More Results On 3-Step Hamiltonicity Of Graphs
And Its Line Graphs*

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Abstract

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A $(p,q)$-graph $G = (V,E)$ is said to be $AL(k)$-traversal if there exists a sequence of vertices $(v_1,v_2,\ldots,v_p)$ such that for each $i = 1, 2, \ldots, p - 1$, the distance between $v_i$ and $v_{i+1}$ is $k$. We call a graph $G$ a $k$-step Hamiltonian graph (or say it admits a $k$-step Hamiltonian cycle) if it has an $AL(k)$-traversal in $G$ and $d(v_p,v_1) = k$. In this paper, we give several construction of some families of graphs and its line graphs which admit a 3-step Hamiltonian cycle.

1 Introduction

Throughout this paper, we will consider only simple undirected graph $G = (V(G), E(G))$. The distance between two vertices $u$ and $v$ in $G$ denoted by $d(u,v)$ is the length of a shortest $u,v$-path in $G$. The line graph $L(G)$ of a graph $G$ has $E(G)$ as its vertex set and two vertices are adjacent in $L(G)$ if and only if they are adjacent as edges in $G$. A matching in a graph $G$ is a set $M \subseteq E(G)$ such that no edges in $M$ have common endpoints. For a vertex $u \in V(G)$, we say $u$ is saturated by a matching $M$ if $u$ is the endpoint of an edge of $M$, otherwise $u$ is unsaturated by $M$. A matching $M$ is called a perfect matching in a graph $G$ if $M$ saturates each vertex of $G$. For terminologies and notations which are not explained here, please refer West [8].

A graph $G$ is said to be Hamiltonian if it contains a Hamiltonian cycle, i.e a spanning cycle that traverses each vertex of $G$ exactly once. Determining whether such cycle exists in a given graph is one of the major classical problems in graph theory. There is no exact characterization to check the existence and non-existence of Hamiltonian cycle for a given graph. A good reference for recent development and open problems related to Hamiltonicity of graphs, please see [2]. This concept of Hamiltonicity is then

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extended by Lau et al. in [3] to $k$-step Hamiltonicity. They introduced the concept of $AL(k)$-traversal and $k$-step Hamiltonian graph as follows: For an integer $k \geq 1$, a $(p,q)$-graph $G$ with $p$ vertices and $q$ edges is said to admit an $AL(k)$-traversal if the $p$ vertices of $G$ can be arranged as $v_1, v_2, \ldots, v_p$ such that $d(v_i, v_{i+1}) = k$ for each $i = 1, 2, \ldots, p-1$. A graph $G$ is $k$-step Hamiltonian (or just $k$-SH) if $G$ admits an $AL(k)$-traversal and $d(v_1, v_p) = k$. The sequence of vertices $v_1, v_2, \ldots, v_p, v_1$ is then called a $k$-SH cycle of $G$. Clearly, 1-SH graphs are Hamiltonian. The distance-$k$ graph, $D_k(G)$, is a graph generated from a graph $G$ such that $V(D_k(G)) = V(G)$ and $uv \in E(D_k(G))$ if and only if $d(u, v) = k$ in $G$. The following important results obtained by Lau et al. in [3] will be needed in our results.

**LEMMA 1.** A graph $G$ is $k$-SH or admits an $AL(k)$-traversal if and only if $D_k(G)$ is Hamiltonian or has a Hamiltonian path, respectively.

**LEMMA 2.** A bipartite graph does not admit a $k$-SH cycle for even $k \geq 2$.

Lau et al. in [4] obtained the following necessary and sufficient condition for cycles $C_n$ to be $k$-SH.

**THEOREM 1.** The cycle graph $C_n$, $n \geq 3$ admits a $k$-SH cycle for $k \geq 2$ if and only if $n \geq 2k + 1$ and gcd$(n, k) = 1$.

Several classes of $k$-SH graphs including trees, tripartite graphs, cycles, grid graphs, cubic graphs and subdivision of cycles, have been studied, see [3, 4, 5, 6, 7]. In [1], the authors investigated some families of graphs and its line graphs which admit a 3-SH cycle. In this paper, we extend the results in [1] and give new construction of some families of graphs and its line graphs which admit a 3-SH cycle.

## 2 Main Results

In [3], we know that the complete bipartite graph $K_{m,n}$ is not $k$-SH for all $m,n$ and $k \geq 2$. Note that the line graph of complete bipartite graph $K_{m,n}$ is a graph obtained from a grid graph $P_m \times P_n$ such that vertices of the same horizontal (respectively vertical) path are also adjacent to each other. We denote $(a,b)$ as the vertex on row $a$ and column $b$ of $P_m \times P_n$ for $1 \leq a \leq m, 1 \leq b \leq n$. Two vertices $(a,b)$ and $(c,d)$ in $L(K_{m,n})$ are of distance 2 if $a \neq c$ and $b \neq d$. Otherwise, they are of distance 1. Therefore, we conclude that $L(K_{m,n})$ is not $k$-SH for all $k \geq 3$.

