Boundary Value Problem For A Coupled System Of Nonlinear Fractional Differential Equations Involving Erdélyi-Kober Derivative*

Yacine Arioua†, Maria Titraoui‡

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Abstract

In this paper, we focus on the boundary value problem (BVP) for a coupled system of nonlinear fractional differential equations (SFDEs) involving the Erdélyi–Kober derivatives on an infinite interval. First, we define the integral solution of the BVP for Erdélyi-Kober (SFDEs). Then, by using the Banach contraction principle and the Leray-Schauder nonlinear alternative fixed point theorem in a special Banach space, existence and uniqueness theorems of the given problem are demonstrated, respectively. Finally, several indispensable examples are presented to illustrate the usefulness of our main results.

1 Introduction

Fractional-order differential equations have been used in the study of models of many phenomena in various fields of science and engineering, such as viscoelasticity, fluid mechanics, electrochemistry, control, porous media, mathematical biology, and electromagnetic bioengineering. More details are available, for instance, in the books by Samko et al. 1993 [27], Podlubny 1999 [25], Kilbas et al. 2006 [12], Sabatier et al. 2007 [26], Das 2008 [9], Diethelm 2010 [10], and Mathai and Haubold 2018 [20].

The classical fractional calculus is based on several definitions for the operators of integration and differentiation of arbitrary order [13]. Among the various definitions of fractional differentiation, the Riemann–Liouville and Caputo fractional derivatives are widely used in the literature. The most useful classical fractional integrals, however, seem to be the Erdélyi–Kober operators. These were introduced by Sneddon (see, for example, [29, 30, 31]), who studied their basic properties and emphasized their useful applications to generalized axially symmetric potential theory and other physical problems, such as in electrostatics and elasticity.

The theory of boundary value problems on infinite intervals arises quite naturally and has many applications [4]; it is important and several authors have done much work on this topic [2, 3, 15, 16, 17, 23, 28, 32, 34, 35, 37, 39]. For instance, in [22], Ahmad and Ntouyas, studied a three point boundary value problem for a coupled system of nonlinear fractional differential equations given by

$$\begin{cases}
D^\alpha_{0+} u(t) = f(t, v(t), D^p_{0+} v(t)), & t \in (0, 1), \\
D^\beta_{0+} v(t) = g(t, u(t), D^q_{0+} u(t)), & t \in (0, 1), \\
u(0) = 0, & u(1) = \gamma u(\eta), \\
v(0) = 0, & v(1) = \gamma v(\eta),
\end{cases}$$

where $1 < \alpha, \beta < 2$, $p, q, \gamma > 0$, $0 < \eta < 1$, $D$ is the standard Riemann–Liouville fractional derivative and $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

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†Department of Mathematics, Laboratory for Pure and Applied Mathematics, University of M’sila, Bp 166 M’sila, Algeria
‡Laboratory for Pure and Applied Mathematics, University of M’sila, Bp 166 M’sila, Algeria
In [36], Zhai and Jiang considered a new coupled system of fractional differential equations with integral boundary conditions

\[
\begin{align*}
\mathcal{D}_t^\alpha u(t) + f(t, v(t)) &= a, \quad 0 < t < 1, \\
\mathcal{D}_t^\beta v(t) + g(t, u(t)) &= b, \quad 0 < t < 1, \\
u(0) &= 0, \quad u(1) = \int_0^1 \phi(t) u(t) dt, \\
v(0) &= 0, \quad v(1) = \int_0^1 \psi(t) v(t) dt,
\end{align*}
\]

where \(1 < \alpha, \beta \leq 2\), \(a, b\) are constants. \(\mathcal{D}\) denotes the usual Riemann–Liouville fractional derivative. \(f, g \in C([0, 1] \times \mathbb{R} \times \mathbb{R})\), \(\phi, \psi \in L^1[0, 1]\).

The aim of this study is to investigate the existence and uniqueness of a positive solution to boundary value problem for a coupled system of nonlinear fractional differential equation involving Erdélyi–Kober differential operators on an infinite interval:

\[
\begin{align*}
\mathcal{D}_t^{\gamma_1, \delta_1} u(t) + f_1(t, u(t), v(t)) &= 0, \quad t > 0, \\
\mathcal{D}_t^{\gamma_2, \delta_2} v(t) + f_2(t, u(t), v(t)) &= 0, \quad t > 0,
\end{align*}
\]

with the boundary conditions

\[
\lim_{t \to 0} t^{\beta(2+\gamma)} \mathcal{T}^{\delta_1+\gamma, 2-\delta_1} u(t) = 0, \quad \lim_{t \to \infty} t^{\beta(1+\gamma)} \mathcal{T}^{\delta_1+\gamma, 2-\delta_1} u(t) = 0, \\
\lim_{t \to 0} t^{\beta(2+\gamma)} \mathcal{T}^{\delta_2+\gamma, 2-\delta_2} v(t) = 0, \quad \lim_{t \to \infty} t^{\beta(1+\gamma)} \mathcal{T}^{\delta_2+\gamma, 2-\delta_2} v(t) = 0,
\]

where \(\mathcal{D}_t^{\gamma, \delta_k}, (k = 1, 2),\) denotes the Erdélyi–Kober fractional derivative operator of order \(\delta_k\) and \(\mathcal{T}^{\delta_k+\gamma, 2-\delta_k}\) is the Erdélyi–Kober fractional integral of order \(2 - \delta_k\), with \(1 < \delta_k \leq 2\), \(-2 < \gamma < -1\), \(\beta > 0\), and \(f_k\) are given continuous functions.

We obtain several existence and uniqueness results for a coupled system of the nonlinear fractional boundary value problem (1)–(3). The methods used in this work are the Leray–Schauder nonlinear alternative fixed point theorem and Banach contraction principle in a special Banach space.

Throughout this paper, we will refer to the following hypotheses:

\textbf{(H1)} \(f_k : (0, \infty) \times \mathbb{R}^2 \rightarrow (0, \infty)\) are continuous functions.

