Riesz Idempotent And Weyl’s Theorem For
$k$-Quasi-$*$-Paranormal Operators*

Ilmi Hoxha†, Naim Latif Braha‡, Agron Tato§

Received 25 February 2018

Abstract

An operator $T$ on $H$ is called a $k$-quasi-$*$-paranormal operator if

$$\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\|,$$

for all $x \in H$, where $k$ is a natural number. First, there will be seen some spectral properties for $k$-quasi-$*$-paranormal operator, examples and inclusions. It will also be seen if $T$ is algebraically $k$-quasi-$*$-paranormal then $T$ has …nite ascent and $T$ is polaroid operator. Second, it will be shown that the Riesz idempotent $P_\mu$ of every $k$-quasi-$*$-paranormal $T$ with respect to each isolated point $\mu \neq 0$ of its spectrum $\sigma(T)$ is self-adjoint and satisfies $P_\mu(H) = \ker(T - \mu) = \ker(T - \mu)^*$, and if $\mu = 0$, then $P_\mu(H) = \ker(T^{k+1})$. Finally, it will be proved the generalized Weyl’s theorem for $f(T)$ for every $f \in Hol(\sigma(T))$, if $T$ is an algebraically $k$-quasi-$*$-paranormal operator. If $T^*$ is an algebraically $k$-quasi-$*$-paranormal then $f(T)$ satisfies $a$-Weyl’s theorem for every $f \in Hol(\sigma(T))$. Moreover, we show that if $T$ is an algebraically $k$-quasi-$*$-paranormal operator, $F$ is algebraic with $TF = FT$, then $f(T + F)$ satisfies the generalized Weyl’s theorem for all $f \in Hol(\sigma(T + F))$.

1 Introduction

Throughout this paper, let $H$ and $K$ be infinite dimensional complex Hilbert spaces with inner product $(\cdot, \cdot)$. We denote by $L(H, K)$ the set of all bounded operators from $H$ into $K$. To simplify, we put $L(H) := L(H, H)$. For $T \in L(H)$, we denote by $\ker T$ the null space and by $T(H)$ the range of $T$. The closure of a set $M$ will be denoted by $\overline{M}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all real numbers by $\mathbb{R}$ and the set of all non–negative integers by $\mathbb{N}$. An operator $T \in L(H)$, is a positive operator, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in H$.

Let $T \in L(H)$. For an operator $T$, as usual, by $T^*$ we mean the adjoint of $T$ and $|T| = (T^*T)^{\frac{1}{2}}$. An operator $T$ is said to be a hyponormal, if $|T|^2 \geq |T^*|^2$. An operator $T$

---

*Mathematics Subject Classifications: 47B20, 47B37, 47B06, 47A10.
†Department of Mathematics and Computer Sciences, University of Prishtina, Avenue "Mother Theresa " 5, Prishtinë, 10000, Kosova
‡Department of Mathematics and Computer Sciences, University of Prishtina, Avenue "Mother Theresa " 5, Prishtinë, 10000, Kosova—Corresponding author
§Department of Mathematics, Polytechnic University of Tirana, Albania
is said to be a paranormal, if \( \|T^2x\| \geq \|Tx\|^2 \) for any unit vector \( x \) in \( H \), [19]. Further, \( T \) is said to be a *-paranormal, if \( \|T^2x\| \geq \|T^*x\|^2 \) for any unit vector \( x \) in \( H \), [7].

T. Furuta, M. Ito and T. Yamazaki [20] introduced a very interesting class of bounded linear Hilbert space operators: class \( \mathcal{A} \) defined by \( |T^2| \geq |T|^2 \), and they showed that the class \( \mathcal{A} \) is a subclass of paranormal operators. I. H. Jeon and I. H. Kim [23] introduced quasi-class \( \mathcal{A} \) (i.e., \( T^*|T^2|T \geq T^*|T|^2T \)) operators as an extension of the notion of a class \( \mathcal{A} \) operators. B. P. Dugall, I. H. Jeon, and I. H. Kim [17], introduced \( \mathcal{A} \)-class \( \mathcal{A} \) operator. An operator \( T \) is said to be a \( \mathcal{A} \)-class \( \mathcal{A} \) operator, if

\[
|T^2| \geq |T^*|^2.
\]

A \( \mathcal{A} \)-class \( \mathcal{A} \) is a generalization of a hyponormal operator, [17, Theorem 1.2], and \( \mathcal{A} \)-class \( \mathcal{A} \) is a subclass of the class of \( \mathcal{A} \)-paranormal operators, [17, Theorem 1.3]. We denote the set of \( \mathcal{A} \)-class \( \mathcal{A} \) by \( \mathcal{A}^* \). J. L. Shen, F. Zuo and S. C. Yang, in [28] introduced a quasi-\( \mathcal{A} \)-class \( \mathcal{A} \) operator: An operator \( T \) is said to be a quasi-\( \mathcal{A} \)-class \( \mathcal{A} \) operator, if

\[
T^*|T^2|T \geq T^*|T^*|^2T.
\]

We denote the set of quasi-\( \mathcal{A} \)-class \( \mathcal{A} \) by \( Q(\mathcal{A}^*) \).

### 2 Definition and Example

**Definition 2.1** ([21]). An operator \( T \in L(H) \) is called a \( k \)-quasi-\( \mathcal{A} \)-paranormal operator if

\[
\|T^*T^kx\|^2 \leq \|T^k+2x\|\|T^kx\|,
\]

for all \( x \in H \), where \( k \) is a natural number.

This class of the operators, also is defined in paper [25]. If \( T \) is \( k \)-quasi-\( \mathcal{A} \)-paranormal operator then \( T \) is a \((k+1)\)-quasi-\( \mathcal{A} \)-paranormal operator. The inverse is not true, as it can be seen below.

**Example 2.2.** Consider the unilateral weighted shift operators as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of a positive numbers \( \alpha : \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots \) (called weights) the unilateral weighted shift \( W_\alpha \) associated with weight \( \alpha \) is the operator on \( H = l_2 \) defined by \( W_\alpha e_n = \alpha_n e_{n+1} \) for all \( n \geq 1 \), where \( \{e_n\}_{n=1}^\infty \) is the canonical orthonormal basis on \( l_2 \).

\[
W_\alpha = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
\alpha_1 & 0 & 0 & 0 & \ldots \\
0 & \alpha_2 & 0 & 0 & \ldots \\
0 & 0 & \alpha_3 & 0 & \ldots \\
0 & 0 & 0 & \alpha_4 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]

It is well known that the following assertions are equivalent:
1. $W_\alpha$ is a \(*\)-paranormal operator,

2. $W_\alpha$ is a \(*\)-class $A$ operator,

3. $\alpha_n^2 \leq \alpha_{n+1}\alpha_{n+2}$ for all $n \geq 1$.

From [21, Proposition 2.1], $W_\alpha$ is a $k$-quasi-$*$-paranormal operator, if and only if,

$$W_\alpha^{*k}(W_\alpha^{*2}W_\alpha^{*2} - 2\lambda W_\alpha W_\alpha^{*} + \lambda^2)W_\alpha^{*k} \geq O$$

for all $\lambda \in \mathbb{R}$.

Let $\text{diag}(\{\alpha_n\}_{n=1}^\infty) = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \ldots)$ denote an infinite diagonal matrix on $l_2$. Then,

$$W_\alpha^{*k} (W_\alpha^{*2}W_\alpha^{2} - 2\lambda W_\alpha W_\alpha^{*} + \lambda^2)W_\alpha^{k}$$

$$= \text{diag}(\{\alpha_n^2 \alpha_{n+1}^2 \ldots \alpha_{n+k-1}^2 \alpha_{n+k+1}^2\}_{n=1}^{\infty}) - 2\lambda \text{diag}(\{\alpha_n^2 \alpha_{n+1}^2 \ldots \alpha_{n+k-2}^2 \alpha_{n+k-1}^2\}_{n=1}^{\infty})$$

$$+ \lambda^2 \text{diag}(\{\alpha_n^2 \alpha_{n+1}^2 \ldots \alpha_{n+k-1}^2\}_{n=1}^{\infty})$$

Thus, $W_\alpha$ is a $k$-quasi-$*$-paranormal operator, if and only if,

$$\alpha_n^2 \alpha_{n+1}^2 \ldots \alpha_{n+k-1}^2 (\alpha_{n+k}^2 \alpha_{n+k+1}^2 - 2\lambda \alpha_{n+k-1}^2 + \lambda^2) \geq 0,$$

for all $\lambda \in \mathbb{R}$, and $n \geq 1$. Equivalently

$$\alpha_{n+k-1}^2 \leq \alpha_{n+k}\alpha_{n+k+1}$$

for all $n \geq 1$.

