On The Asymptotic Expansion Of A Generalized Smith’s Determinant*

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Abstract

In this paper we study the generalized Smith’s determinant \( \Delta_s(n) := \det [(\gcd(i,j))^s]_{1 \leq i,j \leq n} \), where \( s \neq 0 \) is fixed real. For large values of \( n \) we obtain asymptotic expansions of \( \log |\Delta_s(n)| \), and for \( s > 1 \) we obtain Stirling type approximations for \( \Delta_s(n) \). Furthermore, we prove that for \( s < 0 \) the sign of \( \Delta_s(n) \) is independent of \( s \), and is same as the sign of \((-1)^{\eta_n} \), where \( \eta_n \) denotes the number of integers \( m \in [1,n] \) having odd number of distinct prime divisors.

1 Introduction

In 1875 Smith [8] considered the determinant of the matrix \( [a_{ij}]_{1 \leq i,j \leq n} \) with elements given by \( a_{ij} = \gcd(i,j) \), greatest common divisor of \( i \) and \( j \). He proved that

\[
\det [\gcd(i,j)]_{1 \leq i,j \leq n} = \prod_{m=1}^{n} \varphi(m),
\]

where \( \varphi(m) \) denotes the Euler function of \( m \), counting the number of positive integers not exceeding \( m \) and coprime to \( m \). The above determinant is known as Smith’s determinant. Since Smith’s work this field has been studied extensively. For a recent account of the theory of gcd-matrices we refer the reader to [4] and the references given there. Also, see [2, p. 123] for some classical generalizations of Smith’s determinant, including the assertion that if \( f \) is an arithmetic function then

\[
\det [f(\gcd(i,j))]_{1 \leq i,j \leq n} = \prod_{m=1}^{n} \sum_{d|m} \mu(d) f \left( \frac{m}{d} \right),
\]

where \( \mu(d) \) denotes the Möbius function of \( d \), which is 1 if \( d = 1 \), is \((-1)^k\) if \( d \) is equal to the product of \( k \) distinct primes, and is 0 otherwise. In this paper we are motivated by the asymptotic growth of generalized Smith’s determinant (1) for \( f(n) = n^s \). The exponent \( s \) is an arbitrary non-zero real. For the case \( s > 0 \) we prove the following result.

**Theorem 1** Let \( s > 0 \) be fixed real and

\[
\Delta_s^+(n) = \det [(\gcd(i,j))^s]_{1 \leq i,j \leq n},
\]

Define the absolute constant \( \alpha_s \) by

\[
\alpha_s = \sum_p \frac{1}{p} \log \left( 1 - \frac{1}{p^s} \right),
\]

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where $p$ runs over all primes. Then, as $n \to \infty$,
\[
\log \Delta_s^+(n) = \begin{cases} 
\alpha_s - s)n + O(n^{1-s}) & (0 < s < 1), \\
\alpha_s - s)n + \frac{1}{2} \log n + O(\log \log n) & (s = 1), \\
\alpha_s - s)n + \frac{1}{2} \log n + s \log \sqrt{2\pi} + O\left(\frac{1}{n^{1-s}}\right) & (1 < s \leq 2).
\end{cases}
\]

Also, for each $s > 2$ the following approximation holds
\[
\log \Delta_s^+(n) = sn \log n + (\alpha_s - s)n + \frac{s}{2} \log n + s \log \sqrt{2\pi} + \sum_{1 \leq j \leq n^{1/2}} \frac{B_{2j}}{(2j)(2j - 1)n^{2j - 1}} + O\left(\frac{1}{n^{s-1}}\right),
\]
where $B_i$ denotes the $i$-th Bernoulli number.

A more sophisticated argument, similar to that used in our paper [3], enables us to consider the case of negative values of exponent.

**Corollary 1** Let $\Delta_s^+(n)$ be the determinant defined by (2). Then, as $n \to \infty$,
\[
\Delta_s^+(n) = \begin{cases} 
\left(\frac{n}{e}\right)^s \beta_s^n \sqrt{(2\pi)n^s} \left(1 + O\left(\frac{1}{n^{1-s}}\right)\right) & (1 < s < 2), \\
\left(\frac{n}{e}\right)^s \beta_s^n \sqrt{(2\pi)n^s} \left(1 + O\left(\frac{1}{n}\right)\right) & (s \geq 2),
\end{cases}
\]

where $\beta_s$ is an absolute constant defined by
\[
\beta_s = \prod_p \left(1 - \frac{1}{p^s}\right)^{\frac{1}{p}},
\]
and $p$ runs over all primes.