\textbf{(H2)} For all \((t, u, v) \in (0, \infty) \times \mathbb{R}^2\),

\[
\begin{align*}
F_1(t, u, v) &= t^{\beta(1+\gamma)-1} f_1 \left( t, \left(1 + t^{-\beta(1+\gamma)}\right) u, \left(1 + t^{-\beta(1+\gamma)}\right) v \right), \\
F_2(t, u, v) &= t^{\beta(1+\gamma)-1} f_2 \left( t, \left(1 + t^{-\beta(1+\gamma)}\right) u, \left(1 + t^{-\beta(1+\gamma)}\right) v \right),
\end{align*}
\]

such that

\[
\begin{align*}
F_1(t, u, v) &\leq \varphi_1(t) \omega_1(|u|) + \psi_1(t) \omega_2(|v|), \\
F_2(t, u, v) &\leq \varphi_2(t) \tilde{\omega}_1(|u|) + \psi_2(t) \tilde{\omega}_2(|v|),
\end{align*}
\]

with \(\omega_k, \tilde{\omega}_k \in C((0, \infty), (0, \infty))\) nondecreasing and \(\varphi_k, \psi_k \in L^1(0, \infty)\), \(k = 1, 2\).

\textbf{(H3)} There exist a positive functions \(q_k, \tilde{q}_k\) with

\[
q_k^* = \int_0^\infty \left(1 + t^{-\beta(1+\gamma)}\right) q_k(t) dt < \infty,
\]

\[
\tilde{q}_k^* = \int_0^\infty \left(1 + t^{-\beta(1+\gamma)}\right) \tilde{q}_k(t) dt < \infty,
\]

such that

\[
I^{\beta(\gamma+1)-1} |f_k(t, u, v) - f_k(t, \tilde{u}, \tilde{v})| \leq q_k(t) |u - \tilde{u}| + \tilde{q}_k(t) |v - \tilde{v}|,
\]

for any \(u, v, \tilde{u}, \tilde{v} \in \mathbb{R}\) and \(t \in (0, \infty)\).
The remainder of this paper is organized as follows. In Section 2, we recall some necessary preliminary facts. In Section 3, we prove our main results, after we have established sufficient conditions for the existence and uniqueness results for the solution to the problem (1)-(3). In Section 4, two examples are presented to explain the application of our main results.

2 Preliminaries

In this section, we present the necessary definitions and lemmas from fractional calculus theory that will be used to derive our main results.

Definition 1 ([18]) The space of functions $C^n_\alpha$, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, consists of all functions $f(t)$, $t > 0$, that can be represented in the form $f(t) = t^p f_1(t)$ with $p > \alpha$ and $f_1 \in C^n([0, \infty))$.

Definition 2 (Erdélyi–Kober fractional integrals [18]) The right-hand Erdélyi-Kober fractional integrals of the orders $\delta$ of the function $u \in C^n$ are defined by

$$I^\gamma_{\beta}u(t) = \beta \int_0^t \frac{t^{\gamma-\delta} - s^{\gamma-\delta}}{r^{\beta}} \left( s^{\delta-1} \right) u(s) ds, \quad \gamma > 0, \quad \beta > 0, \quad \gamma, \delta \in \mathbb{R},$$

where $\Gamma$ is the Euler gamma function.

Definition 3 (Erdélyi–Kober fractional derivatives [18]) Let $n - 1 < \delta \leq n$, $n \in \mathbb{N}$. The right-hand Erdélyi-Kober fractional derivative of the order $\delta$ of the function $u \in C^n$ is defined by

$$D^\gamma_{\beta}u(t) = \prod_{j=1}^n \left( \gamma + j + \frac{1}{\beta} \frac{d}{dt} \right) I^\gamma_{\beta+n-\delta}u(t),$$

where

$$\prod_{j=1}^n \left( \gamma + j + \frac{1}{\beta} \frac{d}{dt} \right) I^\gamma_{\beta+n-\delta}u = \left( \gamma + 1 + \frac{1}{\beta} \frac{d}{dt} \right) \cdots \left( \gamma + n + \frac{1}{\beta} \frac{d}{dt} \right) I^\gamma_{\beta+n-\delta}u.$$

Remark 1 Let $\delta, \beta > 0$ and $\gamma \in \mathbb{R}$. Then we have

$$D^\gamma_{\beta}t^p = \frac{\Gamma \left( \gamma + \frac{p}{\beta} + 1 \right)}{\Gamma \left( \gamma + \frac{p}{\beta} + 1 \right)} t^p, \quad p + \beta (\gamma + 1) > 0.$$

In particular,

$$D^\gamma_{\beta}t^{-\beta(\gamma-i)} = 0, \text{ for each } i \in \{1, 2, ..., n\}.$$

Definition 4 Let $E$ be a Banach space; a subset $P$ in $C(E)$ is called equicontinuous if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall u, v \in E, \forall A \in P, \|u - v\| < \delta \Rightarrow |A(u) - A(v)| < \varepsilon.$$

Theorem 1 (Ascoli–Arzela [7]) Let $E$ be a compact metric space. If $P$ is an equicontinuous, bounded subset of $C(E)$, then $P$ is relatively compact.

Definition 5 ([4]) Let $E$ be a Banach space. A map $A : X \subseteq E \rightarrow E$ is said to be completely continuous if $A(Y)$ is relatively compact for all bounded sets $Y \subseteq X$.

The following fixed-point theorems are fundamental in the proofs of our main results.
Theorem 2 (Leray–Schauder Nonlinear Alternative theorem [14]) Let $E$ be a Banach space, and $\Omega$ a bounded open subset of $E$ with $0 \in \Omega$. Then, every completely continuous map $A : \overline{\Omega} \to E$ has at least one of the following two properties:

(i) $A$ has a fixed point in $\overline{\Omega}$.

(ii) There is an $x \in \partial \Omega$ and $\lambda \in (0,1)$ with $x = \lambda Ax$.

Theorem 3 (Banach’s fixed point theorem [1]) Let $E$ be a Banach space, $D$ be closed subset of $E$, and $A : D \to D$ be a strict contraction, i.e.,

$$\|Au - Av\| \leq k \|u - v\| \text{ for some } k \in (0,1) \text{ and all } u, v \in D.$$ 

Then $A$ has a unique fixed point.

3 Main Results

In this section, we prove a preparatory lemma for the boundary value problem of nonlinear fractional differential equations with an Erdélyi-Kober derivative.