It is interesting to know about the $k$-step Hamiltonicity of the complete bipartite graph $K_{m,n}$ if some edges are deleted. But, from Lemma 2, we know that the graph, say $G$ obtained from $K_{m,n}$ by deleting some edges is not $k$-SH for even $k \geq 2$ and the $k$-step Hamiltonicity of $G$ for odd $k \geq 3$ is not studied yet.

We now check the 3-step Hamiltonicity of some graphs obtained from the complete bipartite graph $K_{m,n}$ by deleting two disjoint perfect matchings $S$ and $T$. But here, we will consider only $K_{n,n}$, $n \geq 2$ since $K_{m,n}$ for $m \neq n$ does not have perfect matching. Let $V = \{a_1, a_2, \ldots, a_n\}$ and $W = \{a_1^*, a_2^*, \ldots, a_n^*\}$ be the partite sets of $K_{n,n}$ such.
that $E(K_{n,n}) = \{a_i a^*_j : 1 \leq j \leq n\}$. We then obtain the following results. Note that all subscripts are to be read modulo $n$.

**LEMMA 3.** For $S = \{a_i a^*_i : 1 \leq i \leq n\}$ and $T = \{a_i a^*_{i+1} : 1 \leq i \leq n\}$, the graph $G = K_{n,n} - \{S,T\}$ is 3-SH if and only if $n \geq 4$.

**PROOF.** It is obvious that $G$ is disconnected when $n = 2$ and $n = 3$ so that $G$ does not admit a 3-SH cycle. For $n \geq 4$, observe that $d(a^*_i, a_{i+1}) = d(a_i, a^*_{i+2}) = 1$ and $d(a_i, a_{i-1}) = d(a^*_i, a^*_{i-1}) = 2$ for $1 \leq i \leq n$. Since $a^*_i$ is not adjacent to $a_i$ and $a_i$ is not adjacent to $a^*_i$, we have $d(a^*_i, a_i) = d(a_i, a^*_{i+1}) = 3$. Therefore, the sequence $a^*_1, a_1, a^*_2, a_2, \ldots, a^*_n, a_n, a^*_1$ is a possible 3-SH cycle of $G$.

**LEMMA 4.** For $S = \{a_i a^*_{i+1} : 1 \leq i \leq n\}$ and $T = \{a_i a^*_{i-1} : 1 \leq i \leq n\}$, the graph $G = K_{n,n} - \{S,T\}$ is 3-SH if and only if $n \geq 5$ is odd.

**PROOF.** We need $n \geq 3$ because when $n = 2$, we have $S = T$. For $n = 3$ and $n = 4$, graph $G$ is disconnected and thus is not 3-SH. For $n \geq 5$ is even, $D_3(G)$ consists of 2 components each of size $n$ so that $D_3(G)$ is not Hamiltonian. By Lemma 1, $G$ is not 3-SH.

Now, consider odd $n \geq 5$. Note that for $1 \leq i \leq n$, $d(a_i, a^*_i) = 1$ and $d(a^*_i, a^*_{i+1}) = d(a_i, a_{i+1}) = 2$. Since $a_i$ is not adjacent to $a^*_i$, $a^*_{i+1}$ and $a^*_i$ is not adjacent to $a_{i+1}$, we have $d(a_i, a^*_{i+1}) = d(a^*_i, a_{i+1}) = 3$. A 3-SH cycle is then given by $a_1, a^*_2, a_3, a^*_4, \ldots, a^*_n, a_n, a^*_1, a_2, a^*_3, \ldots, a^*_n, a_n, a^*_1$.

**LEMMA 5.** For $S = \{a_i a^*_i : 1 \leq i \leq n\}$ and $T = \{a_i a^*_{i+3} : 1 \leq i \leq n\}$, the graph $G = K_{n,n} - \{S,T\}$ is 3-SH if and only if $n \geq 4$, $n \not\equiv 0 \pmod{3}$.

