HYPERSURFACES OF A RIEMANNIAN MANIFOLD WITH A RICCI-QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. In this paper we study hypersurfaces of a Riemannian manifold endowed with a Ricci-quarter symmetric metric connection. We prove that the induced connection is also a Ricci-quarter symmetric metric connection. We consider the total geodesicness, the total umbilicity and the minimality of a hypersurface of a Riemannian manifold endowed with the Ricci-quarter symmetric metric connection. We obtain the Gauss, Weingarten and Codazzi equations with respect to the Ricci-quarter symmetric metric connection. The relation between the sectional curvatures of $M^n$ and $M^{(n+1)}$ with respect to the Ricci-quarter symmetric metric connection has been also given.

1. Introduction

In 1975, Golab [5] introduced the notion of a quarter-symmetric linear connection in a differentiable manifold. Later Misra and Pandey [7] considered a quarter symmetric F-connection and studied some of its properties. They considered especially the case of Kaehlerian structure and introduced the notion of a Ricci-quarter symmetric metric connection. Kamilya and De [6] studied some properties of a Ricci-quarter symmetric metric connection. In [4], Quasi Einstein manifolds admitting a Ricci-quarter symmetric metric connection were considered. In 1982, Yano and Imai [10] studied some curvature conditions for quarter symmetric metric connections in Riemannian, Hermitian and Kaehlerian manifolds. In [3], De and Mondal considered hypersurfaces of Kenmotsu manifolds with a quarter symmetric non-metric connection. In [1], Ahmad, Jun and Haseeb investigated some properties of invariant hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection.

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In the present paper, we have studied hypersurfaces of a Riemannian manifold endowed with a Ricci-quarter symmetric metric connection. The paper is organized as follows: In Section 2, we have given some properties of the Ricci-quarter symmetric metric connection; in Section 3, some necessary information about a hypersurface of a Riemannian manifold endowed with the Ricci-quarter symmetric metric connection has been given and we have proved that the induced connection is also a Ricci-quarter symmetric metric connection. We have also considered the total geodesicness, the total umbilicity and the minimality of a hypersurface of a Riemannian manifold endowed with the Ricci-quarter symmetric metric connection. In Section 4, we have obtained the Gauss, Weingarten, and Codazzi equations with respect to the Ricci-quarter symmetric metric connection. The relation between the sectional curvatures of $M^n$ and $M^{(n+1)}$ with respect to the Ricci-quarter symmetric metric connection has been also found.

2. Preliminaries

Let $M$ be an $(n+1)$ dimensional Riemannian manifold with a Riemannian metric $g$, and let $\nabla$ be a linear connection on $M$. The linear connection $\nabla$ in Riemannian manifold $M$ is said to be a quarter symmetric connection if its torsion tensor $T$ satisfies \[ (1) \quad T(X,Y) = w(Y)LX - w(X)LY, \]
where $w$ is a 1-form associated with a non-zero vector field $\rho$ by $w(X) = g(X, \rho)$ and $L$ is a tensor field of type $(1, 1)$.

A linear connection $\nabla$ is called a metric connection if \[ \nabla g = 0. \]

In (1), if a tensor field $L$ is a $(1, 1)$-Ricci tensor of a Riemannian manifold $M$, then the linear connection $\nabla$ of a Riemannian manifold $M$ is called a Ricci-quarter symmetric connection. Such a connection together with the metric condition is said to be a Ricci-quarter symmetric metric connection \[ (2) \quad \nabla_X Y = \nabla^*_X Y + w(Y)LX - S^*(X,Y)\rho, \]
where $L$ is a Ricci tensor of type $(1, 1)$ defined by $S^*(X,Y) = g(LX,Y)$, where $S^*$ is the Ricci tensor of $M$. 
We denote the curvature tensor of $M$ with respect to the Ricci-quarter symmetric metric connection $\nabla$ by $R$. So we have

