SEMI-SYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract. We show that semi symmetric and pseudo symmetric generalized Sasakian-space-forms are Einstein when \( (0,6) \)-tensors satisfy \( R \cdot R = 0 \), \( R \cdot R = LRQ(g, R) \), \( R \cdot C = 0 \), \( R \cdot C = LCQ(g, C) \), and \( C \cdot C = 0 \), where \( C \) is Quasi conformal curvature tensor. Further we discuss about Ricci solitons.

1. Introduction

Let \((M, g)\) be a \((2n + 1)\) dimensional Riemannian manifold and let \(\nabla\) be its Levi-Civita connection. The endomorphism \(R(X,Y)Z\) of the Lie algebra of vector fields on \(M\), named the curvature operator is defined by

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [X, Y] Z,
\]

where \(X, Y, \) and \(Z\) are vector fields on \(M\) and \([X, Y]\) denotes the Lie bracket of \(X\) and \(Y\). We also denote \(R(X,Y)Z\) as the derivation induced by the curvature operator. For a symmetric \((0,2)\)-tensor \(A\) and any vector fields \(X, Y,\) and \(Z\) on \(M\), we define the endomorphism \((X \wedge_A Y)\) of \(M\) by

\[
(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y.
\]

The Riemannian Christoffel curvature tensor \(R\) and the \((0,4)\)-tensor \(G\) are defined by

\[
R(X, Y, W, Z) = g(R(X, Y)W, Z),
G(X, Y, W, Z) = g((X \wedge_g Y)Z, W).
\]

respectively.

The \((0,6)\)-tensor \(R \cdot R\), obtained by the action of the curvature operators \(R(X, Y)\) on the \((0,4)\)-curvature tensor \(R\), is given by [9]

\[
(R \cdot R)(U, V, W, Z; X, Y) = -R(R(X, Y)U, V, W, Z)
\]

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where $U, V, W, Z, X, Y \in M$.

The tensor $R \cdot R$ has the following algebraic properties:

\[
(R \cdot R)(U, V, W, Z; X, Y) = -(R \cdot R)(V, U, W, Z; X, Y)
\]

\[
= -(R \cdot R)(U, V, Z, W; X, Y),
\]

\[
(R \cdot R)(U, V, W, Z; X, Y) + (R \cdot R)(U, W, Z, V; X, Y)
\]

\[
+ (R \cdot R)(U, Z, V, W; X, Y) = 0,
\]

\[
(R \cdot R)(U, V, W, Z; X, Y) = -(R \cdot R)(U, V, W, Z; Y, X),
\]

\[
(R \cdot R)(U, V, W, Z; X, Y) + (R \cdot R)(W, Z, X, Y; U, V)
\]

\[
+ (R \cdot R)(X, Y, U, V; W, Z) = 0.
\]

The simplest $(0, 6)$-tensor having the same symmetry properties as $R \cdot R$ may well be the Tachibana tensor $Q(g, R)$ defined by [9]

\[
\]

\[
+ R(U, V, (X \wedge Y)W, Z) + R(U, V, W, (X \wedge Y)Z)
\]


2. Preliminaries

A $(2n + 1)$ dimensional Riemannian manifold is called an almost contact metric manifold if the following results hold [4]

\[
\phi^2 X = -X + \eta(X)\xi,
\]

\[
\eta(\xi) = 1,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

\[
g(\phi X, Y) = -g(X, \phi Y),
\]

\[
\eta(\phi X) = 0,
\]

\[
g(\phi X, X) = 0,
\]

\[
(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y).
\]
For a $(2n + 1)$ dimensional generalized Sasakian-space-forms we have [1]

\[(3) \quad R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} \]
\[+ f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \]
\[+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \]

For any vector fields $X, Y, Z$ on $M$, where $R$ denotes the curvature tensor of $M$,

\[
S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) + (3f_2 - (2n - 1)f_3)\eta(X)\eta(Y), \]
\[QX = (2nf_1 + 3f_2 - f_3)X + (3f_2 - (2n - 1)f_3)\eta(X)\xi, \]
\[r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3 \]

are valid, where $f_1, f_2, f_3$ are differentiable functions on $M$ and $X, Y, Z$ are vector fields on $M$. In such case we will write the manifold as $M(f_1, f_2, f_3)$. This kind of manifolds appears as natural generalization of the Sasakian-space-forms by taking $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where $c$ denotes constant $\phi$-sectional curvature. The $\phi$-sectional curvature of generalized Sasakian-space-forms $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$. Moreover, cosymplectic space-forms and Kenmotsu space-forms are also particular case of generalized Sasakian-space-forms.