Furthermore, the sign of $\Delta_s^+(n)$ is independent of $s$, and is same as the sign of $(-1)^{\eta_n}$, where $\eta_n$ denotes the number of integers $m \in [1, n]$ having odd number of distinct prime divisors.

**Theorem 2** Let $s > 0$ be fixed real and
\[
\Delta_s^-(n) = \det \left((\gcd(i, j))^{-s}\right)_{1 \leq i, j \leq n}.
\]

Then, for any positive integer $r$ there exist computable constants $c_1, \ldots, c_r$ such that as $n \to \infty$,
\[
\log |\Delta_s^-(n)| = (\alpha_s + \gamma + E) + s \sum_{j=1}^r \frac{c_j n}{\log^r n} + O\left(\frac{n}{\log^{r+1} n}\right),
\]
where $\alpha_s$ is defined by (3), $\gamma$ is Euler’s constant, and $E$ is the constant in Mertens’ approximation given by
\[
E = \lim_{x \to \infty} \sum_{p \leq x} \frac{\log p}{p} - \log x.
\]

Furthermore, the sign of $\Delta_s^-(n)$ is independent of $s$, and is same as the sign of $(-1)^{\eta_n}$, where $\eta_n$ denotes the number of integers $m \in [1, n]$ having odd number of distinct prime divisors.
2 Proof of Theorem 1

Proof. For \( f(n) = n^s \), we conclude from (1) that

\[
\Delta^+_n(n) = \prod_{m=1}^{n} m^s g_s(m) = n!^s \prod_{m=1}^{n} g_s(m),
\]

where \( g_s(m) = \sum_{d|m} \mu(d)d^{-s} \). Since \( g_s \) is multiplicative, we get

\[
g_s(m) = \prod_{p^a|m} g_s(p^a) = \prod_{p^a|m} \left(1 - \frac{1}{p^a}\right) = \prod_{p|n} \left(1 - \frac{1}{p^a}\right).
\]

Thus,

\[
\Delta^+_n(n) = n!^s \prod_{m=1}^{n} \prod_{p|m} \left(1 - \frac{1}{p^a}\right),
\]

and

\[
\log \Delta^+_n(n) = s \log n! + \sum_{m=1}^{n} \sum_{p|m} \log \left(1 - \frac{1}{p^a}\right). \quad (5)
\]

Stirling’s approximation [7, p. 294] for \( \log n! \) asserts that given any positive integer \( r \), as \( n \to \infty \),

\[
\log n! = n \log n - n + \frac{1}{2} \log n + \log \sqrt{2\pi} + \sum_{j=1}^{r} \frac{B_{2j}}{(2j)(2j-1)n^{2j-1}} + O\left(\frac{1}{n^{2r+1}}\right). \quad (6)
\]

To approximate the double sum in (5), we change the order of summations. Thus,

\[
\sum_{m=1}^{n} \sum_{p|m} \log \left(1 - \frac{1}{p^a}\right) = \sum_{p \leq n} \log \left(1 - \frac{1}{p^a}\right) \sum_{m \leq n} \log \left(1 - \frac{1}{p^a}\right) \left\lfloor \frac{n}{p} \right\rfloor
\]

\[
= \sum_{p \leq n} \log \left(1 - \frac{1}{p^a}\right) \left(\frac{n}{p} + O(1)\right)
\]

\[
= n \sum_{p \leq n} \frac{1}{p} \log \left(1 - \frac{1}{p^a}\right) + O \left(\sum_{p \leq n} \log \left(1 - \frac{1}{p^a}\right)\right)
\]

\[
= \alpha_s n + n \sum_{p > n} \frac{1}{p} \log \left(1 - \frac{1}{p^a}\right) + O \left(\sum_{p \leq n} \log \left(1 - \frac{1}{p^a}\right)\right).
\]

