A Remark On The Crux Problem 1052**

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Abstract

An interesting family of infinite series is evaluated exactly using standard methods from complex analysis.

1 Introduction and Results

The following problem was given to students in an examination paper at Trinity College, Cambridge, in June 1901: Prove that

\[ \frac{1}{1^2 \cdot 3^2 \cdot 5^2} - \frac{1}{3^2 \cdot 5^2 \cdot 7^2} + \frac{1}{5^2 \cdot 7^2 \cdot 9^2} - \cdots = \frac{1}{9} - \frac{\pi}{2^6} - \frac{\pi^3}{2^9}. \]  

(1)

The problem appeared as Problem 1052* in Crux [3]. Szekeres gave three proofs of the statement in [4]. In this note, we generalize the problem in the following way: For \( k \in \mathbb{Z} \), we consider the family of infinite series

\[ S(k) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n - 2k + 1)^2(2n + 1)^3(2n + 2k + 1)^2}. \]  

(2)

We derive a closed-form for the series. We recommend the article [2], its content is related to the present problem. It is obvious that \( S(1) \) corresponds to the LHS of equation (1). Also, we have the symmetry relation \( S(k) = S(-k) \). Finally, we mention that \( S(0) \) can be evaluated using Dirichlet’s beta function (or L-function) as follows:

\[ S(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n + 1)^3} = 1 - \beta(7) = 1 - \frac{61}{184320} \pi^7, \]

where \( \beta(s) \) is given by (see [1] or [5])

\[ \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^s}, \quad Re(s) > 1, \]

and where we have used the relation between \( \beta(s) \) and Euler numbers

\[ \beta(2p + 1) = \frac{(-1)^p E_{2p} \pi^{2p+1}}{4^{p+1}(2p)!}, \quad p \in \mathbb{N}. \]

Our main result is the following theorem.

**Theorem 1** Let \( k \in \mathbb{Z} \) and let \( S(k) \) be defined as in (2). Then,

\[ S(k) = \begin{cases} \frac{1}{(2k-1)^2(2k+1)^2} - \frac{\pi^3}{2^9 \pi^7}, & k \text{ even} \\ \frac{1}{(2k-1)^2(2k+1)^2} - \frac{\pi^3}{2^9 \pi^7} - \frac{\pi^3}{2^9 \pi^7}, & k \text{ odd} \end{cases} \]
The result in (1) is obtained from $S(1)$. We shall state two more special cases:

$$S(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-3)^2(2n+1)^3(2n+5)^2} = \frac{1}{225} - \frac{\pi^3}{243}$$

and

$$S(3) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-5)^2(2n+1)^3(2n+7)^2} = \frac{1}{1225} - \frac{\pi}{6} - \frac{\pi^3}{3432}.$$

2 The Proof

We will give a proof mainly based on the theory of residues, extending the arguments of Szekeres. To do so, the following technical lemma will be needed, which seems to be a familiar result. To keep this note self contained, we provide a proof below.

**Lemma 1** Let $z \in \mathbb{C}$. Let further $N \in \mathbb{N}$ and $C$ be the square in the complex plane with corners $(\pm N, \pm N)$. Then, $|\cos(\pi z)| \geq 1$ for all $z$ on the square $C$.

**Proof.** Let $z = x + iy$. Then, from the definition of the complex cosine function,

$$\cos(\pi z) = \cos(\pi x) \cosh(\pi y) - i \sin(\pi x) \sinh(\pi y).$$

Using the half angle formulas

$$\cos(2z) = 2\cos^2(z) - 1 \quad \text{and} \quad \cosh(2z) = 2\cosh^2(z) - 1,$$

we get

$$|\cos(\pi z)|^2 = \cos^2(\pi x) \cosh^2(\pi y) + \sin^2(\pi x) \sinh^2(\pi y) = \frac{\cos(2\pi x)}{2} + \frac{\cosh(2\pi y)}{2}.$$

On the vertical sides of $C$, $z = \pm N + iy$ and

$$|\cos(\pi z)|^2 = \frac{\cos(\pm 2\pi N)}{2} + \frac{\cosh(2\pi y)}{2} = 1 + \frac{\cosh(2\pi y)}{2} \geq 1.$$

Finally, on the horizontal sides of $C$ we have that $z = x \pm iN$ and

$$|\cos(\pi z)|^2 = 1 + \frac{1}{2} \sum_{n=1}^{\infty} ((-1)^n x^{2n} + N^{2n}) \frac{(2\pi)^{2n}}{(2n)!} \geq 1.$$

The proof of Theorem 1 follows.