If $\alpha_{k+1} \leq \alpha_{k+2} \leq \alpha_{k+3} \leq \alpha_{k+4} \leq \ldots$ and $\alpha_k > \alpha_{k+1}$, then $W_\alpha$ is a $(k+1)$-quasi-$*$-paranormal but is not a $k$-quasi-$*$-paranormal operator.

We write $r(T) = \lim_{n \to \infty} \|T^n\|^\frac{1}{n}$ for the spectral radius. It is well known that $r(T) \leq \|T\|$, for every $T \in L(H)$. The operator $T$ is called normaloid operator if $r(T) = \|T\|$. It is well known that a \(*\)-paranormal operator is normaloid [7, Theorem 1.1], and a quasi-$*$-paranormal is normaloid, but a $k$-quasi-$*$-paranormal operator for $k \geq 2$ is not normaloid operator: if $\alpha_1 > \alpha_2$ and $\alpha_2 = \alpha_3 = \ldots = \alpha_k = \alpha_{k+1} = \ldots$, then

$$\|T\| = \alpha_1 \text{ and } r(T) = \lim_{n \to \infty} \|T^n\|^\frac{1}{n} = \alpha_2.$$

THEOREM 2.3. Let $T \in L(H)$ be a $k$-quasi-$*$-paranormal operator for a positive integer $k$. Then the following assertions hold.

1. $\|T^nT^n\|^2 \leq \|T^{n+2}\|\|T^n\|$ for all positive integers $n \geq k$,

2. If $\|T^n\| = \|T\|^n$ for some positive $n \geq k$, then $T$ is normaloid operator.

PROOF. 1). Since $k$-quasi-$*$-paranormal operators are $(k+1)$-quasi-$*$-paranormal operators, we only need to prove the case $n = k$. It is clear by the definition of $k$-quasi-$*$-paranormal operators.
2). Let \( n \geq k \). From 1) and using \( \|T^n\| = \|T\|^n \) we have

\[
\|T\|^4n = \|T^n\|^4 = \|T^{*(n-1)}T^*T^n\|^2 \leq \|T^{*(n-1)}\|^2\|T^*T^n\|^2 \leq \|T\|^{2(n-1)}\|T^{n+2}\|\|T^n\| = \|T\|^{2(n-1)}\|T^{n+2}\|\|T^n\|. 
\]

Therefore, \( \|T\|^{n+2} \leq \|T^{n+2}\| \) so \( \|T\|^{n+1} = \|T^{n+1}\| \). Thus by induction we have \( \|T\|^{n} = \|T^n\| \), for all \( n \in \mathbb{N} \), hence \( T \) is normaloid operator.

DEFINITION 2.4. An operator \( T \) is called an algebraically \( k \)-quasi-\( * \)-paranormal operator, if there exists a nonconstant complex polynomial \( h(z) \) such that \( h(T) \) is a \( k \)-quasi-\( * \)-paranormal.

If \( T \) is a \( k \)-quasi-\( * \)-paranormal operator, then \( T \) is an algebraically \( k \)-quasi-\( * \)-paranormal operator. But the inverse is not true, as shown by the example below.

EXAMPLE 2.5. Let

\[
T = \begin{pmatrix} I & O \\ I & I \end{pmatrix} \in L(l_2 \oplus l_2).
\]

Since \( T^* = \begin{pmatrix} I & I \\ O & I \end{pmatrix} \),

\[
T^* \left( T^2T^2 - 2\lambda TT^* + \lambda^2 \right) T^2 = \begin{pmatrix} (17 - 26\lambda + 5\lambda^2)I & (4 - 10\lambda + 2\lambda^2)I \\ (4 - 10\lambda + 2\lambda^2)I & (1 - 4\lambda + \lambda^2)I \end{pmatrix}.
\]

For \( \lambda = 1 \), \( (17 - 26\lambda + 5\lambda^2)I \) is not a positive operator, thus

\[
T^* \left( T^2T^2 - 2\lambda TT^* + \lambda^2 \right) T^2 \not\geq O
\]

for all \( \lambda \in \mathbb{R} \). Therefore \( T \) is not a 2-quasi-\( * \)-paranormal operator.

On the other hand, consider the non constant complex polynomial \( h(z) = (z - 1)^2 \). Then \( h(T) = O \), and hence \( T \) is an algebraically 2-quasi-\( * \)-paranormal operator.

LEMMA 2.6 ([13, Holder-McCarthy inequality]). Let \( T \) be a positive operator. Then, the following inequalities hold for all \( x \in H \):

1. \( \langle T^r x, x \rangle \leq \langle Tx, x \rangle^r \|x\|^{2(1-r)} \) for \( 0 < r < 1 \),

2. \( \langle T^r x, x \rangle \geq \langle Tx, x \rangle^r \|x\|^{2(1-r)} \) for \( r \geq 1 \).

LEMMA 2.7. If \( T \) is a \( k \)-quasi-\( * \)-class \( \mathcal{A} \) operator, then \( T \) is a \( k \)-quasi-\( * \)-paranormal operator.
PROOF. Let $T$ be a $k$-quasi-$*$-class $A$ operator. From Holder-McCarthy inequality we have
\[
\|T^*T^k x\|^2 = \langle T^k|T^*|^2T^k x, x \rangle \leq \langle |T^2|T^k x, T^k x \rangle \\
\leq \langle |T^2|^2T^k x, T^k x \rangle^{\frac{1}{2}} \|T^k x\| \|T^{k+2} x\| \|T^k x\|.
\]
So $T$ is a $k$-quasi-$*$-paranormal operator.

LEMMA 2.8. Let $S = \oplus_{n=1}^{\infty} H_n$, where $H_n \cong \mathbb{R}^2$. For given positive operators $A, B$ on $\mathbb{R}^2$ and for any fixed $n \in \mathbb{N}$, the operator $T = T_{A,B}$ on $S$ is defined as follows:
\[
T(x_1, x_2, \ldots) = (0, Ax_1, Bx_2, Bx_3, Bx_4, \ldots),
\]
and the adjoint operator of $T$ is
\[
T^*(x_1, x_2, \ldots) = (Ax_2, Bx_3, Bx_4, Bx_5, \ldots).
\]
Then
1. The operator $T_{A,B}$ is a quasi-$*$-class $A$ operator, if and only if,
\[
AB^2 A \geq A^4,
\]
2. The operator $T_{A,B}$ is a quasi-$*$-paranormal operator, if and only if,
\[
A(B^4 - 2\lambda A^2 + \lambda^2) A \geq O, \text{ for all } \lambda \in \mathbb{R}.
\]


Take $A$ and $B$ as
\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2}} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}^{\frac{1}{2}}.
\]
Then
\[
A(B^2 - A^2) A = \begin{pmatrix} -0.3359... & -0.2265... \\ -0.2265... & 0.8244... \end{pmatrix} \not\geq O,
\]
hence $T_{A,B}$ is not a quasi-$*$-class $A$ operator. But,
\[
A(B^4 - 2\lambda A^2 + \lambda^2) A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2}} \left( \begin{pmatrix} 1 - \lambda \\ 2(1 - \lambda) \end{pmatrix} \begin{pmatrix} 2(1 - \lambda) & \lambda^2 - 4\lambda + 8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2}} \right) \geq O,
\]
so, $T_{A,B}$ is a quasi-$*$-paranormal operator.
3 Spectral Properties

We write \( \sigma(T) \), \( \sigma_p(T) \) and \( \sigma_a(T) \) for the spectrum, point spectrum and approximate point spectrum, respectively. Sets of isolated points and accumulation points of \( \sigma(T) \) are denoted by \( \text{iso}(T) \) and \( \text{acc}(T) \), respectively.

A complex number \( \mu \) is said to be in the point spectrum \( \sigma_p(T) \) of \( T \) if there is a nonzero \( x \in H \) such that \((T - \mu)x = 0 \). If in addition, \((T - \mu)^*x = 0 \), then \( \mu \) is said to be in the joint point spectrum \( \sigma_{jp}(T) \) of \( T \). Clearly \( \sigma_{jp}(T) \subseteq \sigma_p(T) \). In general \( \sigma_{jp}(T) \neq \sigma_p(T) \).

**Lemma 3.1** ([21, Proposition 3.1]). If \( T \) is a \( k \)-quasi-\( s \)-paranormal operator and \((T - \mu)x = 0 \), then \((T - \mu)^*x = 0 \) for all \( \mu \neq 0 \).