Since \( -\log(1-t) \sim t \) as \( t \to 0 \), we get

\[
\sum_{p > n} \frac{1}{p} \log \left(1 - \frac{1}{p^a}\right)^{-1} \ll \sum_{p > n} \frac{1}{p^{s+1}} \ll \int_{n}^{\infty} \frac{dx}{x^{s+1}} \ll \frac{1}{n^s}.
\]

Also, by using the approximation \( \sum_{p \leq n} \frac{1}{p} \ll \log \log n \) we obtain

\[
\sum_{p \leq n} \log \left(1 - \frac{1}{p^a}\right) \ll \sum_{p \leq n} \frac{1}{p^a} \ll \begin{cases} \int_{2}^{n} \frac{dx}{x^a} \ll \frac{1}{n^{s+1}} & (s \neq 1), \\ \log \log n & (s = 1). \end{cases}
\]
Hence,
\[ \sum_{m=1}^{n} \sum_{p|m} \log \left( 1 - \frac{1}{p^s} \right) = \alpha_s n + O \left( \left\{ \begin{array}{c} n^{1-s} \log \log n \quad (s \neq 1) \\ s \quad (s = 1) \end{array} \right. \right). \]  
(7)

We let \( r = [s] \) in (6). Note that \( 2[s] + 1 \geq s - 1 \). Therefore, by considering (5) and (7) we conclude the proof.

3 Proof of Theorem 2

Proof. Let \( s > 0 \). We conclude from (1) that
\[ \Delta_s^-(n) = \prod_{m=1}^{n} m^{-s} h_s(m) = n^{1-s} \prod_{m=1}^{n} h_s(m), \]
where \( h_s(m) = \sum_{d|m} \mu(d)d^s \). Since \( h_s \) is multiplicative, we get
\[
\begin{align*}
\prod_{p^a|m} h_s(p^a) &= \prod_{p^a|m} \left( 1 - \frac{1}{p^a} \right) \\
&= (-1)^{\omega(m)} \prod_{p|m} (p^s - 1) = (-1)^{\omega(m)} \kappa(m)^s \prod_{p|m} \left( 1 - \frac{1}{p^s} \right),
\end{align*}
\]
where \( \omega(m) \) counts the number of distinct prime factors of \( m \), and \( \kappa(m) \) denotes the product of distinct prime factors of \( m \). Thus,
\[ \Delta_s^-(n) = (-1)^{\sum_{m=1}^{n} \omega(m)} n^{1-s} \prod_{m=1}^{n} \kappa(m)^s \prod_{m=1}^{n} \prod_{p|m} \left( 1 - \frac{1}{p^s} \right). \]
(8)

This relation implies that the sign of \( \Delta_s^-(n) \) depends on the value of \( \sum_{m=1}^{n} \omega(m) \), which is independent of \( s \). Moreover, the sign of \( \Delta_s^-(n) \) is same as the sign of \( (-1)^{\eta_n} \), where
\[ \eta_n = \sum_{\substack{1 \leq m \leq n \\ \omega(m) \text{ is odd}}} 1. \]
denoting the number of integers \( m \in [1, n] \) which have odd number of distinct prime divisors. Furthermore, we conclude from (8) that
\[ \log |\Delta_s^-(n)| = -s \log n! + s \sum_{m=1}^{n} \log \kappa(m) + \sum_{m=1}^{n} \sum_{p|m} \log \left( 1 - \frac{1}{p^s} \right). \]

To approximate \( \sum_{m=1}^{n} \log \kappa(m) \) we recall the notion of the index of composition of \( n \), which is defined by
\[ \lambda(n) = \frac{\log n}{\log \kappa(n)}, \]
for each integer \( n \geq 2 \). Note that \( \lambda(n) \) “somehow” measures how much the integer \( n \geq 2 \) is composite! For \( n \) square-free it takes the value \( \lambda(n) = 1 \), and for integers \( n \) having square factors in heart, it takes the value \( \lambda(n) > 1 \). De Koninck and Kátai [1] proved that given any positive integer \( r \), there exist computable constants \( d_1, \ldots, d_r \) such that
\[ v(x) := \sum_{k \leq x} \frac{1}{\lambda(k)} = x + \sum_{j=1}^{r} d_j \frac{x}{\log^j x} + O \left( \frac{x}{\log^{r+1} x} \right). \]  
(9)
By using Abel summation we get
\[ \sum_{k=1}^{n} \log \kappa(k) = \sum_{k=2}^{n} \frac{1}{\lambda(k)} \log k = v(n) \log n - v(2) \log 2 - \int_{2}^{n} \frac{v(t)}{t} \, dt. \]