**PROOF.** We consider only $n = 2$ and $n \geq 4$ because when $n = 3$, we have $S = T$. It is obvious that $G$ is disconnected when $n = 2$ and thus $G$ is not 3-SH. Suppose $n \geq 6$, $n \equiv 0 \pmod{3}$. We can observe that $D_3(G)$ consists of 3 components each of size $\frac{2n}{3}$ and so $D_3(G)$ is not Hamiltonian. By Lemma 1, $G$ is not 3-SH. Suppose now $n \geq 4$, $n \not\equiv 0 \pmod{3}$. Note that $d(a^*_i, a_{i+1}) = d(a_i, a_{i+2}) = 1$ and $d(a_i, a_{i-1}) = d(a^*_i, a^*_{i+2}) = 2$ for $1 \leq i \leq n$. Since $a^*_i$ is not adjacent to $a_i$ and $a_i$ is not adjacent to $a^*_{i+3}$, we have $d(a^*_i, a_i) = d(a_i, a^*_{i+3}) = 3$. Then, $G$ is 3-SH by choosing the sequence $a^*_1, a_1, a^*_2, a_4, \ldots, a^*_n, a_{n-3}, a_{n-3}, a_n, a^*_n, a_{n-3}, a^*_n, a_{n-3}, a_{n-2}, a_{n-2}, a^*_1$ for $n \equiv 1 \pmod{3}$ and the sequence $a^*_1, a_1, a^*_2, a_4, \ldots, a^*_n, a_{n-3}, a_n, a^*_n, a_{n-3}, a_{n-2}, a_{n-2}, a^*_1, a_{n-3}, a_{n-3}, a^*_n, a_{n-3}, a^*_n, a_{n-3}, a^*_n, a_{n-3}, a_{n-2}, a_{n-2}, a^*_1$ for $n \equiv 2 \pmod{3}$ as the 3-SH cycle.

**LEMMA 6.** For $S = \{a_i a^*_i : 1 \leq i \leq n\}$ and $T = \{a_i a^*_{i+4} : 1 \leq i \leq n\}$, the graph $G = K_{n,n} - \{S,T\}$ is 3-SH if and only if $n \geq 5$ is odd.

**PROOF.** We consider only $n = 3$ and $n \geq 5$ because when $n = 2$ and $n = 4$, we have $S = T$. It is also obvious that $G$ is disconnected when $n = 3$ so that $G$ is not 3-SH. Suppose $n \geq 6$ is even. Observe that for $n \equiv 0 \pmod{4}$, $D_3(G)$ consists of 4 components each of size $\frac{n}{2}$ and for $n \equiv 2 \pmod{4}$, $D_3(G)$ consists of 2 components each
of size $n$. Therefore, for each case $D_3(G)$ is not Hamiltonian and thus by Lemma 1, $G$ is not 3-SH. Suppose now $n \geq 5$ is odd. In Figure 1 and Figure 2, we give a labeling of Hamiltonian cycle for graph $D_3(G)$ when $n = 7$ and $n = 9$, respectively. Note that for all odd $n \geq 5$ such that $n \equiv 1 \pmod{4}$, a Hamiltonian cycle of $D_3(G)$ can be obtained in a similar way to the labeling in Figure 2 and for all odd $n \geq 7$ such that $n \equiv 3 \pmod{4}$, a labeling for Hamiltonian cycle follows those in Figure 1. By Lemma 1, we know that all these graphs $G$ are 3-SH such that the Hamiltonian cycle in $D_3(G)$ is a 3-SH cycle of $G$.

As we can see from these 4 lemmas, we can get a 3-SH graph from the complete bipartite graph $K_{n,n}$ by deleting a set of edges. It is difficult to solve the 3-step Hamiltonicity of $G = K_{n,n} - \{S, T\}$ in general because there are $n!$ perfect matchings of $K_{n,n}$. There are a lot more cases that should be considered. We then propose the following problems.

PROBLEM 1. Solve the 3-step Hamiltonicity of $G = K_{n,n} - \{S, T\}$ for all cases of $S$ and $T$.

PROBLEM 2. Study the 3-step Hamiltonicity of complete bipartite graph $K_{m,n}$ with more edges deleted.

Next, consider a graph $G$ with $n$ vertices. The corona product of $G$ and any graph $H$, denoted by $G \circ H$, is a graph obtained by taking one copy of $G$ and $n$ copies $H_1, H_2, \ldots, H_n$ of $H$, and then joining the $i$-th vertex of $G$ to every vertex in $H_i$.

Suppose $G$ is a graph of order $n$ that admits a Hamiltonian cycle given by the sequence $u_1, u_2, \ldots, u_n, u_1$ and 3-SH cycle given by $v_1, v_2, \ldots, v_n, v_1$ such that $v_1 = u_1$.
and $v_n = u_{n-2}$.

THEOREM 2. The corona product of graph $G$ described above and empty graph $O_m$ of order $m$ is 3-SH for all $m \geq 1$.

PROOF. We know that the graph $G \odot O_m$ is obtained from $G$ by adding $nm$ more vertices and $nm$ more edges. Without loss of generality, we let the $nm$ pendant vertices be $u_{i,1}, u_{i,2}, \ldots, u_{i,m}$ such that the added edges are $u_i u_{i,1}, u_i u_{i,2}, \ldots, u_i u_{i,m}$ for $i = 1, \ldots, n$. We can see that the sequence $v_1 = u_1, v_2, \ldots, v_n = u_{n-2}, u_{n-1,1}, u_{1,1}, u_{2,1}, \ldots, u_{n-1,1}, u_{n,2}, u_{1,2}, u_{2,2}, \ldots, u_{n-1,2}, u_{n,3}, \ldots, u_{n,m}, u_{1,m}, u_{2,m}, \ldots, u_{n-1,m}, u_1$ is a 3-SH cycle of $G \odot O_m$.