Lemma 1 ([5]) Let $1 < \delta \leq 2$, $-2 < \gamma < -1$, $\beta > 0$ and $y \in C^2_{\alpha}$, with $\int_0^1 s^{\beta(\gamma+m)-1}y(s)ds < \infty$, $m = \{1,2\}$. Then, the fractional differential equation

$$D_{\delta}^{\gamma}\frac{\partial}{\partial t} u(t) + y(t) = 0, \quad t > 0,$$  

with the conditions

$$\lim_{t \to 0} t^{\beta(2+\gamma)}I^{\delta+\gamma,2-\delta}u(t) = 0,$$  

$$\lim_{t \to \infty} t^{\beta(1+\gamma)}I^{\delta+\gamma,2-\delta}u(t) = 0,$$

has a unique solution given by

$$u(t) = \int_0^\infty G_\delta(t,s)s^{\beta(\gamma+1)-1}y(s)ds,$$  

where

$$G_\delta(t,s) = \begin{cases} \frac{1}{\Gamma(\delta)} \left[ t^{-\beta(\gamma+1)} - t^{-\beta(\delta+\gamma)} (t^\beta - s^\beta)^{\delta-1} \right], & 0 < s \leq t < \infty, \\ \frac{1}{\Gamma(\delta)} t^{-\beta(\gamma+1)}, & 0 < t \leq s < \infty, \end{cases}$$

is called the Green function of the boundary value problem (6)–(8).

Now, we present some properties of Green’s function that form the basis of our main work.

Lemma 2 ([5]) For $1 < \delta \leq 2$, $-2 < \gamma < -1$, and $\beta > 0$, the function $G_\delta(t,s)$ in Lemma 1 satisfies the following conditions:

1. $G_\delta(t,s) > 0$, $\forall t, s \in (0, \infty)$.

2. $\frac{G_\delta(t,s)}{t^{\beta(1+\gamma)}} \leq \frac{\beta}{\Gamma(\delta)}$, $\forall t, s \in (0, \infty)$.

3. For all $0 < \frac{\tau}{\lambda} \leq t \leq \tau$ and $\forall s > \frac{\tau}{\lambda}$, where $\lambda > 1$, $\tau > 0$, we have

$$\frac{G_\delta(t,s)}{1 + t^{-\beta(1+\gamma)}} \geq \frac{\beta (\delta - 1) \tau^{-\beta(1+\gamma)}}{\Gamma(\delta) \lambda^{\beta(1-\gamma)} (1 + \tau^{-\beta(1+\gamma)})} = \frac{\beta}{\Gamma(\delta)} p(\tau).$$
Based on the previous Lemma 1, we will define the integral solution of the problem (1)–(3).

**Lemma 3** Let $0 < \delta_1, \delta_2 \leq 1$. We give $f_k \in C^2_{\alpha}(k = 1, 2)$ with

$$
\int_0^\infty s^{\beta(\gamma+m)-1} f_k(s, u(s), v(s)) ds < \infty, \quad m = \{1, 2\}.
$$

Then the problem (1)–(3) has a unique solution given by

$$
u(t) = \int_0^\infty G_{\delta_1}(t, s) s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s)) ds,
$$

$$
v(t) = \int_0^\infty G_{\delta_2}(t, s) s^{\beta(\gamma+1)-1} f_2(s, u(s), v(s)) ds,
$$

where

$$
G_{\delta_1}(t, s) = \left\{ \begin{array}{ll}
\frac{\beta}{\Gamma(\beta)} \left[ t^{-\beta(\gamma+1)} - t^{-\beta(\delta_1+\gamma)} (t^\beta - s^\beta)^{\delta_1-1} \right], & 0 < s \leq t < \infty, \\
\frac{\beta}{\Gamma(\beta)} t^{-\beta(\gamma+1)}, & 0 < t \leq s < \infty,
\end{array} \right.
$$

$$
G_{\delta_2}(t, s) = \left\{ \begin{array}{ll}
\frac{\beta}{\Gamma(\beta)} \left[ t^{-\beta(\gamma+1)} - t^{-\beta(\delta_2+\gamma)} (t^\beta - s^\beta)^{\delta_2-1} \right], & 0 < s \leq t < \infty, \\
\frac{\beta}{\Gamma(\beta)} t^{-\beta(\gamma+1)}, & 0 < t \leq s < \infty,
\end{array} \right.
$$

**Proof.** By Lemma 1, for $\delta = \delta_1$, and $\delta = \delta_2$, respectively. ■

We now turn to the question of existence for the boundary value problem (1)–(3).

In this work, we use the space $E = X \times Y$, with the norm $\|(u, v)\|_E = \|u\|_X + \|v\|_Y$. Obviously $(E, \|(u, v)\|_E)$, is a Banach space, where

$$
X = \left\{ u \in C((0, \infty), \mathbb{R}) \mid \lim_{t \to 0} \frac{u(t)}{1 + t^{-\beta(1+\gamma)}} \text{ and } \lim_{t \to +\infty} \frac{u(t)}{1 + t^{-\beta(1+\gamma)}} \text{ exist} \right\},
$$

with the norm

$$
\|u\|_X = \sup_{t > 0} \left| \frac{u(t)}{1 + t^{-\beta(1+\gamma)}} \right|.
$$

and

$$
Y = \left\{ v \in C((0, \infty), \mathbb{R}) \mid \lim_{t \to 0} \frac{v(t)}{1 + t^{-\beta(1+\gamma)}} \text{ and } \lim_{t \to +\infty} \frac{v(t)}{1 + t^{-\beta(1+\gamma)}} \text{ exist} \right\},
$$

with the norm

$$
\|v\|_Y = \sup_{t > 0} \left| \frac{v(t)}{1 + t^{-\beta(1+\gamma)}} \right|.
$$

Define an integral operator $A : E \to E$ by

$$
A(u, v)(t) = \left( \begin{array}{c}
A_1(u, v)(t) \\
A_2(u, v)(t)
\end{array} \right),
$$

(15)

where

$$
A_1(u, v)(t) = \int_0^\infty G_{\delta_1}(t, s) s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s)) ds,
$$

$$
A_2(u, v)(t) = \int_0^\infty G_{\delta_2}(t, s) s^{\beta(\gamma+1)-1} f_2(s, u(s), v(s)) ds,
$$

with $G_{\delta_k}(t, s), (k = 1, 2)$ is defined by (13) and (14).

Clearly, from Lemma 3, the fixed points of the operator $A$ coincide with the solutions of the problem (1)–(3).
Lemma 4 If (H1)–(H2) hold, then \( \mathcal{A} : E \to E \) is completely continuous.

