On Ryser’s Conjecture: Modulo 2 Approach*

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Abstract

We prove the nonexistence of circulant Hadamard matrices $H$ of order $n > 4$ under the truth of some congruences $(\text{mod } 2)$ extending a result of Brualdi. The new idea consists of exploiting modular properties of a related circulant weighing matrix of order $n/2$.

1 Introduction

A matrix of order $n$ is a square matrix with $n$ rows. A circulant matrix $A = \text{circ}(a_1, \ldots, a_n)$ of order $n$ is a matrix of order $n$ of first row $[a_1, \ldots, a_n]$ in which each row after the first is obtained by a cyclic shift of its predecessor by one position. For example, the second row of $A$ is $[a_n, a_1, \ldots, a_{n-1}]$. As usual, $J$ is the matrix of order $n$ with all its entries equal to 1 (i.e., $J = \text{circ}(1, \ldots, 1)$). A Hadamard matrix $H$ of order $n$ is a matrix of order $n$ with entries in $\{-1, 1\}$ such that $\frac{H}{\sqrt{n}}$ is an orthogonal matrix. A circulant Hadamard matrix of order $n$ is a circulant matrix that is Hadamard. The 10 known circulant Hadamard matrices are $H_1 = \text{circ}(1), H_2 = -H_1, H_3 = \text{circ}(1, -1, -1), H_4 = -H_3, H_5 = \text{circ}(-1, -1, -1), H_6 = H_5, H_7 = \text{circ}(-1, -1, -1, -1), H_8 = -H_7, H_9 = \text{circ}(-1, -1, -1, 1), H_{10} = -H_9.

If $H = \text{circ}(h_1, \ldots, h_n)$, is a circulant Hadamard matrix of order $n$ then its representer polynomial is the polynomial $R(x) = h_1 + h_2 x + \cdots + h_n x^{n-1}$.

No one has been able, despite several deep computations (see [9]), to discover any other circulant Hadamard matrices of order $n > 4$. Ryser [2, p. 97], [15] proposed in 1963 the conjecture of the non-existence of these matrices when $n > 4$. Preceding work on the conjecture includes [3, 4, 6, 7, 8, 11, 13, 14, 16].

Ryser’s conjecture (there is no circulant Hadamard matrices of order $> 4$) has been studied by several different methods. Brualdi [1] proved in 1965 the first special, and important, case of the conjecture, in which all eigenvalues of a circulant Hadamard matrix $H = \text{circ}(h_1, \ldots, h_n)$ of order $n > 4$, are real; i.e., we assume that $H$ is symmetric. We relax in this paper the symmetry condition, by asking just a condition of symmetry modulo 2 of related matrix.

Assume the existence of a circulant Hadamard matrix $H$ of order $n > 4$. The present paper proves that this is impossible when the matrix $H_2 = (H + J)/2 \text{ reduced modulo 2}$ is a symmetric matrix. It is also impossible when an $n/4 \times n/4$ related matrix reduced (mod 2) is symmetric. The result follows, essentially, from a result of MacWilliams [10, Corollary 1.8] (see Lemma 5).

In order to be more precise, we define some sub-matrices of a given circulant matrix of even order. Let $M$ be a circulant matrix of even order $2k$. Observe that $M$, having even order $2k$, can be partitioned in four blocks $M_1, M_2, M_3, M_4$, each of size $k \times k$, as follows

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}.$$  

Since $M$ is circulant, we have $M_4 = M_1$, and $M_3 = M_2$. We are thus associating to the $2k \times 2k$ circulant matrix $M$ the square $k \times k$ matrices $M_1$ and $M_2$ (see exact details in Lemma 4), in order to have

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}.$$ 

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Our main result is as follows:

**Theorem 1** There is no circulant Hadamard matrix $H$ of order $n > 4$ provided

(a) the matrix $S_2 = (H + J)/2 \pmod{2}$ is symmetric, or

(b) both matrices $C$ and $D$, defined below, are symmetric.

Write $H$ as

$$H = \begin{bmatrix} H_1 & H_2 \\ H_2 & H_1 \end{bmatrix}$$

where the $n/2 \times n/2$ matrices $H_1$ and $H_2$ are defined in Lemma 4 applied to $H$. Put $T = (H_1 + H_2)/2$. Observe that $T$ is circulant, and define $n/4 \times n/4$ matrices $T_1$ and $T_2$ as above, by using again Lemma 4, this time applied to $T$. Namely, write $T$ as

$$T = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}.$$  

Finally, we define $C = T_1 \pmod{2}$, and $D = T_2 \pmod{2}$.

Section 2 contains the main tools necessary for the proof of the theorem. Section 3 contains the proof of Theorem 1. Throughout the paper, we let $A^*$ denote the transpose conjugate of a matrix $A$, and the identity matrix of order $k$ is denoted by $I_k$. We let $\mathbb{F}_2 = \{0, 1\}$ denote, as usual, the binary finite field. A binary matrix is a matrix with all its entries in $\mathbb{F}_2$.