The corona product $C_n \odot K_1$, in particular, is the graph consisting of a cycle $C_n$, $n \geq 3$ (with edges $u_1 u_2, u_2 u_3, \ldots, u_{n-1} u_n, u_n u_1$), $n$ more pendant vertices $v_1, v_2, \ldots, v_n$ and $n$ more edges $u_i v_i$ for $i = 1, 2, \ldots, n$. We call this graph the sun graph $S_n$.

THEOREM 3. The sun graph $S_n$ is 3-SH if and only if $n \geq 5$.

PROOF. Observe that all $u_i$ are isolated in $D_3(S_n)$ if $n = 3$ and of degree 1 if $n = 4$ so that $D_3(S_n)$ cannot be Hamiltonian and thus $S_3$ and $S_4$ are not 3-SH. Suppose $n \geq 5$. We consider 2 cases.

Case 1. $n \equiv 0 \pmod{3}$.
A 3-SH cycle is given by the sequence $v_1, u_3, u_6, \ldots, u_n, v_2, u_4, u_7, \ldots, u_{n-2}, u_1, v_3, u_5, u_8, \ldots, u_{n-1}, u_2, v_4, v_5, \ldots, v_n, v_1$. In Figure 3, we give a 3-SH cycle for $S_9$.

![Figure 3: A 3-step Hamiltonian cycle for $S_9$.](image)

Case 2. $n \not\equiv 0 \pmod{3}$.
If $n = 5$, the sequence of vertices $v_1, u_3, v_5, u_2, v_4, u_1, v_3, u_5, v_2, u_4, v_1$ is a possible 3-SH cycle in $S_5$. For $n \geq 7$, since cycle $C_n$ is 3-SH by Theorem 1, a possible 3-SH cycle in $S_n$ is given in the proof of Theorem 2.

This completes the proof.
THEOREM 4. The line graph of $S_n$ is 3-SH if and only if $n \geq 6$.

PROOF. We denote the vertices of $G = L(S_n)$ by $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$. Then, the edge set is $\{u_iu_{i+1}, u_nv_1 : i = 1, \ldots, n-1\} \cup \{u_iv_i : i = 1, \ldots, n\} \cup \{v_iv_{i+1}, v_nv_1 : i = 1, \ldots, n-1\}$. See Figure 4 for graph $L(S_n)$.

Clearly, if $n = 3$, every vertex of $G$ is a distance at most 2 from each other so that $G$ is not 3-SH. Note that for $n = 4$ and $n = 5$, there exist isolated or pendant vertices in $D_3(G)$. Hence $D_3(G)$ is not Hamiltonian and thus $G$ is not 3-SH. Next we assume $n \geq 6$. We consider 2 cases.

**Case 1.** $n$ is odd. We consider 2 subcases.

(i) $n \equiv 0 \pmod{3}$.

A 3-SH cycle is given by $v_1, v_3, \ldots, v_{n-2}, u_1, u_4, \ldots, u_{n-2}, v_n, u_3, u_6, \ldots, u_n, v_2, u_5, u_8, \ldots, u_{n-1}, u_2, v_4, v_6, \ldots, v_{n-1}, v_1$.

(ii) $n \not\equiv 0 \pmod{3}$.

A 3-SH cycle is given by $v_1, v_3, v_5, \ldots, v_n, v_2, v_4, \ldots, v_{n-1}$ followed by $u_2, u_5, \ldots, u_{n-1}$ such that $\{2, 5, 8, \ldots, n-1\} \pmod{n}$ is a set of distinct integers and it is clear that $u_{n-1}$ is a distance 3 to $v_1$.

**Case 2.** $n$ is even. We consider 3 subcases.

(i) $n \equiv 0 \pmod{3}$.

A 3-SH cycle is given by $v_1, v_3, \ldots, v_{n-3}, u_n, v_2, v_4, \ldots, v_{n-2}, u_1, u_4, \ldots, u_{n-2}, v_n, u_3, u_6, \ldots, u_{n-3}, u_{n-1}, u_2, u_5, \ldots, u_{n-1}, v_1$. Figure 5 shows the graph $L(S_6)$ with a 3-SH labeling in it.

(ii) $n \equiv 1 \pmod{3}$.

A 3-SH cycle is given by $v_1, v_3, \ldots, v_{n-1}, u_2, u_5, \ldots, u_{n-2}, u_1, u_4, \ldots, u_n, v_2, v_4, \ldots, v_n, u_3, u_6, \ldots, u_{n-1}, v_1$.

(iii) $n \equiv 2 \pmod{3}$.