Proof. First, for all \((u, v) \in E\), we have
\[
\| \mathcal{A}_1(u, v)(t) \|_X = \sup_{t > 0} \frac{\| \mathcal{A}_1(u, v)(t) \|}{1 + t^{-\beta(1+\gamma)}}
\]
\[
= \sup_{t > 0} \left| \int_0^\infty \frac{G_{\delta_1}(t, s)}{1 + t^{-\beta(1+\gamma)}} s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s)) \, ds \right|
\]
\[
\leq \frac{\beta}{\Gamma(\delta)} \int_0^\infty \left| s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s)) \right| \, ds;
\]

Together with conditions (H1) and (H2), it then follows that
\[
\int_0^\infty \left| s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s)) \right| \, ds
\]
\[
= \int_0^\infty \left| s^{\beta(1+\gamma)-1} f_1 \left( s, \frac{(1 + s^{-\beta(1+\gamma)}) u(s)}{1 + s^{-\beta(1+\gamma)}}, \frac{(1 + s^{-\beta(1+\gamma)}) v(s)}{1 + s^{-\beta(1+\gamma)}} \right) \right| \, ds
\]
\[
= \int_0^\infty F_1 \left( s, \frac{u(s)}{1 + s^{-\beta(1+\gamma)}}, \frac{v(s)}{1 + s^{-\beta(1+\gamma)}} \right) \, ds
\]
\[
\leq \int_0^\infty \left( \varphi_1(s) \omega_1 \left( \frac{|u(s)|}{1 + s^{-\beta(1+\gamma)}} \right) + \psi_1(s) \omega_2 \left( \frac{|v(s)|}{1 + s^{-\beta(1+\gamma)}} \right) \right) \, ds
\]
\[
\leq \omega_1(\|u\|_X) \int_0^\infty \varphi_1(s) \, ds + \omega_2(\|v\|_Y) \int_0^\infty \psi_1(s) \, ds < \infty.
\]

Similarly, one can find that
\[
\| \mathcal{A}_2(u, v)(t) \|_Y \leq \tilde{\omega}_1(\|u\|_X) \int_0^\infty \varphi_2(s) \, ds + \tilde{\omega}_2(\|v\|_Y) \int_0^\infty \psi_2(s) \, ds < \infty.
\]

Hence, \( \mathcal{A} : E \to E \) is well-defined. Let \( \Omega = \{(u, v) \in E, \|(u, v)\|_E \leq r, \ r > 0\} \) be a bounded subset of \( E \). In the following, we divide the proof into several steps.

Step 1: \( \mathcal{A} \) is continuous.

Let \((u_n, v_n)_{n \in \mathbb{N}} \in E\) be a convergent sequence to \((u, v)\) in \( E \) such that \( \|(u, v)\|_E \leq r \), from Lemma 2, we obtain that
\[
\| \mathcal{A}_1(u_n, v_n) - \mathcal{A}_1(u, v) \|_X = \sup_{t \in (0, \infty)} \frac{\| \mathcal{A}_1(u_n, v_n)(t) - \mathcal{A}_1(u, v)(t) \|}{1 + t^{-\beta(1+\gamma)}}
\]
\[
\leq \frac{\beta}{\Gamma(\delta_1)} \sup_{t \in (0, \infty)} \left| \int_0^\infty s^{\beta(1+\gamma)-1} f_1(s, u_n(s), v_n(s)) \, ds - \int_0^\infty s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s)) \, ds \right|.
\]

By the condition (H2), we obtain
\[
\left| s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s)) \right| \leq \omega_1(r) \varphi_1(s) + \omega_2(r) \psi_1(s) \in L^1(0, \infty).
\]

Together with the continuity of the function \( s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s)) \), the Lebesgue dominated convergence theorem (Theorem 12.12, page 199 in [6]) yields \((u, v) \to \int_0^\infty s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s)) \, ds\), is continuous and it follows that
\[
\int_0^\infty s^{\beta(1+\gamma)-1} f_1(s, u_n(s), v_n(s)) \, ds \to \int_0^\infty s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s)) \, ds \text{ as } n \to \infty.
\]
Therefore,
\[ \|A_1(u_n,v_n) - A_1(u,v)\|_X \to 0, \text{ as } n \to \infty. \]

Similarly, one can find that
\[ \|A_2(u_n,v_n) - A_2(u,v)\|_Y \to 0, \text{ as } n \to \infty. \]

Thus
\[ \|A(u_n,v_n) - A(u,v)\|_E \to 0, \text{ as } n \to \infty. \]

**Step 2:** \(A(\Omega)\) is relatively compact.
First, we show that \(A(\Omega)\) is uniformly bounded. Let \((u,v) \in \Omega\), by the condition (H2), we obtain
\[
\frac{|A_1(u,v)(t)|}{1 + t^{-\beta(1+\gamma)}} = \left| \int_0^\infty \frac{G_{\delta_1}(t,s)}{1 + t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) ds \right| 
\leq \frac{\beta}{\Gamma(\delta_1)} \int_0^\infty |s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))| ds 
\leq \frac{\beta}{\Gamma(\delta_1)} \omega_1(r) \int_0^\infty \varphi_1(s) ds + \frac{\beta}{\Gamma(\delta_1)} \omega_2(r) \int_0^\infty \psi_1(s) ds < \infty,
\]
consequently,
\[ \|A_1(u,v)\|_X \leq \frac{\beta}{\Gamma(\delta_1)} \omega_1(r) \int_0^\infty \varphi_1(s) ds + \frac{\beta}{\Gamma(\delta_1)} \omega_2(r) \int_0^\infty \psi_1(s) ds < \infty, \tag{16} \]

Similarly, one can find that
\[ \|A_2(u,v)\|_Y \leq \frac{\beta}{\Gamma(\delta_2)} \omega_1(r) \int_0^\infty \varphi_2(s) ds + \frac{\beta}{\Gamma(\delta_2)} \omega_2(r) \int_0^\infty \psi_2(s) ds < \infty, \tag{17} \]
for all \((u,v) \in \Omega\). Hence, \(A(\Omega)\) is uniformly bounded.