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

$$= R^*(X,Y)Z - M(Y,Z)LX + M(X,Z)LY$$

$$- S^*(Y,Z)QX + S^*(X,Z)QY + \pi(Z)[(\nabla_X L)(Y) - (\nabla_Y L)(X)]$$

$$- [(\nabla_X S^*) (Y, Z) - (\nabla_Y S^*)(X, Z)] \rho,$$

where

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]}Z$$

is the curvature tensor of the manifold with respect to the Levi-Civita connection $\nabla^*$ and $M$ is a tensor of type $(0,2)$ defined by

$$M(X,Y) = g(QX,Y) = (\nabla_X w)(Y) - w(Y)w(LX) + \frac{1}{2} w(\rho)S^*(X,Y),$$

and $Q$ is a tensor field of type $(2,1)$ defined by [7]

$$QX = \nabla_X \rho - w(LX)\rho + \frac{1}{2} w(\rho)LX.$$

3. Hypersurfaces

Let $\overline{M}$ be an $n$ dimensional hypersurface immersed in $M$ by the immersion $i : \overline{M} \to M$. If $B$ denotes the derivative of $i$, then any vector field $\overline{X} \in T(\overline{M})$ implies $B\overline{X} \in T(M)$. We denote the objects belonging to $\overline{M}$ by $\overline{X}$, $\overline{L}$, etc.

Let $N$ be an oriented unit normal vector field on $\overline{M}$. Then the induced $\overline{g}$ on $\overline{M}$ is $\overline{g}(\overline{X}, \overline{Y}) = g(\overline{X}, \overline{Y})$. Then we have [2]

$$g(\overline{X}, N) = 0 \quad \text{and} \quad g(N, N) = 1.$$

Let $\overline{\nabla}^*$ be the induced connection on a hypersurface from $\nabla^*$ with respect to the unit normal $N$, then the Gauss equation is given by

$$\nabla^*_X \overline{Y} = \overline{\nabla}_X \overline{Y} + h(\overline{X}, \overline{Y})N,$$

where $h$ is the second fundamental tensor

$$h(\overline{X}, \overline{Y}) = h(\overline{Y}, \overline{X}) = \overline{g}(H\overline{X}, \overline{Y}),$$

and $H$ is a tensor field of type $(1,1)$ of $\overline{M}$.

If $\overline{\nabla}$ is the induced connection on the hypersurface from the Ricci-quarter symmetric metric connection $\nabla$ with respect to the unit normal $N$, then we have

$$\nabla_X \overline{Y} = \overline{\nabla}_X \overline{Y} + m(\overline{X}, \overline{Y})N.$$

Now every vector field $X$ on $M$ is decomposed as

$$X = \overline{X} + l(X)N,$$
where \( l \) is a 1-form on \( M \). For any tangent vector field \( \overline{X} \) on \( \overline{M} \) and normal \( N \) we have

\[
\begin{align*}
LN &= \overline{N} + KN, \\
L\overline{X} &= \overline{L} \overline{X} + b(\overline{X})N, \\
\rho &= \overline{\rho} + \lambda N,
\end{align*}
\]

where \( \overline{L} \) is a Ricci tensor field of type \((1,1)\) on the hypersurface \( \overline{M} \), \( b \) is a 1-form, \( K \) and \( \lambda \) are scalar functions on \( \overline{M} \).

Using (2), (6), and (7), we have

\[
\nabla_X Y = \nabla^*_{\overline{X}} Y + w(Y) \{ \overline{L} X + b(\overline{X})N \} - S^*(\overline{X}, Y) \{ \overline{\rho} + \lambda N \},
\]

where \( w(Y) = \overline{w}(Y) \). Using (3) and (4) in (8) yields

\[
\begin{align*}
\nabla_X Y + m(X,Y)N &= \nabla^*_{\overline{X}} Y + h(\overline{X},Y)N + w(Y) \{ \overline{L} X + b(\overline{X})N \} \\
&\quad - S^*(\overline{X}, Y) \{ \overline{\rho} + \lambda N \}.
\end{align*}
\]

