \) satisfies the condition

\[
\lim_{n \to \infty} \sup \omega \left( \frac{1}{n}, p([\log_2 n]) \log_2 p([\log_2 n]) \right) = c_0 > 0,
\]

then there exists a function in the class \( H^\omega \cap BO (p (n) \uparrow \infty) \) for which the Walsh-Fourier series diverges at some point.

The theorem of Tevzadze [19] implies that if \( p < \frac{1}{\alpha} \) and \( f \in BF_p \cap C_\omega \), then the Cesàro mean \( \sigma_n^{-\alpha} (f) \) of Walsh-Fourier series uniformly converges to the
function $f$. On the other hand, for $p = \frac{1}{\alpha}$ there exists a continuous function $f$ for which $\sigma_n^{-\alpha}(f, 0)$ diverges. On the basis of the above facts the following problems arise naturally:

Let $f \in BO\left(p(n) \uparrow \frac{1}{\alpha}\right)$, $0 < \alpha < 1$. Under what condition on the sequence \{p(n) : n \geq 1\} the uniform convergence of Cesàro $(C, -\alpha)$ means of Walsh-Fourier series of the function $f$ holds?

The following theorem is true.

**Theorem 3.** Let $f \in C_w([0, 1]) \cap BO\left(p(n) \uparrow \frac{1}{\alpha}\right)$, $0 < \alpha < 1$, $2^k \leq n \leq 2^{k+1}$. Then

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_c \leq c(\alpha) \left\{ \sum_{r=0}^{k} 2^{r-k} \omega\left(\frac{1}{2^r}, f\right) + \frac{(\omega\left(\frac{1}{2^r}, f\right))^{1-\alpha p(k)}}{1 - \alpha p(k)} \right\}.$$  

**Corollary 1.** Let $f \in C_w([0, 1]) \cap BO\left(p(n) \uparrow \frac{1}{\alpha}\right)$, $0 < \alpha < 1$ and

$$\frac{(\omega\left(\frac{1}{2^r}, f\right))^{1-\alpha p(k)}}{1 - \alpha p(k)} \to 0 \text{ as } k \to \infty.$$

Then

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_c \to 0.$$

In order to prove Theorem 3 we need the following lemmas proved by Goginava in [9, 8].

**Lemma 1** (Goginava [9]). Let $f \in C_w([0, 1])$. Then for every $\alpha \in (0, 1)$ the following estimation holds

$$\frac{1}{A_n^{-\alpha}} \left\| \int_0^{1/2^{k-1}} \sum_{\nu=0}^{2^{k-1}-1} A_{n-k}^{-\alpha} w_\nu(u) [f (\cdot \oplus u) - f (\cdot)] du \right\|_c \leq c(\alpha) \sum_{r=0}^{k-1} 2^{r-k} \omega\left(1/2^r, f\right)_p,$$

where $2^k \leq n < 2^{k+1}$.

**Lemma 2** (Goginava [8]). Let $f \in C_w([0, 1])$ and $2^k \leq n < 2^{k+1}$. Then for every $\alpha \in (0, 1)$ the following estimations hold

$$\frac{1}{A_n^{-\alpha}} \left\| \int_0^{1/2^{k-1}} \sum_{\nu=0}^{2^{k-1}-1} A_{n-k}^{-\alpha} w_\nu(u) [f (\cdot \oplus u) - f (\cdot)] du \right\|$$

$$\leq c(\alpha) \left( \sum_{j=1}^{2^{k-1}-1} \frac{1}{j^{1-\alpha}} \left| f \left( x \oplus \frac{2j}{2^k} \right) - f \left( x \oplus \frac{2j + 1}{2^k} \right) \right| \right),$$

$$\frac{1}{A_n^{-\alpha}} \left\| \int_0^{1/2^k} \sum_{\nu=2^k}^{n} A_{n-k}^{-\alpha} w_\nu(u) [f (\cdot \oplus u) - f (\cdot)] du \right\|$$

$$\leq c(\alpha) \left( \sum_{j=1}^{2^k} \frac{1}{j^{1-\alpha}} \left| f \left( x \oplus \frac{2j}{2^{k+1}} \right) - f \left( x \oplus \frac{2j + 1}{2^{k+1}} \right) \right| \right).$$
Proof of Theorem 3. We can write

\[
\sigma_{n}^{-\alpha}(f, x) - f(x) = \frac{1}{A_{n}^{-\alpha}} \int_{0}^{1} \sum_{\nu=0}^{n} A_{n-\nu}^{-\alpha} w_{\nu}(x) [f(x + u) - f(x)] du
\]

\[
= \frac{1}{A_{n}^{-\alpha}} \int_{0}^{1} \sum_{\nu=0}^{2k-1} A_{n-\nu}^{-\alpha} w_{\nu}(x) [f(x + u) - f(x)] du
\]

\[
+ \frac{1}{A_{n}^{-\alpha}} \int_{0}^{1} \sum_{\nu=2k-1}^{2k-1} A_{n-\nu}^{-\alpha} w_{\nu}(x) [f(x + u) - f(x)] du
\]

\[
+ \frac{1}{A_{n}^{-\alpha}} \int_{0}^{1} \sum_{\nu=2k}^{n} A_{n-\nu}^{-\alpha} w_{\nu}(x) [f(x + u) - f(x)] du
\]

\[= I + II + III. \tag{1}\]

From Lemmas 1 and 2 we have

\[
\|I\| \leq c(\alpha) \sum_{\nu=0}^{k-1} 2^{r-k} \omega \left( \frac{1}{2^{r}}, f \right), \tag{2}\]

\[
|II| \leq c(\alpha) \left( \sum_{j=1}^{2k-1} \frac{1}{j^{1-\alpha}} \left| f \left( x + \frac{2j}{2k} \right) - f \left( x + \frac{2j+1}{2k} \right) \right| \right)
\]

and

\[
|III| \leq c(\alpha) \left( \sum_{j=1}^{2k-1} \frac{1}{j^{1-\alpha}} \left| f \left( x + \frac{2j}{2k+1} \right) - f \left( x + \frac{2j+1}{2k+1} \right) \right| \right).
\]