For generalized Sasakian-space-forms we also have

\[
R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \]
\[
R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \]
\[
(4) \quad R(\xi, X)\xi = (f_1 - f_3)[\eta(X)\xi - X], \]
\[
R(\xi, \xi)X = 0, \]
\[
S(X, \xi) = 2n(f_1 - f_3)\eta(X). \]

3. Semi symmetric generalized Sasakian-space-forms

Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ dimensional semi symmetric generalized Sasakian-space-forms. Then from (1) we have

\[
(R \cdot R)(U, V, W, Z; X, Y) = 0, \]
\[
(R(X, Y) \cdot R)(U, V, W, Z) = 0, \]
\[
-R(R(X, Y)U, V, W, Z) - R(U, R(X, Y)V, W, Z) \]
\[
-R(U, V, R(X, Y)W, Z) - R(U, V, W, R(X, Y)Z) = 0, \]
\[
(6) \quad R(R(X, Y)U, V, W, Z) + R(U, R(X, Y)V, W, Z) \]
\[+ R(U, V, R(X, Y)W, Z) + R(U, V, W, R(X, Y)Z) = 0. \]

In view of (4) and (5), for $X = U = \xi$, (6) yields

\[(f_1 - f_3)[R(Y, V, W, Z) + (f_1 - f_3)g(Y, W)g(V, Z) - (f_1 - f_3)g(Y, Z)g(V, W)] = 0. \]
Since \((f_1 - f_3) \neq 0\), we have
\[(7) \quad R(Y, V, W, Z) = (f_1 - f_3)g(Y, Z)g(V, W) + (f_1 - f_3)g(Y, W)g(V, Z).\]

Let \(\{e_1, e_2, \ldots, e_{2n+1}\}\) be an orthonormal basis of the tangent space at each point of the manifold. Putting \(Y = Z = e_i\) in (7) and taking summation over \(i, (1 \leq i \leq (2n + 1))\), we get
\[(8) \quad S(V, W) = 2n(f_1 - f_3)g(V, W).\]

Therefore, \(M(f_1, f_2, f_3)\) is an Einstein manifold. Hence we state the following.

**Theorem 1.** Let \(M(f_1, f_2, f_3)\) be a \((2n + 1)\) dimensional generalized Sasakian-space-forms. If \(M(f_1, f_2, f_3)\) is semi symmetric then \(M(f_1, f_2, f_3)\) is an Einstein manifold.

**Corollary 1.** \((g, V, \lambda)\) is Ricci soliton in semi symmetric generalized Sasakian space forms if and only if \(V\) is conformal killing vector field.

**Proof.** From Theorem 1 and by the definition of Ricci soliton \([6]\), we have
\[
(L_V g)(V, W) + 2S(V, W) + 2\lambda g(V, W) = 0,
\]
where \(\lambda\) is some constant. From (8) we get
\[
(9) \quad (L_V g)(V, W) + 4n(f_1 - f_3)g(V, W) + 2\lambda g(V, W) = 0.
\]

Let \(\{e_1, e_2, \ldots, e_{2n+1}\}\) be an orthonormal basis of the tangent space at each point of the manifold. Putting \(V = W = e_i\) in (9) and taking summation over \(i, (1 \leq i \leq (2n + 1))\), we get
\[
(L_V g)(e_i, e_i) + 4n(2n + 1)(f_1 - f_3) + 2(2n + 1)\lambda = 0.
\]

Since \([e_i, e_j] = 0\), for all \(1 \leq i, j \leq (2n + 1)\), we obtain
\[
\lambda = -2n(f_1 - f_3).
\]

Thus, Ricci soliton in semi symmetric generalized Sasakian-space forms is shrinking if \(f_1 > f_3\), i.e., \(\lambda < 0\), steady if \(f_1 = f_3\), i.e., \(\lambda = 0\), and expands if \(f_1 < f_3\), i.e., \(\lambda > 0\).

\[
\Box
\]