Now taking tangential and normal parts from both sides, we have

\[
\begin{align*}
\overline{\nabla}_X \overline{Y} &= \overline{\nabla}^*_{\overline{X}} \overline{Y} + w(\overline{Y}) \overline{L} \overline{X} - S^*(\overline{X}, \overline{Y}) \overline{\rho} \\
\text{and} \\
m(\overline{X}, \overline{Y}) &= h(\overline{X}, \overline{Y}) + w(\overline{Y}) b(\overline{X}) - \lambda S^*(\overline{X}, \overline{Y}).
\end{align*}
\]

From (9), it follows that

\[
\overline{T}(\overline{X}, \overline{Y}) = w(\overline{Y}) \overline{L} \overline{X} - w(\overline{X}) \overline{L} \overline{Y},
\]

and also using (4), we have

\[
(\overline{\nabla}_{\overline{X}} g)(\overline{Y}, \overline{Z}) = (\overline{\nabla}_{\overline{Y}} g)(\overline{Y}, \overline{Z}).
\]

Thus we get the following.

**Theorem 1.** The connection induced on a hypersurface of a Riemannian manifold with a Ricci-quarter symmetric metric connection is also a Ricci-quarter symmetric metric connection.

### 3.1. Totally geodesic and totally umbilic hypersurfaces

Let \( \{ \overline{E}_1, \ldots, \overline{E}_n \} \) be \( n \) orthonormal vector fields in \( \overline{M} \). Then the function

\[
\frac{1}{n} \sum_{i=1}^{n} h(\overline{E}_i, \overline{E}_i)
\]

is the mean curvature of \( \overline{M} \) with respect to the Levi-Civita connection \( \overline{\nabla}^* \) and

\[
\frac{1}{n} \sum_{i=1}^{n} m(\overline{E}_i, \overline{E}_i)
\]

is called the mean curvature of \( \overline{M} \) with respect to the Ricci-quarter symmetric metric connection \( \overline{\nabla} \).

From this we have the following definitions.
Definition 1. If \( h \) vanishes, we call \( \overline{M} \) a totally geodesic hypersurface of \( M \) with respect to the Levi-Civita connection \( \nabla^* \).

Definition 2. The hypersurface \( \overline{M} \) is called totally umbilical with respect to the connection \( \nabla^* \) if \( h \) is proportional to the metric tensor \( g \).

If we replace \( h \) by \( m \) in the above definitions we get a totally geodesic hypersurface and a totally umbilical hypersurface with respect to the Ricci-quarter symmetric connection \( \nabla \).

Thus we get the following theorem.

Theorem 2. In order that the mean curvature of \( \overline{M} \) with respect to \( \nabla^* \) coincides with that of \( \overline{M} \) with respect to \( \nabla \), it is necessary and sufficient that \( \rho \) and \( L\overline{X} \) are tangent to \( M \). Hence \( \overline{M} \) is minimal with respect to the Levi-Civita connection if and only if it is minimal with respect to the Ricci-quarter symmetric metric connection.

Proof. In view of (10), we get
\[
m(E_i, \overline{E}_i) = h(E_i, \overline{E}_i) + w(E_i)b(\overline{E}_i) - \lambda S^*(E_i, \overline{E}_i),
\]
summing up for \( i = 1, 2, \ldots, n \) and dividing by \( n \), we obtain that
\[
\frac{1}{n} \sum_{i=1}^{n} m(E_i, \overline{E}_i) = \frac{1}{n} \sum_{i=1}^{n} h(E_i, \overline{E}_i)
\]
if and only if \( \lambda = 0 \) and \( b = 0 \). Hence, from (6) and (7), it follows that
\[
\rho = \overline{\rho} \quad \text{and} \quad L\overline{X} = \overline{L} \overline{X}.
\]

Thus \( \rho \) and \( L\overline{X} \) are in a tangent space of \( M \). Moreover, it is clear from (11) that \( \overline{M} \) is minimal with respect to the Levi-Civita connection if and only if it is minimal with respect to the Ricci-quarter symmetric metric connection. \( \square \)

Theorem 3. Let \( \rho \) and \( L\overline{X} \) be tangent to \( M \). Then the hypersurface \( \overline{M} \) is totally umbilical with respect to the Levi-Civita connection \( \nabla^* \) if and only if it is totally umbilical with respect to the Ricci-quarter symmetric metric connection \( \nabla \).