Next, letting \(V = \left\{ \frac{A(u,v)}{1 + t^{-\beta(1+\gamma)}} \mid (u,v) \in \Omega \right\}\), we show that \(V\) is equicontinuous on any compact interval of \(\mathbb{R}_+\). For all \((u,v) \in \Omega, t_1, t_2 \in [a,b], 0 < a < b < \infty,\) and \(t_1 \leq t_2\), we can find
\[
\left| \frac{A_1(u,v)(t_2)}{1 + t_2^{-\beta(1+\gamma)}} - \frac{A_1(u,v)(t_1)}{1 + t_1^{-\beta(1+\gamma)}} \right| \leq \int_0^\infty \left| \frac{G_{\delta_1}(t_2,s)}{1 + t_2^{-\beta(1+\gamma)}} - \frac{G_{\delta_1}(t_1,s)}{1 + t_1^{-\beta(1+\gamma)}} \right| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) ds 
\leq \int_0^\infty \left| \frac{G_{\delta_1}(t_2,s)}{1 + t_2^{-\beta(1+\gamma)}} - \frac{G_{\delta_1}(t_1,s)}{1 + t_1^{-\beta(1+\gamma)}} \right| \left( t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)} \right) \left( 1 + t_2^{-\beta(1+\gamma)} \right) \left( 1 + t_1^{-\beta(1+\gamma)} \right) \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds 
\leq \int_0^\infty \left| \frac{G_{\delta_1}(t_2,s)}{1 + t_2^{-\beta(1+\gamma)}} - \frac{G_{\delta_1}(t_1,s)}{1 + t_1^{-\beta(1+\gamma)}} \right| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) ds 
\leq \int_0^\infty \frac{G_{\delta_1}(t_1,s)}{1 + t_2^{-\beta(1+\gamma)}} \left( s^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)} \right) \left( 1 + t_2^{-\beta(1+\gamma)} \right) \left( 1 + t_1^{-\beta(1+\gamma)} \right) \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds 
\to 0, \text{ uniformly as } t_1 \to t_2 \text{ for all } (u,v) \in \Omega. \]
Similarly, observing that for any $t_1, t_2 \geq T, 0 < s \leq L$, it holds that

$$
\begin{align*}
&\frac{A_2(u, v) (t_2)}{1 + t_2^{-\beta(1+\gamma)}} - \frac{A_2(u, v) (t_1)}{1 + t_1^{-\beta(1+\gamma)}} \\
\leq & \int_0^\infty \frac{|G_{\delta_2} (t_2, s) - G_{\delta_2} (t_1, s)|}{1 + t_2^{-\beta(1+\gamma)}} |s^{\beta(\gamma+1)-1} f_2(s, u(s), v(s))| \, ds \\
&+ \int_0^\infty \frac{G_{\delta_2} (t_1, s) \left( t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)} \right)}{\left(1 + t_2^{-\beta(1+\gamma)}\right) \left(1 + t_1^{-\beta(1+\gamma)}\right)} |s^{\beta(\gamma+1)-1} f_2(s, u(s), v(s))| \, ds.
\end{align*}
$$

Hence, $V$ is locally equicontinuous on $(0, \infty)$.

Finally, we show that $V$ is equiconvergent at $\infty$. We know that

$$
A_1(u, v)(t) = \frac{\beta}{\Gamma(\delta_1)} t^{-\beta(1+\gamma)} \int_0^\infty s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s)) ds = \frac{-\beta}{\Gamma(\delta_1)} t^{-\beta(1+\delta_1)} \int_0^t (t^\beta - s^\beta)^{\delta_1-1} s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s)) ds,
$$

observing that for any $(u, v) \in \Omega$, the condition (H2) gives

$$
\int_0^\infty |s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s))| \, ds < \infty,
$$

for a given $\varepsilon > 0$, there exists a constant $L > 0$, such that

$$
\int_L^\infty |s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s))| \, ds < \varepsilon.
$$

However, because $\lim_{t \to \infty} \frac{t^{-\beta(1+\gamma)}}{1 + t^{-\beta(1+\gamma)}} = 1$, there exists a constant $T_1 > 0$, such that for any $t_1, t_2 \geq T_1$, and we obtain

$$
\begin{align*}
&\frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} - \frac{t_2^{-\beta(1+\gamma)}}{1 + t_2^{-\beta(1+\gamma)}} \\
\leq & 1 - \frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} + 1 - \frac{t_2^{-\beta(1+\gamma)}}{1 + t_2^{-\beta(1+\gamma)}} < \varepsilon.
\end{align*}
$$

Similarly, $\lim_{t \to \infty} \frac{t^{-\beta(1+\gamma)}}{1 + t^{-\beta(1+\gamma)}} = 1$, and thus there exists a constant $T_2 > L > 0$, such that for any $t_1, t_2 \geq T_2$ and $0 < s \leq L$, it holds that

$$
\begin{align*}
&\left| \frac{t_2^{-\beta(\delta_1+\gamma)} \left( t_2^\beta - s^\beta \right)^{\delta_1-1}}{1 + t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(\delta_1+\gamma)} \left( t_1^\beta - s^\beta \right)^{\delta_1-1}}{1 + t_1^{-\beta(1+\gamma)}} \right| \\
\leq & 1 - \frac{t_1^{-\beta(\delta_1+\gamma)} \left( t_1^\beta - s^\beta \right)^{\delta_1-1}}{1 + t_1^{-\beta(1+\gamma)}} + 1 - \frac{t_2^{-\beta(\delta_1+\gamma)} \left( t_2^\beta - s^\beta \right)^{\delta_1-1}}{1 + t_2^{-\beta(1+\gamma)}} \\
\leq & 1 - \frac{t_1^{-\beta(\delta_1+\gamma)} \left( t_1^\beta - L^\beta \right)^{\delta_1-1}}{1 + t_1^{-\beta(1+\gamma)}} + 1 - \frac{t_2^{-\beta(\delta_1+\gamma)} \left( t_2^\beta - L^\beta \right)^{\delta_1-1}}{1 + t_2^{-\beta(1+\gamma)}} < \varepsilon.
\end{align*}
$$
Now, we choose $T > \max\{T_1, T_2\}$, for all $t_1, t_2 \geq T$, by (18), we can obtain

\[
\left| \frac{A_1(u, v)(t_2)}{1 + t_2^{-\beta(1+\gamma)}} \right| + \gamma \left( 1 + t_2^{-\beta(1+\gamma)} \right) \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds
\]

\[
\leq \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_2^{-\beta(1+\gamma)}}{1 + t_2^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_2^{-\beta(1+\gamma)}}{1 + t_2^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds.
\]

a direct calculation yields

\[
\left| \frac{A_1(u, v)(t_2)}{1 + t_2^{-\beta(1+\gamma)}} \right| \leq \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_2^{-\beta(1+\gamma)}}{1 + t_2^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_2^{-\beta(1+\gamma)}}{1 + t_2^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_1^{-\beta(1+\gamma)}}{1 + t_1^{-\beta(1+\gamma)}} \right| \int_0^\infty |s^{\beta(1+\gamma)-1} f_1(s, u(s), v(s))| \, ds.
\]
Assume that hypotheses (H1)–(H2) hold, and now, to prove the first existence result, we use the Leray-Schauder nonlinear alternative fixed point theorem.