### 4. PSEUDOSYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

Let \(M(f_1, f_2, f_3)\) be a \((2n + 1)\) dimensional generalized Sasakian-space-forms. Then from (1) and (2) we have
\[
(R \cdot R)(U, V, W, Z; X, Y) = L_R Q(g, R)(U, V, W, Z; X, Y),
(R(X, Y) \cdot R)(U, V, W, Z) = -L_R ((X \wedge Y) \cdot R)(U, V, W, Z),
(10)\
+ R(U, V, (X \wedge Y)W, Z) + R(U, V, W, (X \wedge Y)Z)]
In view of (4) and (5), for \( X = U = \xi \), (10) yields
\[
(f_1 - f_3)[R(Y, V, W, Z) + (f_1 - f_3)g(Y, W)g(V, Z) - (f_1 - f_3)g(Y, Z)g(V, W)]
= -L_R[R(Y, V, W, Z) + (f_1 - f_3)g(Y, W)g(V, Z) - (f_1 - f_3)g(Y, Z)g(V, W)],
\]
\[
[L_R + (f_1 - f_3)][R(Y, V, W, Z) + (f_1 - f_3)g(Y, W)g(V, Z) - (f_1 - f_3)g(Y, Z)g(V, W)] = 0.
\]
Therefore either \( L_R = -(f_1 - f_3) \) or
\[
R(Y, V, W, Z) = (f_1 - f_3)g(Y, Z)g(V, W) - (f_1 - f_3)g(Y, W)g(V, Z).
\]
Let \( \{e_1, e_2, ..., e_{2n+1}\} \) be an orthonormal basis of the tangent space at each point of the manifold. Putting \( Y = Z = e_i \) in (11) and taking summation over \( i \), \( 1 \leq i \leq (2n+1) \), we get
\[
S(V, W) = 2n(f_1 - f_3)g(V, W).
\]
Therefore, \( M(f_1, f_2, f_3) \) is an Einstein manifold. Hence we state the following.

**Theorem 2.** Let \( M(f_1, f_2, f_3) \) be a \((2n + 1)\) dimensional generalized Sasakian-space-forms. If \( M(f_1, f_2, f_3) \) is pseudosymmetric then \( M(f_1, f_2, f_3) \) is an Einstein manifold provided that \( L_R \neq -(f_1 - f_3) \).

**Corollary 2.** \((g, V, \lambda)\) is Ricci soliton in pseudo symmetric generalized Sasakian space forms if and only if \( V \) is conformal killing vector field provided \( L_R \neq -(f_1 - f_3) \).

**Proof.** Proof follows from Theorem 2 and the definition of Ricci Soliton. \( \square \)

5. **Quasi conformal semi symmetric generalized Sasakian-space-forms**

For a \((2n + 1)\) dimensional almost contact metric manifold the Quasi conformal curvature tensor \( C \) is given by
\[
C(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
\]
\[
- \frac{r}{2n + 1} \left[ \frac{a}{2n} + 2b \right] (g(Y, Z)X - g(X, Z)Y).
\]
Using equation (3) in (12) yields
\[
C(X, Y)\xi = D[\eta(Y)X - \eta(X)Y],
\]
\[
C(\xi, X)Y = D[g(X, Y)\xi - \eta(X)Y],
\]
\[
C(\xi, X)\xi = D[\eta(X)\xi - X],
\]
where
\[
D = a(f_1 - f_3) + 2nb(f_1 - f_3) + b(2nf_1 + 3f_2 - f_3) - \frac{r}{2n + 1} \left[ \frac{a}{2n} + 2b \right].
\]
Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ dimensional Quasi conformal semi symmetric generalized Sasakian-space-forms. Then, from (1), we have

\[
(R(X, Y) \cdot C)(U, V, W, Z) = 0,
\]

\[
-C(R(X, Y)U, V, W, Z) - C(U, R(X, Y)V, W, Z)
\]

\[
-C(U, V, R(X, Y)W, Z) - C(U, V, W, R(X, Y)Z) = 0.
\]

In view of (4), (5), and (13), for $X = U = \xi$, (14) yields

\[
(f_1 - f_3)\{C(Y, V, W, Z) + D[g(Y, W)g(V, Z) - g(Y, Z)g(V, W)]\} = 0.
\]

Since $(f_1 - f_3) \neq 0$, we have

\[
C(Y, V, W, Z) = D[g(Y, Z)g(V, W) - g(Y, W)g(V, Z)].
\]

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (15) and taking summation over $i$, $(1 \leq i \leq (2n + 1))$, using equation (12), we get

\[
S(V, W) = D'g(V, W),
\]

where

\[
D' = \frac{2n[(a + 2nb)(f_1 - f_3) + b(2nf_1 + 3f_2 - f_3)] - br}{a + b(2n + 1)}.
\]

**Theorem 3.** Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ dimensional generalized Sasakian-space-forms. If $M(f_1, f_2, f_3)$ is Quasi conformal semisymmetric then $M(f_1, f_2, f_3)$ is an Einstein manifold.

