A NOTE ON ZERO DIVISOR GRAPH WITH RESPECT TO ANNIHILATOR IDEALS OF A RING

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ABSTRACT. The zero divisor graph has been investigated in general for a commutative ring $R$. We consider a (not necessarily commutative) ring as a right module over itself. We consider annihilators of right ideals of $R$ and define a graph related to these annihilators. Let $R$ be a ring and $I$ be an ideal of $R$. We denote the annihilator of $I$ (viewed as a right $R$-module) by $\text{Ann}(I)$. We define a graph with respect to $\text{Ann}(I)$ as follows and denote it by $\Gamma_{A(I)}(R)$:

$$\Gamma_{A(I)}(R) = \{E = (a, b) \mid a \in I, b \in \text{Ann}(I)\}.$$ 

With this we prove that for a right ideal of a ring $R$ if $I^* \cap \text{Ann}(I)^* = \emptyset$, then $\Gamma_{A(I)}(R)$ is bipartite, where $K^* = K \setminus \{0\}$ for any subset $K \subseteq R$.

1. INTRODUCTION

The concept of zero divisor graph has been an active area of research since the notion was introduced by Beck [5]. Different aspects of zero divisor graph have been studied and investigations are on.

In most of the cases zero divisor graph of a commutative ring has been investigated.

Let $R$ be a commutative ring with identity $1 \neq 0$. Let $Z(R)$ be the set of zero divisors of $R$ and $Z(R)^* = Z(R) \setminus \{0\}$. Two elements $a, b \in Z(R)^*$ are adjacent if and only if $ab = 0 = ba$. The zero divisor graph of $R$ is denoted by $\Gamma(R)$.

For any graph $G = (V, E)$, the set of vertices shall be denoted by $V(G)$ and the set of edges shall be denoted by $E(G)$.

The zero divisor graph has been studied for the ring of continuous functions by Azarpanah and Motamedi [4], for a semiprime Gifand ring by Samei [10]. A relation of $\Gamma(R)$ for a reduced ring has been given with respect to the prime radical of $R$ (Samei [10, Theorem 3.1]).

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For more details and results the reader is referred to Anderson et al. [1, 2, 3]. Some more treatment could be found in Levy and Shepiro [6], Akbari et al. [1], Maimani et al. [7] and Samei [10].

2. Zero divisor graph with respect to annihilators

We continue the above investigation. We relate zero divisor graph of a (not necessarily commutative) ring to annihilator primes. Let $R$ be a ring and $I$ be a right ideal of $R$. Let $A = \{a \in R \mid a$ is regular\}, $B = \{b \in R \mid b$ is a zero divisor\}$, and $B^* = B \setminus \{0\}$. Furthermore, $\text{Ann}(I) = \{r \in R \mid ar = 0,$ for all $a \in I\}$. We know that $\text{Ann}(I)$ is an ideal of $R$. Let $\text{Ann}(I)^* = \text{Ann}(I) \setminus \{0\}$ and $I^* = I \setminus \{0\}$. We introduce zero divisor graph with respect to $I$ in the following way. Let $a \in I^*$, $b \in B^*$. Then $a, b$ are adjacent if and only if $ab = 0$. We denote the graph by $\Gamma_{\text{Ann}(I)}(R)$. We note that $\Gamma_{\text{Ann}(I)}(R)$ is a directed graph and $V(\Gamma_{\text{Ann}(I)}(R)) \subseteq B^*$ and $\text{Ann}(I)^* \subseteq B^*$. If $b \in \text{Ann}(I)^*$, then obviously $ab = 0$ for all $a \in I^*$. With this we prove the following:

**Theorem 1.** Let $R$ be a ring and $I$ be a right ideal of $R$. If $I^* \cap \text{Ann}(I)^* = \emptyset$ and we ignore isolated vertices, then $\Gamma_{\text{Ann}(I)}(R)$ is bipartite.

**Proof.** $I^* \cap \text{Ann}(I)^* = \emptyset$ implies that if $E = (a, b) \in \Gamma_{\text{Ann}(I)}(R)$, then $a \in I^*$, $a \notin \text{Ann}(I)^*$ and $b \in \text{Ann}(I)^*$, $b \notin I^*$.