3.1 Existence of At Least One Solution

From (19), (20), (21), (22) and for \( t_1, t_2 \to \infty \), we obtain

\[
\frac{|A_1(u, v)(t_2)|}{1 + t_2^{-\beta(1+\gamma)}} - \frac{|A_1(u, v)(t_1)|}{1 + t_1^{-\beta(1+\gamma)}} < \frac{\beta}{\Gamma(\delta_1)} \varepsilon \int_0^\infty |s^{\beta(1+1)} f_1(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_1)} \varepsilon \int_0^L |s^{\beta(1+1)} f_1(s, u(s), v(s))| \, ds + \frac{2\beta}{\Gamma(\delta_1)} \varepsilon.
\]

Analogously, we can obtain

\[
\frac{|A_2(u, v)(t_2)|}{1 + t_2^{-\beta(1+\gamma)}} - \frac{|A_2(u, v)(t_1)|}{1 + t_1^{-\beta(1+\gamma)}} < \frac{\beta}{\Gamma(\delta_2)} \varepsilon \int_0^\infty |s^{\beta(1+1)} f_2(s, u(s), v(s))| \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_2)} \varepsilon \int_0^L |s^{\beta(1+1)} f_2(s, u(s), v(s))| \, ds + \frac{2\beta}{\Gamma(\delta_2)} \varepsilon.
\]

Hence, \( V \) is equiconvergent at \( \infty \). Consequently, \( V \) is relatively compact. Therefore, \( A : E \to E \) is completely continuous.

3.1 Existence of At Least One Solution

Now, to prove the first existence result, we use the Leray-Schauder nonlinear alternative fixed point theorem.

**Theorem 4** Assume that hypotheses (H1)–(H2) hold, and

\[
\frac{\beta}{\Gamma(\delta_1)} \omega_1(r) \int_0^\infty \varphi_1(s) \, ds + \frac{\beta}{\Gamma(\delta_1)} \omega_2(r) \int_0^\infty \psi_1(s) \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_2)} \bar{\omega}_1(r) \int_0^\infty \varphi_2(s) \, ds + \frac{\beta}{\Gamma(\delta_2)} \bar{\omega}_2(r) \int_0^\infty \psi_2(s) \, ds < r.
\]

Then, the fractional boundary value problem (1)–(3) has at least one solution \((u, v) \in \Omega\).

**Proof.** From Lemma 4, we know that \( \mathcal{A} \) is a completely continuous operator. We apply the nonlinear alternative of Leray-Schauder to prove that \( \mathcal{A} \) has at least one nontrivial solution in \( \Omega \). Let \((u, v) \in \partial \Omega\), such that \((u, v) = \lambda \mathcal{A}(u, v), \lambda \in (0, 1)\), from (16) and (17), we obtain

\[
\| (u, v) \|_E = \| u \|_X + \| v \|_Y
\]

\[
= \lambda \| \mathcal{A}_1(u, v) \|_X + \lambda \| \mathcal{A}_2(u, v) \|_Y
\]

\[
\leq \frac{\beta}{\Gamma(\delta_1)} \omega_1 r \int_0^\infty \varphi_1(s) \, ds + \frac{\beta}{\Gamma(\delta_1)} \omega_2 r \int_0^\infty \psi_1(s) \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_2)} \bar{\omega}_1 r \int_0^\infty \varphi_2(s) \, ds + \frac{\beta}{\Gamma(\delta_2)} \bar{\omega}_2 r \int_0^\infty \psi_2(s) \, ds,
\]

and thus

\[
r \leq \frac{\beta}{\Gamma(\delta_1)} \omega_1(r) \int_0^\infty \varphi_1(s) \, ds + \frac{\beta}{\Gamma(\delta_1)} \omega_2(r) \int_0^\infty \psi_1(s) \, ds
\]

\[
+ \frac{\beta}{\Gamma(\delta_2)} \bar{\omega}_1(r) \int_0^\infty \varphi_2(s) \, ds + \frac{\beta}{\Gamma(\delta_2)} \bar{\omega}_2(r) \int_0^\infty \psi_2(s) \, ds,
\]

which contradicts (23). By Theorem 2 and Lemma 4, the boundary value problem (1)–(3) has at least one solution \((u, v) \in \Omega\).
3.2 Uniqueness of Solution

The last result of the existence is based on the Banach contraction principle theorem.

**Theorem 5** Assume that hypotheses \((H1), (H2)\) and \((H3)\) hold. If

\[
\frac{\beta(q_1^* + q_1^*)}{\Gamma(\delta_1)} + \frac{\beta(q_2^* + q_2^*)}{\Gamma(\delta_2)} < 1,
\]

then the boundary value problem \((1)–(3)\) has a unique solution.

**Proof.** We shall show that the operator \(A\) defined by \((15)\) is a contraction mapping. Let \((u, v), (\bar{u}, \bar{v}) \in E\); from Lemma \(2\) and by the condition \((H3)\), we can obtain that

\[
\left| A_1(u, v)(t) - A_1(\bar{u}, \bar{v})(t) \right| \leq \int_0^\infty \frac{G_{\delta_1}(t, s)}{1 + t^{-\beta(1+\gamma)}} \frac{\beta}{\Gamma(\delta_1)} \int_0^\infty q_1(s) |f_1(s, u(s), v(s)) - f_1(s, \bar{u}(s), \bar{v}(s))| ds
\]

from Lemma \(2\) and by the condition \((H3)\), we can obtain that

\[
\left| A_2(u, v)(t) - A_2(\bar{u}, \bar{v})(t) \right| \leq \int_0^\infty \frac{G_{\delta_2}(t, s)}{1 + t^{-\beta(1+\gamma)}} \frac{\beta}{\Gamma(\delta_2)} \int_0^\infty q_2(s) |f_2(s, u(s), v(s)) - f_2(s, \bar{u}(s), \bar{v}(s))| ds
\]

this implies that

\[
\|A_1(u, v) - A_1(\bar{u}, \bar{v})\|_X \leq \frac{\beta}{\Gamma(\delta_1)} \|u - \bar{u}\|_X \int_0^\infty q_1(s) \left(1 + s^{-\beta(1+\gamma)}\right) ds
\]

this implies that

\[
\|A_1(u, v) - A_1(\bar{u}, \bar{v})\|_X \leq \frac{\beta}{\Gamma(\delta_1)} \|u - \bar{u}\|_X \int_0^\infty q_1(s) \left(1 + s^{-\beta(1+\gamma)}\right) ds
\]

then

\[
\|A_1(u, v) - A_1(\bar{u}, \bar{v})\|_X \leq \frac{\beta(q_1^* + q_1^*)}{\Gamma(\delta_1)} \|u - \bar{u}\|_X + \|v - \bar{v}\|_Y.
\]