### 6. Quasi conformal pseudo symmetric generalized Sasakian-space-forms

Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ dimensional Quasi conformal pseudo symmetric generalized Sasakian-space-forms. Then, from (1) and (2), we have

\[
(R \cdot C)(U, V, W, Z; X, Y) = L_CQ(g, C)(U, V, W, Z; X, Y),
\]

\[
(R(X, Y) \cdot C)(U, V, W, Z) = -L_C((X \wedge Y) \cdot R)(U, V, W, Z),
\]

(16)

\[
\]

\[
\]

\[
\]

In view of (4), (5), and (13), for $X = U = \xi$, (16) yields

\[
[L_C + (f_1 - f_3)]\{C(Y, V, W, Z) + D[g(Y, W)g(V, Z) - g(Y, Z)g(V, W)]\} = 0.
\]

Therefore, either $L_C = -(f_1 - f_3)$ or

\[
C(Y, V, W, Z) = D[g(Y, Z)g(V, W) - g(Y, W)g(V, Z)].
\]
Let \( \{e_1, e_2, \ldots, e_{2n+1}\} \) be an orthonormal basis of the tangent space at each point of the manifold. Putting \( Y = Z = e_i \) in (17) and taking summation over \( i, (1 \leq i \leq (2n + 1)) \), using equation (12), we get

\[
(18) \quad S(V,W) = D'g(V,W).
\]

**Theorem 4.** Let \( M(f_1, f_2, f_3) \) be a \((2n + 1)\) dimensional generalized Sasakian-space-forms. If \( M(f_1, f_2, f_3) \) is Quasi conformal pseudo symmetric then \( M(f_1, f_2, f_3) \) is an Einstein manifold provided that \( L_C \neq -(f_1 - f_3) \).

7. **Generalized Sasakian-space-forms satisfies the condition**

\[
C \cdot C = 0
\]

Let \( M(f_1, f_2, f_3) \) be a \((2n + 1)\) dimensional generalized Sasakian-space-forms. Let \( C \cdot C \) be a \((0, 6)\)-tensor and \( C \cdot C = 0 \). Then

\[
(C(X, Y) \cdot C)(U, V, W, Z) = 0,
\]

\[
-C(C(X, Y)U, V, W, Z) - C(U, C(X, Y)V, W, Z)
\]

\[
(19) \quad -C(U, V, C(X, Y)W, Z) - C(U, V, W, C(X, Y)Z) = 0.
\]

In view of (13) and (14), for \( X = U = \xi \), (19) yields

\[
-D[C(Y, V, W, Z) + D\{g(Y, W)g(V, Z) - g(Y, Z)g(V, W)\}] = 0.
\]

Since \( D \neq 0 \), we have

\[
(20) \quad C(Y, V, W, Z) = D[g(Y, Z)g(V, W) - g(Y, W)g(V, Z)].
\]

Let \( \{e_1, e_2, \ldots, e_{2n+1}\} \) be an orthonormal basis of the tangent space at each point of the manifold. Putting \( Y = Z = e_i \) in (20) and taking summation over \( i, (1 \leq i \leq (2n + 1)) \), using equation (12), we get

\[
S(V,W) = D'g(V,W).
\]

**Theorem 5.** Let \( M(f_1, f_2, f_3) \) be a \((2n + 1)\) dimensional generalized Sasakian-space-forms. If \((0, 6)\)-tensor \( C \cdot C = 0 \) holds on \( M(f_1, f_2, f_3) \), then \( M(f_1, f_2, f_3) \) is an Einstein manifold.

**Corollary 3.** Let \( M(f_1, f_2, f_3) \) be a \((2n + 1)\) dimensional Quasi conformal semi symmetric generalized Sasakian-space-forms. Then \( R \cdot C = C \cdot C \) holds on \( M(f_1, f_2, f_3) \).

**References**


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