A 3-SH cycle is given by $v_1, v_3, \ldots, v_{n-1}, u_2, u_5, \ldots, u_{n-2}, u_1, u_4, \ldots, u_n, v_2, v_4, \ldots, v_n, u_3, u_6, \ldots, u_{n-2}, u_1, u_4, \ldots, u_{n-1}, v_1$. 
Figure 5: A 3-step Hamiltonian cycle for $L(S_6)$.  

This completes the proof.

**THEOREM 5.** The corona product $C_n \odot P_2$ is 3-SH if and only if $n \geq 4$.

**PROOF.** Let the vertex set and edge set of $C_n \odot P_2$ be $\{u_1, u_{i,1}, u_{i,2} : 1 \leq i \leq n\}$ and $\{u_1 u_n, u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_{i,1} u_{i,2}, u_i u_{i-1}, u_{i,2} u_i : 1 \leq i \leq n\}$, respectively. If $n = 3$, it is obvious that all $u_i$ are a distance at most 2 from all other vertices of $C_n \odot P_2$ so that $C_n \odot P_2$ is not 3-SH. We now assume that $n \geq 4$. In Figure 6, we give a 3-SH labeling for graphs $C_4 \odot P_2$ and $C_5 \odot P_2$. For $n \geq 6$, we consider 2 cases:

**Case 1.** $n \equiv 0(\text{mod } 3)$.
A sequence of vertices $u_{1,1}, u_{2,1}, \ldots, u_{n,1}, u_2, u_5, \ldots, u_{n-1}, u_{1,2}, u_3, u_6, \ldots, u_n, u_{2,2}, u_4, u_7, \ldots, u_{n-2}, u_{1}, u_{3,2}, u_{4,2}, \ldots, u_{n,2}, u_{1,1}$ is a 3-SH cycle of graph $C_n \odot P_2$.

**Case 2.** $n \not\equiv 0(\text{mod } 3)$.
A possible 3-SH cycle is given by $u_{1,1}, u_{2,1}, \ldots, u_{n,1}, u_{1,2}, u_{2,2}, \ldots, u_{n,2}$ followed by $u_2, u_5, u_8, \ldots, u_{n-1}$ such that $\{2, 5, 8, \ldots, n-1\}(\text{mod } n)$ is a set of distinct integers and we can see that $d(u_{1,1}, u_{n-1}) = 3$.

This completes the proof.
THEOREM 6. The line graph of the corona product $C_n \odot P_2$ is 3-SH if and only if $n \geq 5$.

PROOF. Let $G = L(C_n \odot P_2)$ with $V(G) = \{u_i, u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq 3\}$ and $E(G) = \{u_1u_n, u_{i,j}u_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_{i,j}u_{i,j+1}, u_{i,1}u_{i,3} : 1 \leq i \leq n, 1 \leq j \leq 2\} \cup \{u_iu_{i+1}, u_{i+1}u_{i+1}, u_{i+1}u_{i,3}, u_{i+1}u_{i,3} : 1 \leq i \leq n \text{ and } i+1 \text{ is taken modulo } n\}$. See Figure 7 for graph $L(C_3 \odot P_2)$. We consider 2 cases:

**Case 1.** $n$ is odd.

For $n = 3$, note that all $u_i$ are of degree 1 in $D_3(G)$ so that $D_3(G)$ is not Hamiltonian and thus $G$ is not 3-SH. For $n = 5$, a 3-SH cycle is given by the sequence $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}$. A 3-SH cycle is given by the sequence $u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, u_{2,3}, u_{3,1}, u_{3,2}, u_{3,3}, u_{4,1}, u_{4,2}, u_{4,3}, u_{5,1}, u_{5,2}, u_{5,3}, u_{6,1}, u_{6,2}, u_{6,3}, u_{7,1}, u_{7,2}, u_{7,3}, u_{8,1}, u_{8,2}, u_{8,3}, u_{9,1}, u_{9,2}, u_{9,3}, u_{10,1}$. We then completed the 3-SH cycle by traversing the vertices of cycle $C_n$ in the sequence $u_3, u_6, u_9, \ldots, u_n$ such that $\{3, 6, 9, \ldots, n\}(\mod n)$ is a set of distinct integers. Clearly the last vertex $u_n$ is a distance 3 from $u_{1,2}$.