## 2 Tools

The following is well known. See, e.g., [5, p. 1193], [12, p. 234], [16, pp. 329-330] for the first lemma and [2, p. 73] for the second.  

First of all, we recall the notion of regular Hadamard matrix.

**Definition 1** An $r$-regular Hadamard matrix is a Hadamard matrix whose row and column sums are all equal to $r$. A regular Hadamard matrix is an $r$-regular Hadamard matrix for some integer $r$.

**Lemma 1** Let $H$ be a regular Hadamard matrix of order $n \geq 4$. Then $n = 4h^2$ for some positive integer $h$. Moreover, if $H$ is circulant then $h$ is odd. Furthermore, either $H$ or $-H$ is $2h$-regular (the other is $(-2h)$-regular) and each row has $2h^2 + h$ positive entries and $2h^2 - h$ negative entries, when $H$ is $2h$-regular; respectively, has $2h^2 - h$ positive entries and $2h^2 + h$ negative entries, when $H$ is $(-2h)$-regular.

**Lemma 2** Let $H$ be a circulant Hadamard matrix of order $n \geq 1$, let $w = \exp(2\pi i/n)$, and let $R(x)$ be its representer polynomial. Then, the set of all eigenvalues of $H$, consists of the set of all $R(v)$ where $v \in \{1, w, w^2, \ldots, w^{n-1}\}$. Moreover, one has $|R(v)| = \sqrt{n}$.

More generally, and in more detail (see [2]), one has

**Lemma 3** Let $C = \text{circ}(c_1, \ldots, c_n)$ be a circulant matrix of order $n > 0$ with representer polynomial $P(t) = c_1 + c_2t + \ldots + c_nt^{n-1}$. Let $\omega$ be the primitive complex $n$-th root of unity with smaller positive argument. The matrix $C$ is diagonalizable and $C = F^* \Delta F$ where $\Delta = \text{diag}(P(1), P(w), \ldots, P(w^{n-1}))$ is a diagonal matrix containing the eigenvalues of $C$, and $F^* = \left(\frac{\omega^{(i-1)(j-1)}}{\sqrt{n}}\right)$ is the conjugate of the Fourier matrix. Moreover, $F$ is unitary.
The following is well known, useful, and easy to check:

**Lemma 4** Let $M$ be a circulant matrix of even order $n$ and with first row $R_1 = [m_1, \ldots, m_n]$. Then

(a) \[ M = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \]

where $M_1, M_2$ are the matrices of order $\frac{n}{2}$ defined by $M_1 = (a_{i,j})$, $M_2 = (b_{k,\ell})$, where $i, j, k, \ell = 1, \ldots, n/2$, and $a_{i,j} = m_{j-i+1}$, $b_{k,\ell} = m_{\ell+n/2-k+1}$, subscripts (mod $n$).

(b) The matrix $M_1 + M_2$ is circulant.

The following result of MacWilliams [10] is crucial.

**Lemma 5** The only circulant, symmetric, and orthogonal matrix, over the binary field $\mathbb{F}_2$, of given order $n$, is the identity matrix $I_n$.

The following “counting” lemma is important for the proof of the second part of the theorem.

**Lemma 6** Let $H$ be a $\sqrt{n}$-regular circulant Hadamard matrix of order $n > 1$. Let $H_1$ and $H_2$ be the $n/2$ square matrices defined in Lemma 4 applied to $H$. Let $M = \frac{H_1 + H_2}{2}$. Let $a = \text{number of } 0 \text{'s in the first row of } M$. Let $b = \text{number of } 1 \text{'s in the first row of } M$, and let $c = \text{number of } -1 \text{'s in the first row of } M$. Then

(i) $a = \frac{n}{4}$,

(ii) $b = \frac{n+2}{8}\sqrt{n}$,

(iii) $c = \frac{n-2}{8}\sqrt{n}$.

**Proof.** Since $H/\sqrt{n}$ is orthogonal, by Lemma 4, we have $H_1H_1^* + H_2H_2^* = nI_{n/2}$, and $H_1H_2^* + H_2H_1^* = 0$. Then, it follows that

\[ MM^* = (n/4)I_{n/2}. \]  \hspace{1cm} (1)

One has

\[ M = \text{circ} \left( \frac{h_1 + h_{n/2+1}}{2}, \ldots, \frac{h_{n/2} + h_n}{2} \right). \]

Observe, from (1), that $n/4$ equals the sum of squares of all entries in row 1 of $M$, and that an entry $\frac{h_i + h_{n/2+i}}{2} = 0$ does not contribute to the sum of squares, while the other entries, i.e., the nonzero ones, each contribute by 1 to the same sum. In other words one has

\[ n/4 = b + c. \]  \hspace{1cm} (2)