Similarly, one can find that

\[
\|A_2(u, v) - A_2(\bar{u}, \bar{v})\|_Y \leq \frac{\beta(q_2^* + q_2^*)}{\Gamma(\delta_2)} \|u - \bar{u}\|_X + \|v - \bar{v}\|_Y.
\]

Thus it follows from \((25)\) and \((26)\), that

\[
\|A(u, v) - A(\bar{u}, \bar{v})\|_E \leq \left[ \frac{\beta(q_1^* + q_1^*)}{\Gamma(\delta_1)} + \frac{\beta(q_2^* + q_2^*)}{\Gamma(\delta_2)} \right] \|(u, v) - (\bar{u}, \bar{v})\|_E.
\]

It follows from the assumption \((24)\) and the preceding estimate that \(A\) is a contraction mapping. Applying Banach's fixed point Theorem \(3\), the operator \(A\) has a fixed point that corresponds to the unique solution of problem \((1)–(3)\). ■
4 Examples

In this section, we present some examples to illustrate the usefulness of our main results.

Example 1 Consider the following boundary value problem for a coupled System:

\[
\begin{align*}
D_1^{-\frac{3}{2}} u(t) + t^2 e^{-t} \sqrt{\frac{u}{1+t^2}} + e^{-t} \ln \left( \arctan \left( \frac{v}{1+t^2} \right) \right) &= 0, \quad t > 0, \\
D_1^{-\frac{3}{2}} v(t) + t^2 \ln \left( \left( \frac{v}{1+t^2} \right)^2 + 1 \right) e^{-2t^2 + 1} + t^2 \ln \left( \frac{u}{1+t^2} \right)^2 e^{-t} &= 0, \quad t > 0, \\
\lim_{t \to 0} t^\frac{3}{2} I_1^{0, \frac{3}{2}} u(t) &= 0, \quad \lim_{t \to \infty} t^{-\frac{1}{2}} I_1^{0, \frac{1}{2}} u(t) = 0, \\
\lim_{t \to 0} t^{\frac{3}{2}} I_1^{0, \frac{3}{2}} v(t) &= 0, \quad \lim_{t \to \infty} t^{-\frac{1}{2}} I_1^{0, \frac{1}{2}} v(t) = 0.
\end{align*}
\]

Here,

\[
\begin{align*}
f_1(t, u, v) &= t^\frac{3}{2} \left( \sqrt{\frac{u}{1+t^2}} + \ln \left( \arctan \left( \frac{v}{1+t^2} \right) \right) \right) e^{-t}, \\
f_2(t, u, v) &= t^\frac{1}{2} \ln \left( \left( \frac{v}{1+t^2} \right)^2 + 1 \right) e^{-2t^2 + 1} + t^2 \ln \left( \frac{u}{1+t^2} \right)^2 e^{-t},
\end{align*}
\]

with \( \delta_1 = \frac{3}{2}, \delta_2 = \frac{7}{6}, \gamma = -\frac{3}{2} \) and \( \beta = 1 \).

(H1) It is easy to show that the functions \( f_1 \) and \( f_2 \) are continuous for any \((t, u) \in (0, \infty) \times \mathbb{R}\).

- From the expression of the function \( f_1 \), it follows that

\[
F_1(t, u, v) = t^{3(1+\gamma)-1} f_1 \left( t, \left( 1 + t^{-\beta(1+\gamma)} \right) u, \left( 1 + t^{-\beta(1+\gamma)} \right) v \right) = \left( \sqrt{|u|} + \ln \left( \arctan \left( |v| \right) \right) \right) e^{-t}.
\]

If we choose \( \omega_1 (u) = \sqrt{u}, \omega_2 (v) = \ln \left( \arctan \left( |v| \right) \right), \varphi_1 (t) = \psi_1 (t) = e^{-t} \), then we obtain

\[
|F_1(t, u, v)| \leq \varphi_1 (t) \omega_1 (|u|) + \psi_1 (t) \omega_2 (|v|), \quad (0, \infty) \times \mathbb{R},
\]

with \( \omega_1, \omega_2 \in C \left( (0, \infty), (0, \infty) \right) \) nondecreasing and \( \varphi_1, \psi_1 \in L^1 (0, \infty) \).

- From the expression of the function \( f_2 \), it follows that

\[
F_2(t, u, v) = t^{3(1+\gamma)-1} f_2 \left( t, \left( 1 + t^{-\beta(1+\gamma)} \right) u, \left( 1 + t^{-\beta(1+\gamma)} \right) v \right) = \ln \left( v^2 + 1 \right) t e^{-2t^2 + 1} + \ln u^2 e^{-t}.
\]

If we choose \( \bar{\omega}_1 (u) = \ln u^2, \bar{\omega}_2 (v) = \ln \left( v^2 + 1 \right), \varphi_2 (t) = e^{-t}, \psi_2 (t) = t e^{-2t^2 + 1} \), then we obtain

\[
|F_2(t, u, v)| \leq \varphi_2 (t) \bar{\omega}_1 (|u|) + \psi_2 (t) \bar{\omega}_2 (|v|), \quad (0, \infty) \times \mathbb{R},
\]

with \( \bar{\omega}_1, \bar{\omega}_2 \in C \left( (0, \infty), (0, \infty) \right) \) nondecreasing and \( \varphi_2, \psi_2 \in L^1 (0, \infty) \). Then, the condition (H2) holds.