A complex number \( \mu \) is said to be in the approximate point spectrum \( \sigma_a(T) \) of \( T \) if there is a sequence \( \{x_m\}_{m=1}^\infty \) of unit vectors satisfying \((T - \mu)x_m \to 0 \), as \( m \to \infty \). If in addition \((T - \mu)^*x_m \to 0 \), as \( m \to \infty \), then \( \mu \) is said to be in the joint approximate point spectrum \( \sigma_{ja}(T) \) of operator \( T \). Clearly \( \sigma_{ja}(T) \subseteq \sigma_a(T) \). In general \( \sigma_{ja}(T) \neq \sigma_a(T) \).

**Theorem 3.2.** Let \( T \) be a \( k \)-quasi-\( s \)-paranormal operator, and \((T - \mu)x_m \to 0 \), as \( m \to \infty \) for \( \mu \neq 0 \). Then \((T - \mu)^*x_m \to 0 \), as \( m \to \infty \).

**Proof.** Let \( T \) be a \( k \)-quasi-\( s \)-paranormal operator and \((T - \mu)x_m \to 0 \), as \( m \to \infty \). We may assume that \( \|x_m\| = 1 \). By the assumption and using

\[
T^k = (T - \mu + \mu)^k = \sum_{i=1}^{k} \binom{k}{i} \mu^{k-i} (T - \mu)^i + \mu^k, \quad \text{for } k \in \mathbb{N},
\]

we have \((T^k - \mu^k)x_m \to 0 \), as \( m \to \infty \). By

\[
\|T^k x_m\| - |\mu|^k \leq \|(T^k - \mu^k)x_m\|
\]

hence

\[
\|T^k x_m\| \to |\mu|^k, \quad \text{as } m \to \infty. \tag{1}
\]

Moreover

\[
\|T^* \mu^k x_m\| - \|T^*(T^k - \mu^k)x_m\| \leq \|T^* T^k x_m\|. \tag{2}
\]

Since \( T \) is a \( k \)-quasi-\( s \)-paranormal operator, we have

\[
\|T^* T^k x_m\| \leq \|T^{2+k} x_m\|^\frac{1}{2} \|T^k x_m\|^\frac{1}{2}. \tag{3}
\]

Then it follows from (1), (2) and (3) that

\[
\limsup_{m \to \infty} \|T^* x_m\| \leq |\mu|.
\]
Since
\[ \| (T - \mu)^* x_m \|^2 = \| T^* x_m \|^2 - 2\Re \langle T^* x_m, \overline{\mu} x_m \rangle + |\mu|^2 \| x_m \|^2 = \| T^* x_m \|^2 - 2\Re \langle x_m, \overline{\mu} T x_m \rangle + |\mu|^2 \| x_m \|^2, \]
we see that
\[ \limsup_{m \to \infty} \|(T - \mu)^* x_m \|^2 \leq |\mu|^2 - |\mu|^2 = 0. \]
This implies $(T - \mu)^* x_m \to 0$, as $m \to \infty$.

**COROLLARY 3.3.** If $T$ is a $k$-quasi-\ast-paranormal operator, then $\sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$.

**LEMMA 3.4** ([5, Corollary 2]). Let $T = U|T|$ be the polar decomposition of $T$, $\mu = |\mu|e^{i\theta} \neq 0$ and $\{x_m\}$ a sequence of vectors. Then the following assertions are equivalent:

1. $(T - \mu)x_m \to 0$ and $(T^* - \overline{\mu})x_m \to 0$, as $m \to \infty$,
2. $(|T| - |\mu|)x_m \to 0$ and $(U - e^{i\theta})x_m \to 0$, as $m \to \infty$,
3. $(|T^*| - |\mu|)x_m \to 0$ and $(U^* - e^{-i\theta})x_m \to 0$, as $m \to \infty$.

**COROLLARY 3.5.** If $T$ is a $k$-quasi-\ast-paranormal operator and $\mu \in \sigma_a(T) \setminus \{0\}$, then $|\mu| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.

**PROOF.** If $\mu \in \sigma_a(T) \setminus \{0\}$, then by Theorem 3, there exists a sequence of unit vectors $\{x_m\}$ such that $(T - \mu)x_m \to 0$ and $(T - \mu)^* x_m \to 0$, as $m \to \infty$. Hence, from Lemma 3 we have $|\mu| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.

**COROLLARY 3.6.** Let $T$ be a $k$-quasi-\ast-paranormal operator and $T = U|T|$ is the polar decomposition of $T$. If $\mu = |\mu|e^{i\theta} \neq 0$ and $\mu \in \sigma_a(T)$, then $e^{i\theta} \in \sigma_{ja}(U)$.

**PROOF.** Let $\mu \in \sigma_a(T)$. From Corollary 3, $\mu \in \sigma_{ja}(T)$. Then there exists a sequence of unit vectors $\{x_m\}$ such that $(T - \mu)x_m \to 0$ and $(T - \mu)^* x_m \to 0$, as $m \to \infty$. From Lemma 3 we have $(U - e^{i\theta})x_m \to 0$ and $(U^* - e^{-i\theta})x_m \to 0$, as $m \to \infty$. Thus $e^{i\theta} \in \sigma_{ja}(U)$.

**LEMMA 3.7** ([21, Proposition 2.4]). Let $T \in L(H)$ be a $k$-quasi-\ast-paranormal operator, the range of $T^k$ not to be dense, and
\[ T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \quad H = \overline{T^k(H)} \oplus \ker T^k. \]

Then, $A$ is a \ast-paranormal on $\overline{T^k(H)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.
THEOREM 3.8. Let $T$ be a $k$-quasi-*$\sigma$-paranormal operator and $\sigma(T) = \{\mu\}$. Then $T = \mu$ if $\mu \neq 0$, and $T^{k+1} = O$ if $\mu = 0$.

PROOF. Let’s suppose that $T$ is a $k$-quasi-*$\sigma$-paranormal operator. We can consider two cases:

Case I: If $\mu \neq 0$, the range of $T^k$ is dense, then it is a $*$-paranormal operator. Hence by [30, Corollary 1], $T = \mu$.

Case II: If $\mu = 0$, $T$ does not have dense range, by Lemma 3 we can represent $T$ as the upper triangular matrix

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

on $H = \overline{h(T)}^{k}(H) \oplus \ker T^{*k}$.

From the assumption, $\sigma(T) = \{0\}$ and from Lemma 3 we have $\sigma(A) = \{0\}$. Since $A$ is a $*$-paranormal operator, $A = O$ and we have

$$T^{k+1} = \begin{pmatrix} O & BC^k \\ O & C^{k+1} \end{pmatrix} = O.$$

THEOREM 3.9. If $T$ is a quasinilpotent algebraically $k$-quasi-*$\sigma$-paranormal operator, then $T$ is a nilpotent operator.

PROOF. Let $T$ be an algebraically $k$-quasi-*$\sigma$-paranormal operator. Then, there exists a nonconstant polynomial $h(z)$ such that $h(T)$ is a $k$-quasi-*$\sigma$-paranormal operator. If $h(T)^k(H)$ is dense, $h(T)$ is a $*$-paranormal operator. Therefore $T$ is an algebraically $*$-paranormal operator and by [35, Theorem 2.6] $T$ is a nilpotent operator. If $h(T)^k(H)$ is not dense, by Lemma 3 we have

$$h(T) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

on $H = \overline{h(T)}^k(H) \oplus \ker h(T)^*k$,

where $A$ is a $*$-paranormal operator on $\overline{h(T)}^k(H)$, $C^k = O$ and $\sigma(h(T)) = \sigma(A) \cup \{0\}$. Since $T$ is a quasinilpotent operator, $\sigma(h(T)) = h(\sigma(T)) = h(0)$. Therefore $\sigma(A) = \{0\}$, thus $\sigma(h(T)) = \{0\}$. Since $h(0) = 0$, we have $h(T) = aT^k \prod_{i=1}^n (T - \mu_i)$ for some natural number $k$ and a complex number $\mu_i$, $i = 1, 2, ..., n$. By Theorem 3 we have

$$a^{k+1}T^{k(k+1)} \prod_{i=1}^n (T - \mu_i)^{k+1} = O.$$

Since $\sigma(T) = \{0\}$, $T - \mu_i$ is an invertible for all $i = 1, 2, ..., n$, we see that

$$T^{k(k+1)} = O.$$

For $T \in L(H)$, the smallest nonnegative integer $p$ such that $\ker T^p = \ker T^{p+1}$ is called the ascent of $T$ and is denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. We say that $T \in L(H)$ is of finite ascent if $p(T - \mu) < \infty$, for all $\mu \in \mathbb{C}$. For $T \in L(H)$, the smallest nonnegative integer $q$, such that $T^q(H) = T^{q+1}(H)$, is called the descent.
Hence \( q(T) \) is a nonconstant polynomial such that \( h(T) \) is a \( k \)-quasi-\( * \)-paranormal operator and we have

\[
h(T) - h(\mu) = a(T - \mu)^k \prod_{i=1}^{n} (T - \mu_i),
\]

where \( a \neq 0 \), \( \mu_i \neq \mu \) and integers \( k \) and \( n \). Let \( x \neq 0 \). We consider two cases:

1. If \( x \in \ker(T - \mu)^{k+1} \) and \( h(\mu) \neq 0 \), we have

\[
(h(T) - h(\mu))x = a(T - \mu)^k \prod_{i=1}^{n} (T - \mu + \mu - \mu_i)x = a \prod_{i=1}^{n} (\mu - \mu_i)(T - \mu)^k x. \tag{4}
\]

Hence

\[
(h(T) - h(\mu))^2 x = a^2 \prod_{i=1}^{n} (\mu - \mu_i)^2 (T - \mu)^{2k} x = 0.
\]

From [21, Proposition 3.1] we have \( x \in \ker(h(T) - h(\mu))^2 = \ker(h(T) - h(\mu)) \). Hence \( (h(T) - h(\mu))x = 0 \), and from relation (4) we have \((T - \mu)^k x = 0 \), so \( x \in \ker(T - \mu)^k \).

2. If \( h(\mu) = 0 \) we have

\[
h(T)^k x = a^k \prod_{i=1}^{n} (\mu - \mu_i)^k (T - \mu)^k x = b^{-k}(T - \mu)^{k^2} x. \tag{5}
\]

and

\[
\| (T - \mu)^{k^2} x \|^4 = \langle (T - \mu)^{k^2} x, (T - \mu)^{k^2} x \rangle^2 = \langle b^k h(T)^k x, b^k h(T)^k x \rangle^2 = |b|^{4k} \langle h(T)^k h(T)^k x, h(T)^{k-1} x \rangle^2 \leq |b|^{4k} \|h(T)^k h(T)^k x\|^2 \|h(T)^{k-1} x\|^2 \leq |b|^{4k} \|h(T)^{k+2} x\|^2 \|h(T)^{k-1} x\|^2 \|h(T)^k x\|^2 = |b|^{k-1} \|(T - \mu)^{k^2 + 2k} x\| \|(T - \mu)^{k^2 - k} x\| \|(T - \mu)^k x\| = 0.
\]

So,

\[
\| (T - \mu)^{k^2} x \|^3 \leq |b|^{k-1} \|(T - \mu)^{k^2 + 2k} x\| \|(T - \mu)^{k^2 - k} x\|.
\]

If \( x \in \ker(T - \mu)^{k^2 + 1} \), therefore \( \ker(T - \mu)^{k^2} = \ker(T - \mu)^{k^2 + 1} \).

Let \( \text{Hol}(\sigma(T)) \) be the space of all analytic functions in an open neighborhood of \( \sigma(T) \). We say that \( T \in L(H) \) has the single valued extension property at \( \mu \in \mathbb{C} \), if for
every open neighborhood $U$ of $\mu$ the only analytic function $f : U \to \mathbb{C}$ which satisfies the equation $\left( T - \mu \right) f(\mu) = 0$, is the constant function $f \equiv 0$. The operator $T$ is said to have SVEP if $T$ has SVEP at every $\mu \in \mathbb{C}$.

**COROLLARY 3.11.** If $T \in L(H)$ is an algebraically $k$-quasi-$*$-paranormal operator, then $T$ has SVEP.

**PROOF.** The proof of the corollary follows directly from Theorem 3 and [1, Theorem 3.39].

The quasinilpotent part $\mathcal{H}_0(T - \mu)$ and analytic core $K(T - \mu)$ of $T - \mu$ are defined by

$$\mathcal{H}_0(T - \mu) = \{ x \in H : \lim_{n \to \infty} \|(T - \mu)^n x\|^\frac{1}{n} = 0 \},$$

and

$$K(T - \mu) = \{ x \in H : \text{there exists a sequence } \{ x_n \} \subset H \text{ and } \delta > 0 \text{ for which } x = x_0, (T - \mu)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \ldots \}. $$

Clearly $\mathcal{H}_0(T - \mu)$ and $K(T - \mu)$ are linear subspaces of $H$, in general $\mathcal{H}_0(T - \mu)$ and $K(T - \mu)$ are non-closed hyperinvariant subspaces of $T - \mu$, such that $\ker(T - \mu) \subseteq \mathcal{H}_0(T - \mu)$.

An operator $T$ is said to be a semi-regular if $T(H)$ is a closed subspace and $\ker T \subseteq \cap_{n \in \mathbb{N}} T^n(H)$. An operator $T$ admits a generalized Kato decomposition, if there exists a pair of $T$-invariant closed subspaces $(M, N)$ such that $H = M \oplus N$, the restriction $T|_M$ is a quasinilpotent and $T|_N$ is a semi-regular operator. If $T|_M$ is a nilpotent, we say $T$ is a Kato type.

An operator $T$ is said to be isolid operator if every isolated point of $\sigma(T)$ is an eigenvalue of $T$, while an operator $T$ is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of $T$. In general, if $T$ is polaroid operator, then $T$ is isolid operator. However, the converse is not true.

**THEOREM 3.12.** If $T$ is an algebraically $k$-quasi-$*$-paranormal operator, then $T$ and $T^*$ are polaroid operator.

**PROOF.** Let $\mu \in \text{iso}\sigma(T)$. From [2, Theorem 3.76] we have $H = \mathcal{H}_0(T - \mu) \oplus K(T - \mu)$, where $\mathcal{H}_0(T - \mu)$ and $K(T - \mu)$ are closed subspaces. By [1, Theorem 1.28], $(T - \mu)(T - \mu) = K(T - \mu)$ is a closed subspace and $\ker(T - \mu) \subseteq \cap_{n \in \mathbb{N}} (T - \mu)^n(K(T - \mu))$, thus $(T - \mu)|_{K(T - \mu)}$ is a semi-regular operator. We have $\sigma(T|_{\mathcal{H}_0(T - \mu)}) = \{ \mu \}$, then $\sigma((T - \mu)|_{\mathcal{H}_0(T - \mu)}) = \{ 0 \}$, thus $(T - \mu)|_{\mathcal{H}_0(T - \mu)}$ is quasinilpotent operator. Therefore $T - \mu$ admits a generalized Kato decomposition. But, $T - \mu$ is an algebraically $k$-quasi-$*$-paranormal operator, by Theorem 3 $(T - \mu)|_{\mathcal{H}_0(T - \mu)}$ is a nilpotent operator, thus $T - \mu$ admits a Kato type. Since $\sigma(T)$ does not cluster at $\mu$, then $T$ and $T^*$ have the SVEP in $\mu$. From [1, Theorem 2.45 and Theorem 2.46] we have $p(T - \mu) < \infty$ and $q(T - \mu) < \infty$. Hence $\mu$ is a pole of the resolvent of $T$, so $T$ is polaroid operator, therefore $T$ is isolid operator. From [5, Theorem 2.5], $T^*$ is polaroid operator.
An operator $T$ is called $a$-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of $T$. An operator $T$ is called $a$-polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of $T$. Clearly, if $T$ is $a$-polaroid, then $T$ is $a$-isoloid. However, the converse is not true.

LEMMA 3.13. Suppose $T^*$ is an algebraically $k$-quasi-$*$-paranormal operator. Then $T$ is $a$-polaroid.

PROOF. Let $\mu$ be an isolated point of $\sigma_a(T)$. Since $T^*$ has SVEP, by [1, Corollary 2.28] $\mu$ is an isolated point of $\sigma(T)$. But, if $T^*$ is polaroid, then $T$ is also polaroid. Therefore, $T$ is $a$-polaroid operator.

4 Riesz Idempotent for $k$-Quasi-$*$-Paranormal Operator

The Riesz idempotent $P_\mu$ of an operator $T$ with respect to an isolated point $\mu$ of $\sigma(T)$ is defined by
\[
P_\mu = \frac{1}{2\pi i} \int_{\partial D_\mu} (z - T)^{-1} \, dz,
\]
where the integral is taken in the positive direction and $D_\mu$ is a closed disk centered at $\mu$ with a small enough radius $r$ such as $D_\mu \cap \sigma(T) = \{\mu\}$. Then, it is well known that $P_\mu^2 = P_\mu$, $TP_\mu = P_\mu T$, $\sigma(T|_{\mathcal{H}}) = \{\mu\}$ and $\sigma(T|_{(I-P_\mu)(\mathcal{H})}) = \sigma(T) \setminus \{\mu\}$.