Using Abel’s transformation, we get

\[
|III| \leq c(\alpha) \left( \sum_{j=1}^{2k-2} \left( \frac{1}{j^{1-\alpha}} - \frac{1}{(j+1)^{1-\alpha}} \right) \right.
\]

\[
\times \sum_{l=1}^{j} \left| f \left( x + \frac{2l}{2k+1} \right) - f \left( x + \frac{2l+1}{2k+1} \right) \right| 
\]

\[
+ \frac{1}{(2k-1)^{1-\alpha}} \sum_{j=1}^{2k-1} \left| f \left( x + \frac{2j}{2k+1} \right) - f \left( x + \frac{2j+1}{2k+1} \right) \right| 
\]

\[= III_{1} + III_{2}. \tag{3}\]
Let $\varepsilon_k := \alpha p_k < 1$, $s_k := \frac{p(k)}{\varepsilon_k}$, $\frac{1}{s_k} + \frac{1}{t_k} = 1$. Then using Hölder’s inequality for $III_2$ we can write

$$III_2 = \frac{1}{(2^k - 1)^{1-\alpha}} \sum_{j=1}^{2^k-1} \left| f \left( x \oplus \frac{2j}{2^{k+1}} \right) - f \left( x \oplus \frac{2j+1}{2^{k+1}} \right) \left( 1-\varepsilon_k \right)^{\frac{1}{k}} \right|$$

$$\leq \frac{c(\alpha)}{2k(1-\alpha)} \left( \sum_{j=1}^{2^k-1} \left| f \left( x \oplus \frac{2j}{2^{k+1}} \right) - f \left( x \oplus \frac{2j+1}{2^{k+1}} \right) \right| \left( \frac{1}{s_k} \right)^{1-\varepsilon_k} \right)$$

$$\leq \frac{c(\alpha)}{2k(1-\alpha)} \left( BO \left( f, p(k) \uparrow \frac{1}{\alpha} \right) \left( \frac{1}{s_k} \right)^{1-\varepsilon_k} \right)$$

$$\leq c(\alpha) \left( BO \left( f, p(k) \uparrow \frac{1}{\alpha} \right) \right)^{\varepsilon_k} \left( \frac{1}{s_k} \right)^{1-\varepsilon_k} \left( \frac{1}{2^k} \right)^{k(\alpha - \frac{1}{\alpha})}$$

$$\leq c(\alpha) \left( BO \left( f, p(k) \uparrow \frac{1}{\alpha} \right) \right)^{\varepsilon_k} \left( \frac{1}{s_k} \right)^{1-\varepsilon_k} \left( \frac{1}{2^k} \right)^{k(\alpha - \frac{1}{\alpha})}$$

as $k \to \infty$.

Fix $m_0(k)$ and define it later

$$III_1 \leq c(\alpha) \sum_{j=1}^{m_0(k)} \frac{1}{j^{2-\alpha}} \sum_{l=1}^{j} \left| f \left( x \oplus \frac{2l}{2^{k+1}} \right) - f \left( x \oplus \frac{2l+1}{2^{k+1}} \right) \right|$$

$$+ \sum_{j=m_0(k)+1}^{2^k-1} \frac{1}{j^{2-\alpha}} \sum_{l=1}^{j} \left| f \left( x \oplus \frac{2l}{2^{k+1}} \right) - f \left( x \oplus \frac{2l+1}{2^{k+1}} \right) \right|$$

$$\leq c(\alpha) \left\{ \sum_{j=1}^{m_0(k)} \frac{1}{j^{2-\alpha}} \left[ \frac{1}{2^k}, f \right] \right\}$$

$$+ \sum_{j=m_0(k)+1}^{2^k-1} \frac{1}{j^{1+1/p(k)-\alpha}} \sum_{l=1}^{j} \left| f \left( x \oplus \frac{2l}{2^{k+1}} \right) - f \left( x \oplus \frac{2l+1}{2^{k+1}} \right) \right| \left( \frac{1}{p(k)} \right)^{\frac{1}{p(k)}}$$

$$\leq c(\alpha) \left\{ \left( m_0(k) \right)^{\alpha} \left( \frac{1}{2^k}, f \right) + \frac{m_0(k)^{\alpha-\frac{1}{p(k)}}}{p(k) - \alpha} BO \left( f, p(k) \uparrow \frac{1}{\alpha} \right) \right\}.$$
Then we have

\[ III_1 \leq c(\alpha) \left\{ \omega \left( \frac{1}{2^k}, f \right)^{1-\alpha p(k)} + \frac{\omega \left( \frac{1}{2^k}, f \right)^{1-\alpha p(k)}}{1 - \alpha p(k)} \right\} \leq c(\alpha) \frac{\omega \left( \frac{1}{2^k}, f \right)^{1-\alpha p(k)}}{1 - \alpha p(k)}. \]

Combining (3) – (4) we have

\[ |III| \leq c(\alpha) \frac{\omega \left( \frac{1}{2^k}, f \right)^{1-\alpha p(k)}}{1 - \alpha p(k)}. \]

Analogously we can prove that

\[ |II| \leq c(\alpha) \frac{\omega \left( \frac{1}{2^k}, f \right)^{1-\alpha p(k)}}{1 - \alpha p(k)}. \]

Combining (1), (2), (5) and (6) we complete the proof of Theorem 3. \(\square\)

**References**


Received August 18, 2014.

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