**Proof.** We start with the observation that

$$S(k) = \frac{1}{2} \left( S(k) + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(2n-2k+1)^2(2n+1)^3(2n+2k+1)^2} \right)$$

$$= \frac{1}{2} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(2n-2k+1)^2(2n+1)^3(2n+2k+1)^2} + \frac{2}{(2k-1)^2(2k+1)^2} \right).$$

To evaluate the sum, we consider for $k \in \mathbb{Z}$ the complex function

$$f(z) = \frac{\pi}{(z-k)^2z(z+k)^2 \cos(\pi z)}.$$
From Lemma 1, it follows that
\[ \lim_{N \to \infty} \int_C f(z)dz = 0 \]
on each square \( C \) with corners \( (\pm N, \pm N) \). This means that
\[ \sum_{j \geq 1} \text{Res}(f; z_j) = 0, \]
where \( z_j \) are the poles of \( f(z) \) inside \( C \). The classification of the residues is easy: \( f(z) \) has infinitely many poles of order one at \( z = n + 1/2 \), \( n \) an integer, a pole of order two at \( z = k \), a pole of order two at \( z = -k \), and a pole of order three at \( z = 0 \). For each \( n \), the residue of \( f(z) \) at \( z = n + 1/2 \) is
\[ \text{Res}(f; z = n + 1/2) = \lim_{z \to n + 1/2} \frac{\pi(z - n - 1/2)}{(z - k)^2 z^3(z + k)^2 \cos(\pi z)} = \frac{128(-1)^{n+1}}{(2n - 2k + 1)^2(2n + 1)^3(2n + 2k + 1)^2}, \]
Next, the residue at \( z = k \) is
\[ \text{Res}(f; z = k) = \lim_{z \to k} \frac{\pi}{d \frac{d^2}{dz^2}(z - k)^2 \cos(\pi z)} = (-1)^{k+1} \frac{\pi}{k^6}. \]
In the same manner,
\[ \text{Res}(f; z = -k) = \lim_{z \to -k} \frac{\pi}{d \frac{d^2}{dz^2}(z - k)^2 \cos(\pi z)} = (-1)^{k+1} \frac{\pi}{k^6}. \]
Finally,
\[ \text{Res}(f; z = 0) = \lim_{z \to 0} \frac{\pi}{d \frac{d^2}{dz^2}(z - k)^2(z + k)^2 \cos(\pi z)} = \pi \left( A(z) + B(z) + C(z) + D(z) \right), \]
The calculation of the second derivative is straightforward but lengthy. The result is
\[ \frac{d^2}{dz^2} \frac{\pi}{(z - k)^2(z + k)^2 \cos(\pi z)} = \pi \left( A(z) + B(z) + C(z) + D(z) \right), \]
with
\[ A(z) = \frac{2\pi \sin(\pi z)((\pi z^2 - \pi k^2) \sin(\pi z) - 4z \cos(\pi z))}{(z - k)^3(z + k)^3 \cos^3(\pi z)}, \]
\[ B(z) = -3((\pi z^2 - \pi k^2) \sin(\pi z) - 4z \cos(\pi z)) \]
\[ C(z) = -3((\pi z^2 - \pi k^2) \sin(\pi z) - 4z \cos(\pi z)) \]
and
\[ D(z) = 6\pi z \sin(\pi z) + \pi(\pi z^2 - \pi k^2) \cos(\pi z) - 4 \cos(\pi z)) \]
\[ \frac{d^2}{dz^2} \frac{\pi}{(z - k)^3(z + k)^3 \cos^2(\pi z)} \]
Hence,
\[ \text{Res}(f; z = 0) = \lim_{z \to 0} \frac{1}{2} \frac{d^2}{dz^2} \frac{\pi}{(z-k)^2(z+k)^2 \cos(\pi z)} = \frac{\pi}{2} \frac{\pi^2 k^2 + 4}{k^6}. \]

Gathering our results, we end with
\[ 0 = 128 \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(2n - 2k + 1)^2(2n + 1)^3(2n + 2k + 1)^2} + (-1)^{k+1} \frac{2\pi}{k^6} + \frac{\pi^2 k^2 + 4}{2k^6}, \]

or equivalently
\[ \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(2n - 2k + 1)^2(2n + 1)^3(2n + 2k + 1)^2} = \frac{1}{128} \cdot \frac{\pi}{k^6} \left( 2((-1)^k - 1) - \frac{\pi^2 k^3}{2} \right), \]

and the proof is completed. ■

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References