In general, it is well known that the Riesz idempotent $P_\mu$ is not an orthogonal projection, and a necessary and sufficient condition for $P_\mu$ to be orthogonal is that $P_\mu$ is self-adjoint, [15]. For a hyponormal operator in [29], Stampfli has shown that the Riesz idempotent for an isolated point of spectrum of $T$ is self-adjoint and $P_\mu(H) = \ker(T - \mu) = \ker(T - \mu)^*$.

In [31], Uchiyama extended this result for the class $\mathcal{A}$ with respect $\mu \neq 0$ and he proved that in general, the Riesz idempotent of the class $\mathcal{A}$ with respect to $0$ is not self-adjoint and $\ker T \neq \ker T^*$. In [22], Jeon and Kim extended this result for $\mu \neq 0$ in quasi-class $\mathcal{A}$. Also, in [24], Mecheri extended this result for $\mu \neq 0$ in $k$-quasi-$*$-class $\mathcal{A}$ operators. In this paper, we extended this result for $k$-quasi-$*$-paranormal operator.

THEOREM 4.1. Let $T \in L(H)$ be a $k$-quasi-$*$-paranormal operator for the positive integer $k$, and let $\mu$ be an isolated point of $\sigma(T)$, and $P_\mu$ the Riesz idempotent for $\mu$. Then, the following assertions hold:

1. If $\mu \neq 0$, $P_\mu(H) = \ker(T - \mu) = \ker(T - \mu)^*$, and $P_\mu$ is self-adjoint.

2. If $\mu = 0$, then $P_\mu(H) = \ker(T^{k+1})$

PROOF. 1). Let $T$ be a $k$-quasi-$*$-paranormal operator and $\mu \neq 0 \in \text{iso}\sigma(T)$. From Theorem 3 $\mu$ is an eigenvalue of $T$, thus $(T - \mu)x = 0$, for every $x \neq 0 \in H$. Then $x \in$
Theorem 3, that $P_{Hoxha et al.}$ $91$

(5) Generalized Weyl’s Theorem for an upper semi-Fredholm, if it has a closed range and $\alpha(T) > \infty$, while $T$ is called a

ker$(T - \mu)^m = P_\mu(H)$, hence ker$(T - \mu) \subseteq P_\mu(H)$. On the other hand, $\sigma(T|_{P_\mu(H)}) = \{\mu\}$.

From [21, Proposition 2.2], $T|_{P_\mu(H)}$ is a $k$-quasi-*-paranormal operator and by Theorem 3, $T|_{P_\mu(H)} = \mu$. If $x \in P_\mu(H)$, then $Tx = \mu x$, hence $x \in$ ker$(T - \mu)$. Therefore $P_\mu(H) = \ker(T - \mu)$.

Next, we show that ker$(T - \mu) = \ker(T - \mu)^*$. Since $P_\mu(H) = \ker(T - \mu)$, we have ker$(T - \mu)$ is a reducing subspace of $T$ and $T$ can be written as follows

$$T = \mu \oplus T_1 \text{ on } H = \ker(T - \mu) \oplus \ker(T - \mu)^\perp,$$

where $T_1$ is a $k$-quasi-**-paranormal operator and $\sigma(T) = \{\mu\} \cup \sigma(T_1)$.

If $\mu \in \sigma(T_1)$ then $\mu$ is an isolated point of $\sigma(T_1)$. Since $T_1$ is a $k$-quasi-**-paranormal operator, $\mu \in \sigma_p(T_1)$, thus ker$(T - \mu) \not\subseteq \{0\}$. From ker$(T_1 - \mu) \subseteq$ ker$(T - \mu)$, and ker$(T_1 - \mu) \subseteq$ ker$(T - \mu)^\perp$, we have:

$$\{0\} \not\subseteq \ker(T_1 - \mu) \subseteq \ker(T - \mu) \cap \ker(T - \mu)^\perp = \{0\},$$

which is a contradiction. Thus $\mu \not\in \sigma(T_1)$ and $T_1 - \mu$ is invertible in ker$(T - \mu)^\perp$. Therefore $(T - \mu)(\ker(T - \mu)^\perp) = \ker(T - \mu)^\perp$, so ker$(T - \mu)^\perp \subseteq (T - \mu)(H)$. By Lemma 3 we have ker$(T - \mu) \subseteq$ ker$(T - \mu)^* = (T - \mu)(H)^\perp$, therefore

$$(T - \mu)(H) \subseteq \ker(T - \mu)^\perp \subseteq (T - \mu)(H).$$

Thus $(T - \mu)(H) = \ker(T - \mu)^\perp$, which implies that

$$\ker(T - \mu)^* = (T - \mu)(H)^\perp = \ker(T - \mu).$$

Finally, we show that $P_\mu$ is a self-adjoint operator. From

$$P_\mu(H) = \ker(T - \mu) = \ker(T - \mu)^*,$$

we have $T|_{P_\mu(H)} = \mu$. Thus, $((z - T)^*)^{-1}P_\mu = (z - \mu)^{-1}P_\mu$ and we have

$$P_\mu^*P_\mu = -\frac{1}{2\pi i} \int_{\partial D_\mu} ((z - T)^*)^{-1}P_\mu dz = -\frac{1}{2\pi i} \int_{\partial D_\mu} (z - \mu)^{-1}P_\mu dz = \frac{1}{2\pi i} \int_{\partial D_\mu} (z - \mu)^{-1}dzP_\mu = P_\mu.$$

So $P_\mu^*P_\mu = P_\mu = P_\mu^2$, thus $P_\mu^* = P_\mu$.

2). Since ker$T^k \subseteq P_0(H)$, we have to prove that $P_0(H) \subseteq ker(T^{k+1})$. It is known that $P_0(H)$ is an invariant subspace of $T$ and $\sigma(T|_{P_0(H)}) = \{0\}$. From Theorem 3 we have $(T|_{P_0(H)})^{k+1} = T^{k+1}|_{P_0(H)} = O$. This implies $P_0(H) \subseteq ker(T^{k+1})$.

5 Generalized Weyl’s Theorem for $k$-Quasi-**-Paranormal Operator

We write $\alpha(T) = \dim$ker$T$ and $\beta(T) = \dim (H/T(H))$. An operator $T \in L(H)$ is called an upper semi-Fredholm, if it has a closed range and $\alpha(T) < \infty$, while $T$ is called a
lower semi-Fredholm if $\beta(T) < \infty$. However, $T$ is called a semi-Fredholm operator, if $T$ is either an upper or a lower semi-Fredholm, and $T$ is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in L(H)$ is semi-Fredholm, then the index is defined by

$$\text{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in L(H)$ is said to be an upper semi-Weyl operator if it is an upper semi-Fredholm and $\text{ind}(T) \leq 0$, while $T \in L(H)$ is said to be a lower semi-Weyl operator if it is a lower semi-Fredholm operator and $\text{ind}(T) \geq 0$. An operator is said to be a Weyl operator if it is a Fredholm operator of index zero. The Weyl spectrum and the essential approximate spectrum are defined by

$$\sigma_w(T) = \{\mu \in \mathbb{C} : T - \mu \text{ is not Weyl}\}$$

and

$$\sigma_{\text{aw}}(T) = \{\mu \in \mathbb{C} : T - \mu \text{ is not upper semi-Weyl}\}.$$

For $T \in L(H)$ and a nonnegative integer $n$ we define $T[n]$ to be the restriction of $T$ to $T^n(H)$ viewed as a map from $T^n(H)$ into $T^n(H)$, (in particular $T[0] = T$.)

**DEFINITION 5.1** ([11]). We say that $T \in L(H)$

1. is B-Fredholm operator [B-Weyl], if for some integer $n \geq 0$ the range space $T^n(H)$ is a closed and $T[n] = T \big|_{T^n(H)} : T^n(H) \to T^n(H)$ is a Fredholm operator [Weyl operator].

2. is upper(lower) semi-B-Fredholm operator if for some integer $n \geq 0$ the range space $T^n(H)$ is a closed and $T[n] = T \big|_{T^n(H)} : T^n(H) \to T^n(H)$ is upper (resp. lower) semi-Fredholm operator.

3. is upper semi-B-Weyl if $T$ is upper semi-B-Fredholm and $\text{ind}(T) \leq 0$.