**Case 2.** $n$ is even.

For $n = 4$, observe that all vertices in $\{u_i, u_{i,2} : 1 \leq i \leq 4\}$ are of degree 2 in $D_3(G)$, which by themselves forming a non-spanning cycle $C_5$, a contradiction. Hence, $D_3(G)$ is not Hamiltonian and thus $G$ is not 3-SH. For $n \geq 6$, we consider 3 subcases:

**Subcase 2.1.** $n \equiv 0(\mod 3)$.

A 3-SH cycle is given by the sequence $u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, u_{2,3}, u_{3,1}, u_{3,2}, u_{3,3}, u_{4,1}, u_{4,2}, u_{4,3}, u_{5,1}, u_{5,2}, u_{5,3}, u_{6,1}, u_{6,2}, u_{6,3}, u_{7,1}, u_{7,2}, u_{7,3}, u_{8,1}, u_{8,2}, u_{8,3}, u_{9,1}, u_{9,2}, u_{9,3}, u_{10,1}, u_{11,1}$. We then completed the 3-SH cycle by traversing the vertices of cycle $C_n$ in the sequence $u_3, u_6, u_9, \ldots, u_n$ such that $\{3, 6, 9, \ldots, n\}(\mod n)$ is a set of distinct integers. Clearly the last vertex $u_n$ is a distance 3 from $u_{1,2}$.

**Subcase 2.2.** $n \equiv 1(\mod 3)$.

A possible 3-SH cycle is started with subsequence $u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, u_{2,3}, u_{3,1}, u_{3,2}, u_{3,3}, u_{4,1}, u_{4,2}, u_{4,3}, u_{5,1}, u_{5,2}, u_{5,3}, u_{6,1}, u_{6,2}, u_{6,3}, u_{7,1}, u_{7,2}, u_{7,3}, u_{8,1}, u_{8,2}, u_{8,3}, u_{9,1}, u_{9,2}, u_{9,3}, u_{10,1}$. We then completed the 3-SH cycle by traversing the vertices of cycle $C_n$ in the sequence $u_3, u_6, u_9, \ldots, u_n$ such that $\{3, 6, 9, \ldots, n\}(\mod n)$ is a set of distinct integers. Clearly the last vertex $u_n$ is a distance 3 from $u_{1,2}$.
Figure 8: A 3-SH cycle for $L(C_6 \odot P_2)$.

$u_3, 3, u_5, 3, \ldots, u_{n-1}, 3, u_2, u_4, 3, u_6, 3, \ldots, u_{n-2}, 3, u_n, 3, u_1, 2$. In Figure 8, we give a 3-SH labeling for $L(C_6 \odot P_2)$.

**Subcase 2.2. $n \equiv 1 \pmod{3}$.**

A 3-SH cycle is given by the sequence $u_{1,2}, u_{2,1}, u_{3,2}, u_{4,1}, \ldots, u_{n-1,2}, u_{n,1}, u_{2,3}, u_{4,3}, u_{6,3}, \ldots, u_{n,3}, u_6, \ldots, u_{n-2,2}, u_{n-1,1}, u_2, u_5, \ldots, u_{n-2}, u_1, u_3, 3, u_5, 3, u_7, 3, \ldots, u_{n-1,3}, u_1, 3, u_4, u_7, \ldots, u_n, u_1, 2$.

**Subcase 2.3. $n \equiv 2 \pmod{3}$.**

A 3-SH cycle is given by the sequence $u_{1,2}, u_{2,1}, u_{3,2}, u_{4,1}, \ldots, u_{n-1,2}, u_{n,1}, u_{2,3}, u_{4,3}, u_6, \ldots, u_{n,3}, u_3, u_6, \ldots, u_{n-2}, u_1, u_2, 2, u_3, 1, u_4, 2, u_5, 1, \ldots, u_{n-1,1}, u_{n-2,2}, u_{n-1,1}, u_1, u_4, u_7, \ldots, u_{n-1,1}, u_2, u_5, \ldots, u_{n-3,3}, u_{n-3,3}, u_{n-1,3}, u_{n-1,3}, u_1, 3, u_3, 3, u_5, 3, \ldots, u_{n-3,3}, u_n, u_1, 2$.

This completes the proof.

Let $G$ be a graph and $G_1, G_2, \ldots, G_n, n \geq 2$ be $n$ copies of graph $G$. Then, the graph obtained by adding an edge from $G_i$ to $G_{i+1}$, $i = 1, 2, \ldots, n-1$ is called path union of $G$ such that the added edges connecting the same pair of vertices from $G_i$ to $G_{i+1}$. We denote path union of $n$ copies of $G$ by $P(G; n)$.

We now consider $n$ copies of cycle $C_m$, $m \geq 3$ with $C_{i,m} = (u_{i,1}, u_{i,2}, \ldots, u_{i,m})$ be the $i$-th copy of $C_m$ for $1 \leq i \leq n$. The path union of $n$ copies of $C_m$ denoted by $P(C_m; n)$, $n \geq 2$ is obtained by joining the first vertex of the $i$-th copy of $C_m$ to the last vertex of the $(i+1)$-th copy of $C_m$ for $i = 1, 2, \ldots, n-1$. See Figure 9 for graph $P(C_6; 2)$. 

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THEOREM 7. For any \( m \geq 3 \) and \( n \geq 2 \), \( P(C_m; n) \) is not 3-SH.