Proof. The proof follows easily from (10). \( \square \)

4. Gauss, Weingarten, and Codazzi equations with respect to Ricci-quarter symmetric metric connection

In this section we shall obtain the Gauss, Weingarten, and Codazzi equations with respect to the Ricci-quarter symmetric metric connection. For the Levi-Civita connection \( \nabla^* \), the Weingarten equations are given by
\[
\nabla^*_X N = -H\overline{X}
\]
for any vector field in $\overline{M}$, where $H$ has the meaning already stated. In view of
the equation (2), we get
\begin{equation}
\nabla_X N = \nabla^*_X N + \lambda L \overline{X} - S^*(\overline{X},N) \rho,
\end{equation}
where $\lambda = w(N)$. From (6), (7) and (13), it follows that
\begin{equation}
\nabla_X N = \nabla^*_X N + \lambda L \overline{X} - b(\overline{X}) \rho.
\end{equation}

Thus, from (12) and (14), we get
\begin{equation}
\nabla_X N = -H \overline{X} + \lambda L \overline{X} - b(\overline{X}) \rho,
\end{equation}
which is the equation of Weingarten with respect to the Ricci-quarter symmetric metric connection.

Let us denote the curvature tensor of $\overline{M}$ with respect to $\overline{\nabla}$ by $\overline{R}$. Then
\begin{equation}
\overline{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\end{equation}
Using (4) and (14) in $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we get
\begin{equation}
\overline{R}(X,Y)Z = \overline{R}(X,Y)Z + m(Y,Z) \{ -H \overline{X} + \lambda L \overline{X} - b(\overline{X}) \rho \}
\end{equation}
\begin{equation}
- m(\overline{X},Z) \{ -H \overline{Y} + \lambda L \overline{Y} - b(\overline{Y}) \rho \}
\end{equation}
\begin{equation}
+ \{ (\nabla_X m)(Y,Z) - (\nabla_Y m)(\overline{X},Z) - m(w(Y) L \overline{X} - w(\overline{X}) L \overline{Y},Z) \} N.
\end{equation}
From (10), it follows that
\begin{equation}
R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + m(\overline{X},Z)m(Y,W) - m(Y,Z)m(\overline{X},\overline{W}),
\end{equation}
where
\begin{equation}
R(X,Y,Z,W) = g(R(X,Y)Z,W), \quad \overline{R}(X,Y,Z,W) = g(\overline{R}(X,Y)Z,W)
\end{equation}
and $\overline{W}$ is a tangent vector field on $\overline{M}$. The equation (16) is the equation of Gauss with respect to the Ricci-quarter symmetric metric connection.

From (15), the normal component of $R(X,Y)Z$ is given by
\begin{equation}
(R(X,Y)Z)^\perp = (\nabla_X m)(Y,Z) - (\nabla_Y m)(\overline{X},Z) - m(w(Y) L \overline{X} - w(\overline{X}) L \overline{Y},Z).
\end{equation}

The equation (17) is the equation of Codazzi with respect to the Ricci-quarter symmetric metric connection.

**Theorem 4.** A totally umbilical hypersurface $\overline{M}$ of $M$ with vanishing curvature tensor with respect to the Ricci-quarter symmetric metric connection is of constant curvature.
Proof. Since $\overline{M}$ is a totally umbilical hypersurface, $m = k\overline{g}$ where $k$ is a scalar. If we put $R = 0$ and $m = k\overline{g}$ in the equation (16), we obtain

$$R(X, Y, Z, W) = k^2 [\overline{g}(Y, Z)\overline{g}(X, W) - \overline{g}(X, Z)\overline{g}(Y, W)].$$

Hence such a hypersurface $\overline{M}$ is of constant curvature. □

We assume that $M$ is a space of constant curvature and $\overline{M}$ is a conformally flat hypersurface. Since $M$ is of constant curvature, hence it is an Einstein manifold and conformally flat.