The $B$-Weyl spectrum is defined by

$$\sigma_{BW}(T) = \{\mu \in \mathbb{C} : T - \mu \text{ is not B-Weyl}\}$$

while the upper semi-B-Weyl spectrum defined by

$$\sigma_{U_{BW}}(T) = \{\mu \in \mathbb{C} : T - \mu \text{ is not upper semi-B-Weyl}\}.$$

For $T \in L(H)$ we write $\Pi_{00}(T) = \{\mu \in \text{iso}(T) : 0 < \alpha(T - \mu)\}$ for the set of all eigenvalues of $T$ which are isolated in $\sigma(T)$, and $\pi_{00}(T) = \{\mu \in \text{iso}(T) : 0 < \alpha(T - \mu) < \infty\}$ for the set of all isolated eigenvalues of finite multiplicity in $\sigma(T)$.

We say that $T$ satisfies the generalized Weyl’s theorem [10] if

$$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_{00}(T),$$

and we say that $T$ satisfies Weyl’s theorem [14], if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$
In [33], H. Weyl proved that Weyl’s theorem holds for hermitian operators. Weyl’s theorem has been extended from hermitian operators to hyponormal operators from Coburn in [14]. M. Berkani investigated the generalized Weyl’s theorem which extends Weyl’s theorem, and proved that the generalized Weyl’s theorem holds for normal operators [10] and hypernormal operators [12].

**THEOREM 5.2.** If \( T \in L(H) \) is an algebraically \( k \)-quasi-\( * \)-paranormal operator, then \( f(T) \) satisfies the generalized Weyl’s theorem for every \( f \in Hol(\sigma(T)) \).

**PROOF.** Let \( \mu \in \Pi_00(T) \). Then \( \mu \) is an isolated point in the spectrum \( \sigma(T) \). Using the spectral projection \( P_\mu = \frac{1}{2\pi i} \int_{D_\mu} (T - \mu)^{-1} d\mu \), where \( D_\mu \) is a closed disk of center \( \mu \) which contains no other points of \( \sigma(T) \), we can represent \( T \) as the direct sum

\[
T = T_1 \oplus T_2, \quad \text{where } \sigma(T_1) = \{ \mu \} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{ \mu \}.
\]

From Theorem 3, \( \mu \) is a pole of the resolvent of \( T \), there exists a positive integer \( p = p(\mu) \) such that \( T_1 - \mu = (T - \mu)|_{p(H) = \text{ker}(T - \mu)^p} \) and \( T_2 - \mu = (T - \mu)|_{\text{ker} p(T - \mu)^p(H)} \). So \((T - \mu)^p(H)\) is a closed subspace. From Theorem 3, \( T - \mu \) has finite ascent for all \( \mu \in \mathbb{C} \), then \((T - \mu)^n(H) = (T - \mu)^p(H)\) is a closed for all integers \( n \geq p \). By [3, Theorem 2.8] \( T \) satisfies the generalized Weyl’s theorem. By [34, Theorem 2.1], \( f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)) \) for all \( f \in Hol(\sigma(T)) \), since \( T \) has SVEP. Since \( T \) is an isoloid operator from [16, Lemma 3.3],

\[
f(\sigma(T) \setminus \Pi_00(T)) = \sigma(f(T)) \setminus \Pi_00(f(T)),
\]

and

\[
\sigma(f(T)) \setminus \Pi_00(f(T)) = f(\sigma(T) \setminus \Pi_00(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)),
\]

which implies that \( f(T) \) satisfies the generalized Weyl’s theorem.

From [9, Theorem 3.9], we know that: generalized Weyl’s theorem \( \implies \) Weyl’s theorem.

**COROLLARY 5.3.** If \( T \) is an algebraically \( k \)-quasi-\( * \)-paranormal then \( f(T) \) satisfies Weyl’s theorem for every \( f \in Hol(\sigma(T)) \).

**THEOREM 5.4.** If \( T^* \in L(H) \) is an algebraically \( k \)-quasi-\( * \)-paranormal operator, then \( f(T) \) satisfies the generalized Weyl’s theorem for every \( f \in Hol(\sigma(T)) \).

**PROOF.** Let \( \mu \in \Pi_00(T) \). So \( \mu \) is an isolated point of \( \sigma(T) \). By Theorem 3, \( T^* \) is polaroid operator, hence \( T \) is polaroid operator. Thus, \( \mu \) is a pole of the resolvent of \( T \). There exists a positive integer \( p = p(\mu) \) such that \( p = p(T - \mu) = q(T - \mu) \). Then \((T - \mu)^p(H) = (T - \mu)^{(p+1)}(H)\) and \((T - \mu)^n(H)\) is closed for every \( n \geq p \). By [3, Theorem 2.8] \( T \) satisfies the generalized Weyl’s theorem. Since \( T^* \) has SVEP, \( f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)) \) for all \( f \in Hol(\sigma(T)) \), from [34, Theorem 2.1]. By Theorem 3, \( T \) is polaroid operator, hence \( T \) is isoloid operator. From [16, Lemma 3.3],

\[
f(\sigma(T) \setminus \Pi_00(T)) = \sigma(f(T)) \setminus \Pi_00(f(T)),
\]
and
\[ \sigma(f(T)) \setminus \Pi_{00}(f(T)) = f(\sigma(T) \setminus \Pi_{00}(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)), \]
which implies that \( f(T) \) satisfies the generalized Weyl’s theorem. For \( T \in L(H) \) we write \( P_{00}(T) = \{ \mu \in \sigma(T) : 0 < p(T - \mu) = q(T - \mu) < \infty \} \) for the set of all pole of resolvent of \( T \), and \( p_{00}(T) = \{ \mu \in P_{00}(T) : \alpha(T - \mu) < \infty \} \) for the set of all pole of finite rank of resolvent of \( T \).

We say that \( T \) satisfies the generalized Browder’s theorem if
\[ \sigma(T) \setminus \sigma_{BW}(T) = P_{00}(T), \]
and we say that \( T \) satisfies Browder’s theorem, if
\[ \sigma(T) \setminus \sigma_w(T) = p_{00}(T). \]

COROLLARY 5.5. If \( T \in L(H) \) is an algebraically \( k \)-quasi-\( * \)-paranormal operator, then \( f(T) \) satisfies the generalized Browder’s theorem for every \( f \in Hol(\sigma(T)) \).

PROOF. Let \( T \) be an algebraically \( k \)-quasi-\( * \)-paranormal operator, then \( f(T) \) has SVEP. From \cite[Theorem 2.9]{16}, it follows \( f(T) \) satisfies the generalized Browder’s theorem for every \( f \in Hol(\sigma(T)) \).

From \cite[Theorem 3.15]{9}, we know that: generalized Browder’s theorem \( \implies \) Browder’s theorem.

Let \( \Pi_{00}^a(T) = \{ \mu \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \mu) \} \) be the set of all eigenvalues of \( T \), which are isolated in the approximate point spectrum, and \( \pi_{00}^a(T) = \{ \mu \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \mu) < \infty \} \) be the set of all eigenvalues of finite multiplicity, which are isolated in the approximate point spectrum of \( T \).

We say that \( T \) satisfies the generalized \( a \)-Weyl’s theorem \cite{9}, if
\[ \sigma_a(T) \setminus \sigma_{U BW}(T) = \Pi_{00}^a(T), \]
and we say that \( T \) satisfies the \( a \)-Weyl’s theorem \cite{27}, if
\[ \sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T). \]

Let \( P_{00}^a(T) = \{ \mu \in \sigma_a(T) : p(T - \mu) < \infty \) and \( (T - \mu)^p(T - \mu)^{p+1}(H) \) is closed \( \} \), the set of all left poles of resolvent of \( T \) and \( p_{00}^a(T) = \{ \mu \in P_{00}^a(T) : \alpha(T - \mu) < \infty \} \), the set of all left poles of finite rank of resolvent of \( T \).

We say that \( T \) satisfies the generalized \( a \)-Browders theorem \cite{9}, if
\[ \sigma_a(T) \setminus \sigma_{U BW}(T) = P_{00}^a(T), \]
and we say that $T$ satisfies the $a$–Browders theorem [27], if

$$\sigma_a(T) \setminus \sigma_{uu}(T) = p_{00}^a(T).$$

**THEOREM 5.6.** Suppose $T^*$ is an algebraically $k$-quasi-$*$-paranormal operator. Then the generalized $a$–Browder’s theorem holds for $f(T)$ for all $f \in Hol(\sigma(T))$.