PROOF. Obviously the vertex set of \( P(C_m; n) \) is \( \bigcup_{i=1}^{n} V(C_{i,m}) \) and the edge set is

\[
\bigcup_{i=1}^{n} E(C_{i,m}) \cup \{u_{i,1}u_{i+1,m} : 1 \leq i \leq n-1\}.
\]

Suppose \( m = 3 \). Note that for all \( n \geq 2 \), any possible 3-SH cycle in \( P(C_m; n) \) must contain the sequence \( u_{1,2}, u_{2,2}, u_{1,3}, u_{2,1}, u_{1,2} \), a contradiction. Thus, \( P(C_m; n) \) is not 3-SH.

Suppose \( 4 \leq m \leq 6 \). Observe that, in \( D_3(P(C_m; n)) \), there exist 2 or 4 pendant vertices so that it does not have any Hamiltonian cycle and thus \( P(C_m; n) \) is not 3-SH.

Suppose \( m \geq 7 \). We consider 2 cases:

\textbf{Case 1.} \( m \equiv 0 \pmod{3} \).

Note that the vertices \( u_{1,4}, u_{1,7}, \ldots, u_{1,n-2} \) and \( u_{n,3}, u_{n,6}, \ldots, u_{n,m-3} \) are of degree 2 in \( D_3(P(C_m; n)) \) so that any possible Hamiltonian cycle in \( D_3(P(C_m; n)) \) necessarily contains the edges \( u_{1,1}u_{1,4}, u_{1,4}u_{1,7}, \ldots, u_{1,n-5}u_{1,n-2}, u_{1,n-2}u_{1,1} \) and \( u_{n,3}u_{n,6}, u_{n,6}u_{n,9}, \ldots, u_{n,m-3}u_{n,m}, u_{n,m}u_{n,3} \), forming 2 different cycles which is a contradiction. So we conclude that \( D_3(P(C_m; n)) \) is not Hamiltonian and thus \( P(C_m; n) \) is not 3-SH.

\textbf{Case 2.} \( m \not\equiv 0 \pmod{3} \).

For all \( n \geq 2 \), the following observations hold:

(i) All the vertices in the sets \( \{u_{1,4}, u_{1,7}, \ldots, u_{1,n-2}\} \), \( \{u_{n,3}, u_{n,6}, \ldots, u_{n,m-3}\} \) and \( \{u_{i,4}, u_{i,5}, \ldots, u_{i,m-3} : i \neq 1, n\} \) (when \( n \geq 3 \)) are of degree 2 in \( D_3(P(C_m; n)) \).

(ii) The vertices \( u_{1,3}, 1 \leq i \leq n-1 \) and \( u_{m-1} \) are of degree 3 in \( D_3(P(C_m; n)) \) with \( u_{1,3} \) and \( u_{m-1} \) having a common neighbor \( u_{2,m} \).

(iii) In any possible Hamiltonian cycle of \( D_3(P(C_m; n)) \), \( u_{1,1} \) and \( u_{m} \) have been traversed and no more visits available. Moreover, in \( D_3(P(C_m; n)) \), each \( u_{i,m} \), \( 1 \leq i \leq n-1 \), is adjacent to both \( u_{i,m} \) (which has one more visit available in any Hamiltonian cycle of \( D_3(P(C_m; n)) \)) and \( u_{i+1,m} \).

From (i), (ii) and (iii), it is clear that \( u_{n,m} \) is not available for \( u_{n-1,3} \) so that the remaining 2 edges incident with \( u_{n-1,3} \) are required to form Hamiltonian cycle in \( D_3(P(C_m; n)) \). The same result is then continuously applied to all other \( u_{i,3} \), \( i = n-2, n-3, \ldots, 1 \). Finally, as vertex \( u_{2,m} \) is no more available for \( u_{1,m-1} \), any possible Hamiltonian cycle in \( D_3(P(C_m; n)) \) must necessarily contain a non-spanning cycle \( u_{1,2}, u_{1,5}, u_{1,8}, \ldots, u_{1,m-2}, u_{1,1}, u_{1,4}, \ldots, u_{1,m}, u_{1,3}, u_{1,6}, \ldots, u_{1,m-1}, u_{1,2} \) for every \( m \equiv 1 \pmod{3} \), or a cycle \( u_{1,2}, u_{1,5}, u_{1,8}, \ldots, u_{1,m}, u_{1,3}, u_{1,6}, \ldots, u_{1,m-2}, u_{1,1}, u_{1,4}, \ldots, u_{1,m-1}, u_{1,2} \) for every \( m \equiv 2 \pmod{3} \), a contradiction. Therefore, \( D_3(P(C_m; n)) \) is not Hamil-
tonian and thus \( P(C_m; n) \) is not 3-SH.

This completes the proof.