Theorem 5. [8] If $V_n$ is a conformally flat hypersurface of a conformally flat space $V_{n+1}$ and $V_n$ is a quasi-umbilical hypersurface, that is, there exists a non-zero vector field $v_i$ such that the second fundamental tensor $h_{ji}$ is given in the form $h_{ji} = \alpha g_{ji} + \beta v_j v_i$ for some functions $\alpha$ and $\beta$ on $V_n$, where $\alpha$ is differentiable.

Using the above theorem, we have

$$h(\overline{X}, \overline{Y}) = \alpha \overline{g}(\overline{X}, \overline{Y}) + \beta \overline{w}(\overline{X})\overline{w}(\overline{Y}),$$

where $\alpha$ and $\beta$ are some functions on $\overline{M}$ such that $\alpha$ is differentiable. From (10) and (18), it follows that

$$m(\overline{X}, \overline{Y}) = \gamma \overline{g}(\overline{X}, \overline{Y}) + \beta \overline{w}(\overline{X})\overline{w}(\overline{Y}) + \overline{w}(\overline{Y})b(\overline{X}),$$

where $\gamma = \alpha - \frac{r}{n+1}\lambda$ if and only if $b = 0$. Then we get

$$m(\overline{X}, \overline{Y}) = \gamma \overline{g}(\overline{X}, \overline{Y}) + \beta \overline{w}(\overline{X})\overline{w}(\overline{Y}).$$

Thus we obtain the following.

Theorem 6. Let $\overline{M}$ be a quasi-umbilical hypersurface of a conformally flat manifold $M$ with respect to the Levi-Civita connection $\nabla^*$. Then $\overline{M}$ is a quasi-umbilical hypersurface of a conformally flat manifold $M$ with respect to the Ricci-quarter symmetric metric connection $\nabla$ if and only if $L\overline{X}$ is tangent to $M$.

Now let $\overline{X}$ and $\overline{Y}$ be orthogonal unit tangent vector fields on $\overline{M}$ and $\pi$ be a subspace of the tangent space spanned by the orthonormal base $\{\overline{X}, \overline{Y}\}$. Then in view of (16) we can write

$$R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}) = R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}) + m(\overline{X}, \overline{Y})m(\overline{Y}, \overline{X}) - m(\overline{Y}, \overline{Y})m(\overline{X}, \overline{X}).$$

Let $K(\pi)$ and $\overline{K}(\pi)$ be the sectional curvatures of $M$ and $\overline{M}$ at a point $p \in M$, respectively, with respect to the Ricci-quarter symmetric metric connection. Then we get

$$K(\pi) = \overline{K}(\pi) + m(\overline{X}, \overline{Y})m(\overline{Y}, \overline{X}) - m(\overline{Y}, \overline{Y})m(\overline{X}, \overline{X}).$$

Let $\gamma$ be a geodesic in $M$ which lies in $\overline{M}$ and $\overline{T}$ be a unit tangent vector field of $\gamma$ in $\overline{M}$. Then $h(\overline{T}, \overline{T}) = 0$ and from (10), it follows that

$$m(\overline{T}, \overline{T}) = w(\overline{T})b(\overline{T}) - \lambda S^*(\overline{T}, \overline{T}).$$
Let $\pi$ be the subspace of the tangent space spanned by $X$ and $T$, and let $\rho$ and $LX$ be tangent to $M$. Then from (10), it follows that $m(T, T) = 0$. Thus using (19), we have
\[ K(\pi) = K(\pi) + m(X, T)m(T, X). \]

Hence we have the following theorem.

**Theorem 7.** Let $\gamma$ be a geodesic in $M$ which lies in $\overline{M}$ and $T$ be a unit tangent vector field of $\gamma$ in $\overline{M}$. Let $\pi$ be a subspace of the tangent space spanned by $X$ and $T$. If $\rho$ and $LX$ are tangent to $M$, then $K(\pi) \leq K(\pi)$ along $\gamma$.

**References**


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