**PROOF.** Since algebraically $k$-quasi-$*$-paranormal operator has finite ascent, then $T^*$ has SVEP. From [6, Theorem 3.2], $f(T)$ satisfies the generalized $a$–Browder’s theorem for all $f \in Hol(\sigma(T))$.

**THEOREM 5.7.** Suppose $T^*$ is an algebraically $k$-quasi-$*$-paranormal operator. Then the generalized $a$–Weyl’s theorem holds for $T$.

**PROOF.** Since algebraically $k$-quasi-$*$-paranormal operator has finite ascent, then $T^*$ has SVEP. Then from Theorem 5, $T$ satisfies the generalized $a$–Browder’s theorem. So, in view of [4], it is sufficient to show that $\Pi_{00}^a(T) = P_{00}^a(T)$. Since the inclusion $P_{00}^a(T) \subseteq \Pi_{00}^a(T)$ always holds true, then it is sufficient to prove this $\Pi_{00}^a(T) \subseteq P_{00}^a(T)$.

Let $\mu$ be an arbitrary point of $\Pi_{00}^a(T)$, then $\mu$ is an isolated point on $\sigma_a(T)$. From Lemma 3, $\mu$ is a pole of the resolvent of $T$, there exists a positive integer $p = p(\mu)$ such that $p(T - \mu) = q(T - \mu) = p < \infty$. Thus, $(T - \mu)^p(H) = (T - \mu)^p(H)$ and $(T - \mu)^p(H)$ is closed, since it coincides with the kernel of the spectral projection associated with $\{\mu\}$. Therefore, $\mu \in P_{00}^a(T)$.

**THEOREM 5.8.** Suppose $T^*$ is an algebraically $k$-quasi-$*$-paranormal operator. Then the generalized $a$–Weyl’s theorem holds for $f(T)$ for all $f \in Hol(\sigma(T))$.

**PROOF.** Suppose that $T^*$ is an algebraically $k$-quasi-$*$-paranormal operator. Then $T^*$ has SVEP, thus $f(T)$ satisfies the generalized $a$–Browder’s theorem. From [4], it is sufficient to show $\Pi_{00}^a(f(T)) \subseteq P_{00}^a(f(T))$. Suppose $\mu \in \Pi_{00}^a(f(T))$. Then $\mu$ is an isolated point of $\sigma_a(f(T))$ and $0 < \alpha(f(T) - \mu)$. Then $\mu \in \sigma_a(f(T))$, and it satisfies the equation:

$$f(T) - \mu = c(T - \mu_1)(T - \mu_2) \cdots (T - \mu_n)g(T) \quad (6)$$

where $c, \mu_1, \mu_2, ..., \mu_n \in \mathbb{C}$, and $g(T)$ is invertible.

Since $\mu$ is an isolated point of $f(\sigma_a(T))$, if $\mu_i \in \sigma_a(T)$, then $\mu_i$ is an isolated point of $\sigma_a(T)$ by relation (6). Since $T$ is $a$–isoloid, $0 < \alpha(T - \mu_i)$ for each $i = 1, 2, ..., n$. Then $\mu_i \in \Pi_{00}^a(T)$ for each $i = 1, 2, ..., n$. From Theorem 5, $T$ satisfies the generalized $a$–Weyl’s theorem, then $T - \lambda_i$ is upper semi B-Fredholm and $\text{ind}(T - \mu_i) \leq 0$ for each $i = 1, 2, ..., n$. Therefore $f(T) - \mu$ is upper semi-B-Fredholm. Since $\mu \in \text{iso} \sigma_a(f(T))$ then $f(T)$ has SVEP in $\mu$, then by [1, Theorem 2.89], $p(f(T) - \mu) < \infty$. Also, since $T^*$ has SVEP, $f(T)^*$ has SVEP in $\mu$, then by [1, Theorem 2.90] $p(f(T) - \mu) = q(f(T) - \mu) = p < \infty$. Thus, $(T - \mu)^{p+1}(H) = (f(T) - \mu)^p(H)$ and $(f(T) - \mu)^p(H)$ is closed, since it coincides with the kernel of the spectral projection associated with $\{\mu\}$. Therefore $\mu \in P_{00}^a(f(T))$. 

From [9, Theorem 3.11], we know that:

generalized $a$–Weyl’s theorem $\iff$ $a$–Weyl’s theorem.

and from [9, Theorem 3.8], we know that:

generalized $a$–Browder’s theorem $\iff$ $a$–Browder’s theorem.

COROLLARY 5.9. If $T^*$ is an algebraically $k$-quasi-$*$-paranormal then $f(T)$ satisfies $a$–Weyl’s theorem for every $f \in Hol(\sigma(T))$.

If $T$ is an algebraically $k$-quasi-$*$-paranormal, then $T$ not satisfies $a$–Weyl’s theorem [1, Example 4.53], consequently $T$ not satisfies generalized $a$–Weyl’s theorem, by [9, Theorem 3.11].

A bounded operator $T \in L(H)$ is said to be hereditarily polaroid, i.e. any restriction to an invariant closed subspace is polaroid. This class of operators has been first considered in [18].

COROLLARY 5.10. Algebraically $k$-quasi-$*$-paranormal operators are hereditarily polaroid.

PROOF. Let $T \in L(H)$ be an algebraically $k$-quasi-$*$-paranormal and $M$ a closed $T$-invariant subspace of $H$. By assumption there exists a nontrivial polynomial $h$ such that $h(T)$ is a $k$-quasi-$*$-paranormal. The restriction of any $k$-quasi-$*$-paranormal operator to an invariant closed subspace is also $k$-quasi-$*$-paranormal, so $h(T)|_M$ is a $k$-quasi-$*$-paranormal. Since $h(T)|_M = h(T)|_M$, $T|_M$ is algebraically $k$-quasi-$*$-paranormal, hence polaroid, from Theorem 3.

Let $\mathcal{K}(H)$ be the space of all compact operators on $H$. Note that $\mathcal{K}(H)$ is a closed ideal of $L(H)$. On the quotient space $L(H)/\mathcal{K}(H)$ it is defined the product $[S][T] = [ST]$, where $[S]$ is the coset $S + \mathcal{K}(H)$. The space $L(H)/\mathcal{K}(H)$ with this additional operation is an algebra, which is called the Calkin Algebra. Let $\pi : L(H) \to L(H)/\mathcal{K}(H)$ be the natural mapping (Calkin homomorphism). If $T \geq O$ then $\pi(T) \geq O$. It is well known the Theorem of Atkinson: $T$ is a Fredholm operator if and only if $\pi(T)$ is an invertible operator in Calkin algebra, thus $\sigma(\pi(T)) = \sigma_c(T)$, where

$$\sigma_c(T) = \{ \mu \in \mathbb{C} : T - \mu \text{ is not Fredholm} \}.$$ 

An operator $T$ is said to be a Riesz operator if $T - \mu$ is a Fredholm operator for all $\mu \in \mathbb{C} \setminus \{0\}$. Thus, $\sigma_c(T) = \{0\}$. Compact operators, also quasinilpotent operators, are Riesz operators.

THEOREM 5.11. If $T \in L(H)$ is a $k$-quasi-$*$-paranormal, $\|T^n\| = \|T\|^n$ for some $n \geq k$, and Riesz operator, then $T$ is a compact operator.

PROOF. Let $T$ be a $k$-quasi-$*$-paranormal operator. Then

$$\pi(T)^* k \left( \pi(T)^* \pi(T)^2 - 2\lambda \pi(T) \pi(T)^* + \lambda^2 \right) \pi(T)^k = \pi \left( T^* k \left( T^* 2 T^2 - 2\lambda TT^* + \lambda^2 \right) T^k \right) \geq O,$$
which shows that \( \pi(T) \) is a \( k \)-quasi-\( * \)-paranormal. Thus \( \pi(T) \) is normaloid operator, Theorem 2. Since \( T \) is a Riesz operator by West Decomposition Theorem [32], we can write \( T = S + Q \) where \( S \) is a compact and \( Q \) is a quasinilpotent operator. From the definition of homomorphism \( \pi \) we have \( \pi(T) = \pi(Q) \), thus \( \sigma(\pi(T)) = \sigma(\pi(Q)) = \sigma_e(Q) = \{0\} \), so \( \pi(T) \) is a quasinilpotent operator. Therefore, \( \|\pi(T)\| = r(\pi(T)) = 0 \), thus \( \pi(T) = O \). Then \( T \) is a compact operator.

COROLLARY 5.12. If \( T \) is a \( k \)-quasi-\( * \)-paranormal operator and if \( \sigma_{BW}(T) = \{0\} \), then \( T \) is normal operator.