From Theorem 1, we know that the cycle \( C_m \), when \( m \not\equiv 0 \pmod{3} \) admits a 3-SH cycle. Therefore, Case 2 in the above theorem shows that the path union of any \( n \) (\( n \geq 2 \)) copies of 3-SH graph is not necessarily 3-SH. But, we can construct a 3-SH graph from two graphs as follows: Suppose \( H_1 \) (respectively \( H_2 \)) is a graph of order \( n \) (respectively \( m \)) with an AL(3)-traversal given by \( u_1, u_2, \ldots, u_n \) (respectively \( v_1, v_2, \ldots, v_m \)) such that \( d(u_1, u_n) = d(v_1, v_m) = 2 \). We join the vertex \( u_1 \) to \( v_1 \) to form a 3-SH graph with the vertex sequence \( u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m, u_1 \) as the 3-SH cycle.

**THEOREM 8.** Let \( G \) be a graph of order \( n \) with an AL(3)-traversal \( u_1, u_2, \ldots, u_n \) such that \( d(u_1, u_n) = 2 \). Then, there exists a path union of two copies of \( G \), \( P(G; 2) \) which admits a 3-SH cycle.

Suppose \( G \) is a graph of order \( p \) with a 3-SH cycle given by \( u_1, u_2, \ldots, u_p, u_1 \) and \( H \) is a graph of order \( q \) with an AL(3)-traversal \( v_1, v_2, \ldots, v_q \) such that \( d(v_1, v_q) = 1 \). Since \( G \) is 3-SH, there exists a \( u_p - u_1 \) path of length 3, say \( u_p, a, b, u_1 \). Denote by \( G_{av_q} \) the graph obtained from \( G \) and \( H \) by joining the vertex \( a \) to \( v_q \).

**THEOREM 9.** The graph \( G_{av_q} \) of order \( p + q \) is 3-SH.

**PROOF.** Observe that \( d(u_p, v_1) = d(v_q, u_1) = 3 \) and thus the vertex sequence \( u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q, u_1 \) is a 3-SH cycle of \( G_{av_q} \).

**THEOREM 10.** Let \( G \) be the line graph of \( P(C_m; n) \), then

(i) \( G \) is not 3-SH for \( 3 \leq m \leq 5 \) and all \( n \geq 2 \);

(ii) \( G \) is not 3-SH for \( m \geq 6 \), \( m \equiv 0 \pmod{3} \) and \( n = 2 \);

(iii) \( G \) is 3-SH for \( m \geq 7 \), \( m \not\equiv 0 \pmod{3} \) and \( n \geq 3 \).

**PROOF.** Let \( V(G) = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n - 1 \} \cup \{u_i, j : 1 \leq i \leq n, 1 \leq j \leq m - 1 \} \) and \( E(G) = \{u_i u_{i+1}, u_i j u_{j+1}, u_i u_{i+m-1} : 1 \leq i \leq n, 1 \leq j \leq m - 2 \} \cup \{v_i, v_{i+1} : 1 \leq i \leq n - 1 \} \cup \{v_i u_{i+1}, v_{i+1} u_{i+m-1} : 1 \leq i \leq n - 1 \} \) Figure 10 shows the line graph \( L(P(C_4; 3)) \).

(i) Suppose \( m = 3 \). Clearly for \( n = 2 \), vertex \( v_1 \) is a distance at most 2 to all other vertices of \( G \) so that \( G \) is not 3-SH. For all \( n \geq 3 \), any possible 3-SH cycle in \( G \) must consist of the subcycle \( u_{1,1}, v_2, u_1, u_{2,1}, u_{1,1} \), a contradiction. Thus, \( G \) is not 3-SH. Suppose \( m = 4 \). For all \( n \geq 2 \), observe that the set of vertices \( \{u_{1,3}, u_2, u_1, u_{2,3}\} \) induce a cycle in any possible 3-SH cycle of \( G \) so that \( G \) is not 3-SH. Suppose \( m = 5 \). For all \( n \geq 2 \), there exist exactly 2 pendant vertices in \( D_3(G) \), from the first and last copy of \( C_m \), respectively. Hence, \( D_3(G) \) is not Hamiltonian and thus \( G \) is not 3-SH.
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(ii) Observe that $v_1$ is a cut-vertex in $D_3(G)$ so that it is not Hamiltonian. Hence, $G$ is not 3-SH.

(iii) A 3-SH labeling for $L(P(C_8; 5))$ and $L(P(C_7; 6))$ are given in Figure 11 and in Figure 12, respectively. For $m \geq 7$ and odd $n \geq 3$, a 3-SH cycle can be constructed in a way similar to that in $L(P(C_8; 5))$ whereas we can get a 3-SH labeling for $m \geq 7$ and even $n \geq 4$ by referring to the labeling pattern in $L(P(C_7; 6))$.

This completes the proof.

From Theorem 10, we pose the following open problem.

PROBLEM 3. Solve the 3-step Hamiltonicity of line graph of $P(C_m; n)$ for all $m \geq 3$ and $n \geq 2$. 
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References