Since $H$ is $\sqrt{n}$-regular, and $2\sqrt{n} > 0$, we have that $M$ is $S$-regular, with $S > 0$. Compute now $S$, i.e., compute the sum of all entries in row 1 of $M$:

\[ S = \sum_{i=1}^{n/2} h_i + h_{n/2+i} = \frac{1}{2} \sum_{i=1}^{n} h_i = \frac{\sqrt{n}}{2}. \]  \hspace{1cm} (3)

But $S = b - c$, since zeros do not contribute to the sum, thus it follows from (3) that

\[ b - c = \frac{\sqrt{n}}{2}. \]  \hspace{1cm} (4)

From (2) and (4) we get (ii) and (iii). Since the total number of entries in the first row of $M$ is equal to $n/2$, we have

\[ n/2 = a + b + c, \]

thereby obtaining also (i). This finishes the proof of the lemma. ■
3 Proof of Theorem 1

Proof. Assume, on the contrary, the existence of a circulant Hadamard matrix $H = \text{circ}(h_1, \ldots, h_n)$ where $n > 4$, such that

(a) for $C_1 = (H + J)/2$, the matrix $S_2 = C_1 \pmod{2}$ is symmetric. Put $I = I_n$. By Lemma 1, $n = 4h^2$ with odd $h > 1$, and we can assume that all the row sums of $H$ equal $2h$ (i.e. $H$ is $2h$–regular).

Observe that $HH^* = 4h^2I$, $HJ = JH^* = 2hJ$, and $J^2 = nJ$. Thus

$$C_1C_1^* = HH^*/4 + (HJ + JH^*)/4 + J^2/4 = h^2I + (h + h^2)J.$$ \hspace{1cm} (5)

Since $h$ is odd, it follows then from (5) that $S_2S_2^* = I$, as a matrix over $\mathbb{F}_2$. In other words, $S_2$ is an orthogonal matrix of order $n$ over $\mathbb{F}_2$. Thus, since we assumed that $S_2$ is symmetric, Lemma 5 implies that $S_2 = I$. In particular, the number of 1’s in the first row of $S_2$ is equal to 1. But, by definition of $S_2$, this says that $C_1$ is a $\{0, 1\}$ matrix, and thus $H$ has also only a single 1 in its first row. By Lemma 1, and since $H$ is $2h$–regular, we know that the number of these 1’s is equal to $2h^2 + h$. We conclude that $2h^2 + h = 1$. This is impossible since $h > 1$. This contradiction proves the result.

(b) Put $E = C + D$. Thus $E$ is symmetric. Apply Lemma 4 to $M = H$ to get matrices $A_1 = H_1, B_1 = H_2$ of order $2h^2$ for which $T = (H_1 + H_2)/2$ is a circulant $\{-1, 0, 1\}$ matrix. Apply again Lemma 4, this time to $M = T$, to get matrices $A_2 = T_1, B_2 = T_2$ of order $h^2$ for which $L = (A_2 + B_2)$ is a circulant matrix with entries in $\{-2, -1, 0, 1, 2\}$. Thus $C = A_2 \pmod{2},$ and $D = B_2 \pmod{2}$. Since $HH^* = nI_n$, we get by block multiplication

$$A_1A_1^* + B_1B_1^* = 4h^2I_{2h^2}, \quad A_1B_1^* + B_1A_1^* = 0$$ \hspace{1cm} (6)

so that, by adding both equations in (6) we get

$$TT^* = h^2I_{2h^2}.$$ \hspace{1cm} (7)

Remember that we have

$$D = D^*, \quad C = C^*.$$ \hspace{1cm} (8)

Put $U = T \pmod{2}$. Reducing (7) (mod 2) one sees that $U$ is orthogonal. Thus, it follows from the definition of $T$, and from (8), that $U$ is also symmetric. Therefore, a new application of Lemma 5 gives

$$U = I_{2h^2}.$$ \hspace{1cm} (9)

But (9) contradicts Lemma 6 since the number of entries equal to $-1$ or to 1 in the first row of $T$ (and thus, the number of 1’s in the first row of $U$) is (with the notation of the lemma) equal to $b+c = h^2 \geq 9,$ and not equal to 1, as is in the matrix $I_{2h^2}$. This contradiction proves the result.

Remark 1 Concerning the proof of part (b) of the theorem. Asking that $E = C + D$ be symmetric, instead of asking that both $C$ and $D$ be symmetric, (in the hypothesis of the theorem), seems too weak, in order to get the same result. Moreover, when both $C$ and $D$ are assumed to be symmetric, it is possible to prove, using again MacWilliams result, that one has $E = I_{2h^2}$ (i.e., that we have $C = I_{2h^2} + D$). However, this alone do not seems to give a contradiction. Thus, we obtained the contradiction (that proved part (b) of the Theorem) by focusing on the $2h^2 \times 2h^2$ matrix $T$, instead.

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References