PROOF. From Theorem 5, \( T \) satisfies the generalized Weyl’s theorem. By assumption, we have \( \sigma(T) \setminus \{0\} = \Pi_0(T) \). So every nonzero point of \( \sigma(T) \) is an isolated point of \( \sigma(T) \) and an eigenvalue. Hence \( \sigma(T) \setminus \{0\} \) is a finite set or a countably infinite set whose only cluster point is 0. Let \( \sigma(T) \setminus \{0\} = \{\mu_n\} \), with \( |\mu_1| \geq |\mu_2| \geq ... > 0 \). Since \( \mu_n \) is isolated point of \( \sigma(T) \), from Theorem 4, \( \ker(T - \mu_n) \) is a reducing subspace of \( T \). Let \( P_n \) be the orthogonal projection onto \( \ker(T - \mu_n) \). Then \( TP_n = P_nT = \mu_n P_n \) and \( P_nP_m = 0 \) if \( n \neq m \). Put \( P = \oplus_n P_n \), and we have

\[
T = T|_{\ker(I-P)} \oplus T|_{(I-P)(H)} = \oplus_n \mu_n P_n \oplus T|_{(I-P)(H)},
\]

with \( \sigma(T|_{(I-P)(H)}) = \sigma(T) \setminus \{\mu_n\} = \{0\} \). Since \( T|_{(I-P)(H)} \) is also \( k \)-quasi-\( * \)-paranormal operator, \( T|_{(I-P)(H)} = 0 \). Hence \( T = \oplus_n \mu_n P_n \), thus \( T \) is normal operator.

6 Generalized Weyl’s Theorem for Perturbations of Algebraically \( k \)-Quasi-\( * \)-Paranormal Operator

A bounded operator \( T \in L(H) \) is said to be algebraic if there exists a non-constant polynomial \( h \) such that \( h(T) = 0 \). Trivially, every nilpotent operator is algebraic and it is well-known that if \( T^n(H) \) has finite dimension for some \( n \in \mathbb{N} \) then \( T \) is algebraic.

THEOREM 6.1. If \( T \) is an algebraically \( k \)-quasi-\( * \)-paranormal operator, \( F \) is algebraic with \( TF = FT \), then \( T + F \) satisfies generalized Weyl’s theorem.

PROOF. Since \( F \) is algebraic operator, \( \sigma(F) = \{\mu_1, \mu_2, ..., \mu_n\} \). Denote by \( P_i \) the spectral projections associated with \( F \) and the spectral set \( \{\mu_i\} \), \( i = 1, 2, ..., n \). We write \( F_i = F|_{P_i(H)} \) and \( T_i = T|_{P_i(H)} \). Clearly, \( \sigma(F_i) = \{\mu_i\} \) for every \( i = 1, 2, ..., n \). Let \( h \) be a nontrivial complex polynomial such that \( h(F) = 0 \). Then \( O = h(F_i) = h(F)|_{P_i(H)} \), and from

\[
\{0\} = h(\sigma(F_i)) = h(\sigma(F_i)) = h(\mu_i),
\]

we obtain that \( h(\mu_i) = 0 \). Write \( h(\mu) = (\mu - \mu_i)^k g(\mu) \) with \( g(\mu_i) \neq 0 \). Then \( O = h(F_i) = (F_i - \mu_i)^k g(F_i) \), where \( g(F_i) \) is invertible. Therefore \( (F_i - \mu_i)^k = O \), hence \( F_i - \mu_i \) is a nilpotent operator for all \( i = 1, 2, ..., n \). Let \( \mu \in \Pi_0(T + F) \). Then \( \mu \) is isolated point in the spectrum \( \sigma(T + F) \). Since \( \sigma(T + F) = \cup_{i=1}^n \sigma(T_i + F_i) \), then \( \mu \in \sigma(T_i + F_i) \), for some \( i = 1, 2, ..., n \) and hence \( \mu - \mu_i \in \text{iso}(T_i + F_i - \mu_i) \). The restriction \( T_i \) to a closed invariant subspace \( P_i(H) \) is also algebraically \( k \)-quasi-\( * \)-paranormal operator,
then $T_i$ is polaroid for all $i = 1, 2, ..., n$. Since $F_i - \mu_i$ is a nilpotent operator for all $i = 1, 2, ..., n$, by [5, Theorem 2.10] $T_i + F_i - \mu_i$ is polaroid for all $i = 1, 2, ..., n$. Then $\mu - \mu_i$ is a pole of the resolvent of $T_i + F_i - \mu_i$. By [2, Theorem 3.74] there exists a positive numbers $m_i$ such that

$$\mathcal{H}_0(T_i + F_i - \mu_i - (\mu - \mu_i)) = \mathcal{H}_0(T_i + F_i - \mu) = \ker(T_i + F_i - \mu)^{m_i},$$

for $i = 1, 2, ..., n$. Taking $\mathcal{H}_0(T_i + F_i - \mu) = \{0\}$ for $\mu \notin \sigma(T_i + F_i)$ and we have

$$\mathcal{H}_0(T + F - \mu) = \oplus_{i=1}^n \mathcal{H}_0(T_i + F_i - \mu) = \oplus_{i=1}^n \ker(T_i + F_i - \mu)^{m_i} = \ker(T + F - \mu)^m,$$

where $m = \max\{m_1, m_2, ..., m_n\}$. Since $\mu \in \text{iso}(T + F)$, we have

$$H = \mathcal{H}_0(T + F - \mu) \oplus K(T + F - \mu) = \ker(T + F - \mu)^m \oplus K(T + F - \mu).$$

Therefore,

$$(T + F - \mu)^m(H) = K(T + F - \mu) \quad \text{and} \quad H = \ker(T + F - \mu)^m \oplus (T + F - \mu)^m(H).$$

From [2, Theorem 3.6] $T + F - \mu$ has finite ascent. So, $(T + F - \mu)^m(H) = (T + F - \mu)^{m+1}(H)$ and $(T + F - \mu)^p(H)$ is closed for every $p \geq m$. By [3, Theorem 2.8] $T + F$ satisfies the generalized Weyl’s theorem.

**THEOREM 6.2.** If $T$ is an algebraically $k$-quasi-$*$-paranormal operator, $F$ is algebraic with $TF = FT$, then $f(T + F)$ satisfies the generalized Weyl’s theorem for all $f \in \text{Hol}(\sigma(T + F))$.

**PROOF.** Let $F$ be an algebraic operator. Then, $\sigma(F) = \{\mu_1, \mu_2, ..., \mu_n\}$, and $F_i - \mu_i$ is nilpotent operator for $i = 1, 2, ..., n$. Since $T$ is an algebraically $k$-quasi-$*$-paranormal, then $T_i + \mu_i$ is also an algebraically $k$-quasi-$*$-paranormal operator. Then $T_i + \mu_i$ has SVEP for $i = 1, 2, ..., n$ and from [2, theorem 2.12] $T_i + \mu_i + F_i - \mu_i = T + \mu_i$ has SVEP. From [2, theorem 2.9] $T + F = \oplus_{i=1}^n (T_i + F_i)$ has SVEP. By [16, Corollary 2.8], $f(\sigma_{BW}(T + F)) = \sigma_{BW}(f(T + F))$ for all $f \in \text{Hol}(\sigma(T + F))$. But, from the above theorem we have that $T + F$ is isolid operator, then from [16, Lemma 3.3],

$$f(\sigma(T + F) \setminus \Pi_{00}(T + F)) = \sigma(T + F) \setminus \Pi_{00}(f(T + F)),$$

and

$$\sigma(f(T + F)) \setminus \Pi_{00}(f(T + F)) = f(\sigma(T + F) \setminus \Pi_{00}(T + F)) = f(\sigma_{BW}(T + F)) = \sigma_{BW}(f(T + F)),$$

which implies that $f(T + F)$ satisfies the generalized Weyl’s theorem.

An operator $T$ is said to be finitely-isolid if every isolated point of $\sigma(T)$ is an eigenvalue of $T$ of the finite multiplicity, so: from $\mu \in \text{iso}(T)$ we have $\mu \in \pi_{00}(T)$.

**COROLLARY 6.3.** If $T$ is finitely-isolid and $T$ is an algebraically $k$-quasi-$*$-paranormal operator, $R$ is Riesz operator with $TR = RT$, then $T + R$ satisfies Weyl’s theorem.

**PROOF.** From Corollary 5, it follows that $T$ satisfies Weyl’s theorem and by [26, Theorem 2.7] $T + R$ satisfies Weyl’s theorem.
References