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![Figure 1](image-url)
faithful as a right \( R \)-module. (Recall that a right module \( M \) over a ring \( R \) is called faithful if \( \text{Ann}(M) = \{0\} \). We note that here \( \Gamma_{A(I)}(R) \) is the null graph.

Recall that \( \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\} \) (where \( n \) is a positive integer) is a ring under addition modulo \( n \) and multiplication modulo \( n \).

**Example 4.** Let \( R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_n \text{ where } n \text{ is some positive integer} \right\} \) and \( J = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_n \right\} \).

Then \( J \) is a right ideal of \( R \) and we see that \( \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), therefore, \( J^* \cap \text{Ann}(J)^* \neq \phi \) which implies that \( \Gamma_{A(J)}(R) \) is not bipartite. We also note that the zero divisor graph of \( J \) as a ring (i.e. \( \Gamma(J) \)) is complete. Here for all \( a, b \in J \) we have \( ab = 0 = ba \).

**Example 5.** Let \( R = M_2(\mathbb{Z}_2) \) and \( I = \left\{ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \).

Then \( I \) is a right ideal of \( R \) and \( \text{Ann}(I)^* = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \).

Therefore, \( I \) is faithful as a right \( R \)-module and \( \Gamma_{A(I)}(R) \) is the null graph.

**Example 6.** Let

\[
R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\},
\]

\[
I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{Z}_2 \right\},
\]

\[
I^* = \left\{ A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},
\]

\[
\text{Ann}(I)^* = \left\{ B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\},
\]

\[
AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
AC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
AD = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

We see that \( \Gamma_{A(I)}(R) = K_{1,3} \) is complete bipartite.
Example 7. Consider \( \mathbb{Z}_3 = \{0, 1, 2\} \). Let \( R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\} \) and \( I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\} \). Then \( I \) is a right ideal of \( R \). Now

\[
I^* = \left\{ A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right\}
\]

and

\[
P = \text{Ann}(I^*) = \left\{ C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \right. \left. G = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, K = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \right\}.
\]

Now \( P \cap I^* = \emptyset \). Therefore, \( \Gamma_{A(I)}(R) \) is bipartite.

Proposition 8. Let \( P \) be defined as in Example 7. Then \( P \cup \{0\} \) is a prime ideal of \( R \).

Proof. Since

\[
ARB \in P = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}
\]

is valid,

\[
\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} R \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \in \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}.
\]

Therefore,

\[
\begin{pmatrix} aRu & arv + bRu \\ 0 & aRu \end{pmatrix} \in \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}.
\]

Thus,

\[
aRu = 0
\]
and hence
\[ a = 0 \text{ or } u = 0. \]

If \( a = 0 \), we have \( A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in P. \)

If \( u = 0 \), we have \( B = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in P. \) Hence \( P \) is a prime ideal of \( R. \) □

3. Zero divisor graph with respect to ideals

The concept of zero divisor graph with respect to an ideal was introduced by Redmond [9]. Let \( R \) be a commutative ring with identity \( 1 \neq 0 \) and \( I \) an ideal of \( R. \) The zero divisor graph with identity respect to \( I \) is denoted by \( \Gamma_I(R) \) and
\[ \Gamma_I(R) = \{a \in R \setminus I \text{ such that } ab \in I \text{ for some } b \in R \setminus I\} \]
with distinct vertices \( a \) and \( b \) adjacent if and only if \( ab \in I. \) Thus if \( I = \{0\}, \) then \( \Gamma_I(R) = \Gamma(R). \) Redmond [9] found a relationship between \( \Gamma_I(R) \) and \( \Gamma(R/I), \) and proved that for a finite ideal of a commutative ring \( R, \) \( \Gamma_I(R) \) contains \( |I| \) distinct subgraphs isomorphic to \( \Gamma(R/I), \) where \( |I| \) denotes the order of \( I. \)

Maimani et al. [7, Theorem 2.2] have proved the following concerning isomorphisms of zero divisor graphs. Let \( R \) and \( S \) be two rings. Let \( I \) be a finite ideal of \( R \) and \( J \) be a finite ideal of \( S \) such that \( \sqrt{I} = I \) and \( \sqrt{J} = J. \) Then the following hold.