To deal with the last integral, we study the functions \( L_j(t) \) defined for each integer \( j \geq 1 \) by the following anti-derivative
\[ L_j(t) := \int \frac{dt}{\log^j t}. \]

Note that \( L_1(t) \) is the logarithmic integral function, which admits the following expansion
\[ L_1(t) = \text{li}(t) = \sum_{i=1}^{r} (i - 1)! \frac{t}{\log^i t} + O\left(\frac{t}{\log^{r+1} t}\right). \] (10)

Integrating by parts gives
\[ L_{j-1}(t) = \int \left( \frac{1}{\log^{j-1} t} \right) (dt) = \frac{t}{\log^{j-1} t} + (j - 1) \int \frac{dt}{\log^j t}. \]

Hence, for \( j \geq 2 \) the functions \( L_j(t) \) satisfy the recurrence
\[ L_j(t) = \frac{1}{j - 1} L_{j-1}(t) - \frac{t}{(j - 1) \log^{j-1} t}. \]

By repeated using this recurrence we deduce that
\[ (j - 1)! L_j(t) = \text{li}(t) - \sum_{i=1}^{j-1} (i - 1)! \frac{t}{\log^i t}. \]

Hence, by using the expansion (10), for \( 1 \leq j \leq r \) we obtain
\[ L_j(t) = \sum_{i=j}^{r} \frac{(i - 1)!}{(j - 1)! \log^i t} + O\left(\frac{t}{\log^{r+1} t}\right). \] (11)

We deduce from the expansion (9) that
\[ \int_{2}^{n} \frac{v(t)}{t} \, dt = \int_{2}^{n} \left( 1 + \sum_{j=1}^{r} d_j \frac{1}{\log^j t} + O\left(\frac{1}{\log^{r+1} t}\right) \right) \, dt \]
\[ = n + \sum_{j=1}^{r} d_j L_j(n) - \left( 2 + \sum_{j=1}^{r} d_j L_j(2) \right) + O\left(\frac{n}{\log^{r+1} n}\right). \]

With \( r \) replaced by \( r + 1 \) in (9), we obtain
\[ v(n) \log n = n \log n + d_1 n + \sum_{j=1}^{r} d_{j+1} \frac{n}{\log^j n} + O\left(\frac{n}{\log^{r+1} n}\right). \]

Combining the above expansions, we obtain
\[ \sum_{k=1}^{n} \log \kappa(k) = n \log n + (d_1 - 1)n + \sum_{j=1}^{r} \left( d_{j+1} \frac{n}{\log^j n} - d_j L_j(n) \right) - C_r + O\left(\frac{n}{\log^{r+1} n}\right), \]
where

\[ C_r = 2 + v(2^-) \log 2 + \sum_{j=1}^{r} d_j L_j(2) \]

is a constant depending only on \( r \). Thus, \( C_r = O_r(1) \). Moreover, we deduce from the expansion (11) that

\[ \sum_{j=1}^{r} \left( d_{j+1} \frac{n}{\log^j n} - d_j L_j(n) \right) = \sum_{j=1}^{r} \left( d_{j+1} \frac{n}{\log^j n} - \sum_{i=j}^{r} d_j (i-1)! \frac{n}{(j-1)! \log^i n} \right) + O\left( \frac{n}{\log^{r+1} n} \right). \]

Note that

\[ \sum_{j=1}^{r} \left( d_{j+1} \frac{n}{\log^j n} - \sum_{i=j}^{r} d_j (i-1)! \frac{n}{(j-1)! \log^i n} \right) = \sum_{j=1}^{r} c_j \frac{n}{\log^j n} + O\left( \frac{n}{\log^{r+1} n} \right), \]

where \( c_j \)'s are computable constants in terms of \( d_j \)'s. Thus, letting \( c_0 = d_1 - 1 \), we obtain

\[ \sum_{m=1}^{n} \log \kappa(m) = n \log n + c_0 n + \sum_{j=1}^{r} c_j \frac{n}{\log^j n} + O\left( \frac{n}{\log^{r+1} n} \right). \]