1. If \( |I| = |J| \) and \( \Gamma(R/I) \cong \Gamma(S/J), \) then \( \Gamma_I(R) \cong \Gamma_J(S). \)
2. If \( \Gamma_I(R) \cong \Gamma_J(S), \) then \( \Gamma(R/I) \cong \Gamma(S/J). \)

Remark 9. Let \( P \) be a prime ideal of a commutative ring \( R. \) Then \( \Gamma_P(R) \) is the null graph.

We take the notion of zero divisor graph with respect to an ideal of a commutative ring further in noncommutative set up in the following way.

Definition 10. Let \( R \) be a (not necessarily commutative) ring with identity \( 1 \neq 0 \) and \( I \) be a right ideal of \( R. \) The zero divisor graph with respect to \( I \) is denoted by \( \Gamma_I(I_R) \) and
\[ \Gamma_I(I_R) = \{a \in R \setminus I \text{ such that } ab \in I, ba \in I \text{ for some } b \in R \setminus I\} \]
with distinct vertices \( a \) and \( b \) adjacent if and only if \( ab \in I \) and \( ba \in I. \)

Example 11. Let \( A = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\} \) and \( R = M_2(A). \) Now \( I = \{0, 3\} \) is an ideal of \( A \) and \( K = M_2(I) \) is an ideal of \( R. \) Let \( U = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix} \in R, \)
\[ V = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \in R, \text{ and } W = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R. \] Then \( U \notin K, \) \( V \notin K \) and
$W \notin K$. Now we see that $UV = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in K$, $VU = \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix} \notin K$.

$UW = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \in K$, $WU = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in K$. Therefore, $U, W \in \Gamma_K(K_R)$.

**Example 12.** Let $S = \mathbb{Z}_2 = \{0, 1\}$ and $R = M_2(S)$. Now

$$I = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a right ideal of $R$. Also

$$R \setminus I = \left\{ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \\
E = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, H = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \\
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, K = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$  

We see that $\Gamma_I(I_R) = \{F, D\}$ as $FD \in I$ and $DF \in I$.

**Remark 13.** Let $P$ be a completely prime ideal of a ring $R$. Then $\Gamma_P(P_R)$ is the null graph.

Recall that an ideal $P$ of a ring $R$ is completely prime if $R/P$ is a domain, i.e., $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$ (McCoy [8]).

**Remark 14.** Let $R$ be a (not necessarily commutative) ring with identity $1 \neq 0$ and $I$ be a right ideal of $R$. Then $\Gamma_I(I_R) \cup \{0\}$ need not be a right ideal of $R$.

We saw in Example 12 that $D + F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \Gamma_I(I_R)$.

We note that in Example 11 there were vertices $U \notin I, V \notin I$ such that $UV \in I$ but $VU \notin I$. Similarly in Example 12 we had $AB \in I, BA \notin I$, $AD \in I, DA \notin I, CA \in I, AC \notin I, \ldots, LE \in I, EL \notin I, LF \in I, FL \notin I$.

This motivates one to define the graph (directed) with respect to a right ideal in the following way. Here we use the notation as in Redmond [9].

**Definition 15.** Let $R$ be a (not necessarily commutative) ring with identity $1 \neq 0$ and $I$ be a right ideal of $R$. The zero divisor graph (directed) with respect to $I$ is denoted by $\Gamma_I(R)$ and is defined as

$$\Gamma_I(R) = \{a \in R \setminus I \text{ such that } ab \in I \text{ for some } b \in R \setminus I \}$$

with distinct vertices $a$ and $b$ adjacent if and only if $ab \in I$.

**Remark 16.** Let $R$ and $I$ be as in Example 12. Then

$$\Gamma_I(R) = \{A, C, D, F, H, J, K, L\}.$$  

We note that $B \notin \Gamma_I(R)$ as there does not exist any element $T \in R \setminus I$ such that $BT \in I$. 
Remark 17. Let $R$ be a (not necessarily commutative) ring with identity $1 \neq 0$ and $I$ be a right ideal of $R$. Then $\Gamma_I(R) \cup \{0\}$ need not be a right ideal of $R$. We saw in Example 12 that $A + C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin \Gamma_I(R)$.

REFERENCES


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