To compute the precise value of \( c_0 \) we write

\[ \sum_{m=1}^{n} \log \kappa(m) = \sum_{m=1}^{n} \log \prod_{p|n} p = \sum_{m=1}^{n} \sum_{p|n} \left\{ \frac{n}{p} \right\} \log p = n \mathcal{M}(n) - \mathcal{R}(n), \]

where

\[ \mathcal{M}(n) := \sum_{p \leq n} \frac{\log p}{p}, \]

and

\[ \mathcal{R}(n) := \sum_{p \leq n} \left\{ \frac{n}{p} \right\} \log p. \]

It is known due to Landau \( [5, \text{p. } 198] \) that

\[ \mathcal{M}(n) = \log n + E + O\left( \frac{1}{\log n} \right), \]

where \( E \) is the constant given by (4). To estimate \( \mathcal{R}(n) \) we let

\[ \mathcal{S}(n) = \sum_{p \leq n} \left\{ \frac{n}{p} \right\}, \]

and

\[ \mathcal{L}(n) = \sum_{p^\alpha \leq n} \left\{ \frac{n}{p^\alpha} \right\}. \]

It is known due to Lee \([6]\) that

\[ \mathcal{L}(n) = (1 - \gamma) \frac{n}{\log n} + O\left( \frac{n}{\log^2 n} \right). \]

We observe that although the summation \( \mathcal{L}(n) \) has the summation \( \mathcal{S}(n) \) in heart, but their difference is not too large in comparison the true size of \( \mathcal{L}(n) \). More precisely,

\[ \mathcal{L}(n) - \mathcal{S}(n) = \sum_{p^\alpha \leq n} \left\{ \frac{n}{p^\alpha} \right\} < \sum_{p^\alpha \leq n} 1 = \sum_{p \leq n} 1 = \sum_{2 \leq \alpha \leq \log n} \pi\left( \frac{n}{\alpha} \right) \]

\[ \ll \sum_{2 \leq \alpha \leq \log n} \frac{n^{1/2}}{\log n^{1/2}} \leq n^{1/2} \sum_{2 \leq \alpha \leq \log n} \alpha \ll \sqrt{n} \log n, \]
where $\pi(t)$ denotes the number of primes $p$ not exceeding $t$, and we use the simple estimate $\pi(t) \ll \frac{t}{\log t}$ in the above argument. Hence,

$$S(n) = (1 - \gamma) \frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right). \quad (14)$$

Let $\varpi(k)$ to be 1 when $k$ is prime and 0 otherwise. By using Abel summation we get

$$R(n) = \sum_{k=2}^{n} \left\{ \frac{n}{k} \right\} \varpi(k) \log k = S(n) \log n - S(2^{-}) \log 2 - \int_{2}^{n} \frac{F_n(t)}{t} \, dt,$$

where

$$F_n(t) = \sum_{p \leq t} \left\{ \frac{n}{p} \right\}.$$

Since $0 \leq F_n(t) \leq \pi(t) \ll \frac{t}{\log t}$, by using the approximation (14) we deduce that

$$R(n) = (1 - \gamma)n + O\left(\frac{n}{\log n}\right) - \int_{2}^{n} O\left(\frac{t}{\log t}\right) \frac{dt}{t} = (1 - \gamma)n + O\left(\frac{n}{\log n}\right).$$

Thus, by substituting (13) and the last approximation in (12) we obtain the truncated approximation

$$\sum_{m=1}^{n} \log \kappa(m) = n \log n + (\gamma + E - 1)n + O\left(\frac{n}{\log n}\right),$$

implying that $c_0 = \gamma + E - 1$. Hence, given any positive integer $r$, there exist computable constants $c_1, \ldots, c_r$ such that

$$\sum_{m=1}^{n} \log \kappa(m) = n \log n + (\gamma + E - 1)n + \sum_{j=1}^{r} c_j \frac{n}{\log^{j+1} n} + O\left(\frac{n}{\log^{r+1} n}\right).$$

By using this approximation and the relations (6) and (7) we conclude the proof. $\blacksquare$

References