If we choose \( r = \frac{1}{2} \), we show that

\[
\frac{2\sqrt{r}}{\sqrt{\pi}} + \frac{2 \ln \left( \arctan \left( r \right) \right)}{\sqrt{\pi}} + \Gamma \left( \frac{7}{6} \right) \ln \left( r^2 + 1 \right) + \frac{e}{4} \Gamma \left( \frac{7}{6} \right) \ln \left( r^2 + 1 \right) < r
\]

therefore, \((23)\) is satisfied. Hence, all the conditions of Theorem 4 hold, and problem \((27)\) has at least one solution.
Example 2 Consider the following boundary value problem for a coupled System:

\[
\begin{align*}
&D_1^{-\frac{3}{2}, \frac{3}{2}} u(t) + t^{\frac{5}{2}} \left( \frac{u}{1+t^2} + 1 \right) - 1 + 2 \ln \left( \frac{u}{1+t^2} + 1 \right) \frac{e^{-2t^2+1}}{20} = 0, \quad t > 0, \\
&D_1^{-\frac{3}{2}, \frac{3}{2}} v(t) + t^{\frac{5}{2}} \arctan \left( \frac{v}{1+t^2} + \frac{1}{3\sqrt{3}} \right) - 2 \ln \left( \frac{v}{1+t^2} + 1 \right) \frac{e^{-2t^2+1}}{20} = 0, \quad t > 0, \\
&\lim_{t \to 0} t^{\frac{5}{2}} I_1^{\frac{3}{2}, \frac{3}{2}} u(t) = 0, \quad \lim_{t \to \infty} t^{-\frac{5}{2}} I_1^{\frac{3}{2}, \frac{3}{2}} u(t) = 0. \\
&\lim_{t \to 0} t^{\frac{5}{2}} I_1^{0, \frac{3}{2}} v(t) = 0, \quad \lim_{t \to \infty} t^{-\frac{5}{2}} I_1^{0, \frac{3}{2}} v(t) = 0.
\end{align*}
\]  

(28)

Here,

\[
\begin{align*}
&f_1(t, u, v) = t^{\frac{3}{2}} \left( \sinh \left( \left( \frac{u}{1+t^2} + 1 \right) - 1 \right) - 1 + 2 \ln \left( \frac{u}{1+t^2} + 1 \right) + 1 \right) \frac{e^{-2t^2+1}}{20}, \\
&f_2(t, u, v) = t^{\frac{5}{2}} \left( \arctan \left( \left( \frac{v}{1+t^2} + \frac{1}{3\sqrt{3}} \right) + 1 \right) \frac{e^{-2t^2+1}}{20}, \right.
\end{align*}
\]

with \( \delta_1 = \frac{3}{3} \), \( \delta_2 = \frac{3}{2} \), \( \gamma = -\frac{3}{2} \) and \( \beta = 1 \).

**H1** It is easy to show that the functions \( f_1 \) and \( f_2 \) are continuous for any \( (t, u) \in (0, \infty) \times \mathbb{R} \).

- From the expression of the function \( f_1 \), it follows that

\[
F_1(t, u, v) = t^{\beta(1+\gamma)-1} f_1 \left( t, \left( 1 + t^{-\beta(1+\gamma)} \right) u, \left( 1 + t^{-\beta(1+\gamma)} \right) v \right)
= (\sinh (|u|+1) -1) e^{-t} + \ln (|v| + 1) \frac{te^{-2t^2+1}}{20}.
\]

If we choose \( \omega_1(u) = \sinh (u+1) - 1, \omega_2(v) = \ln (v+1), \varphi_1(t) = \frac{e^{-t}}{20}, \psi_1(t) = \frac{te^{-2t^2+1}}{20} \), then we obtain

\[
|F_1(t, u, v)| \leq \varphi_1(t) \omega_1(|u|) + \psi_1(t) \omega_2(|v|), \quad \text{on} \ (0, \infty) \times \mathbb{R},
\]

with \( \omega_1, \omega_2 \in C ((0, \infty), (0, \infty)) \) nondecreasing and \( \varphi_1, \psi_1 \in L^1 (0, \infty) \).

- From the expression of the function \( f_2 \), it follows that

\[
F_2(t, u, v) = t^{\beta(1+\gamma)-1} f_2 \left( t, \left( 1 + t^{-\beta(1+\gamma)} \right) u, \left( 1 + t^{-\beta(1+\gamma)} \right) v \right)
= \left( \arctan \left( |v| + \frac{1}{3\sqrt{3}} \right) + \sinh (|u|) \right) \frac{te^{-2t^2+1}}{20}.
\]

If we choose \( \bar{\omega}_1(u) = \sinh u, \bar{\omega}_2(v) = \arctan \left( v + \frac{1}{3\sqrt{3}} \right), \varphi_2(t) = \psi_2(t) = \frac{te^{-2t^2+1}}{20} \), then we obtain

\[
|F_2(t, u, v)| \leq \varphi_2(t) \bar{\omega}_1(|u|) + \psi_2(t) \bar{\omega}_2(|v|), \quad \text{on} \ (0, \infty) \times \mathbb{R},
\]

with \( \bar{\omega}_1, \bar{\omega}_2 \in C ((0, \infty), (0, \infty)) \) nondecreasing and \( \varphi_2, \psi_2 \in L^1 (0, \infty) \). Then, the condition (H2) holds.

- We have

\[
\left| t^{-\frac{3}{2}} f_1 (t, u, v) - t^{-\frac{3}{2}} f_1 (t, \bar{u}, \bar{v}) \right| \leq \frac{e^{-t}}{20 \left( 1 + t^2 \right)} |u - \bar{u}| + \frac{te^{-2t^2+1}}{20 \left( 1 + t^2 \right)} |v - \bar{v}|.
\]
If we put \( q_1(t) = \frac{e^{-t}}{20(1+t^2)} \), \( \bar{q}_1(t) = \frac{te^{-2t^2+1}}{20(1+t^2)} \) then we obtain
\[
q_1^* = \int_0^{+\infty} \left(1 + t^2\right)^{-\frac{1}{2}} \frac{\exp(-t)}{20(1+t^2)} dt = \frac{1}{20} < \infty.
\]
\[
\bar{q}_1^* = \int_0^{+\infty} \left(1 + t^2\right)^{-\frac{1}{2}} \frac{te^{-2t^2+1}}{20(1+t^2)} dt = \frac{e}{280} < \infty.
\]

- We have
\[
\left| t^{-\frac{3}{2}} f_2(t, u, v) - t^{-\frac{3}{2}} f_2(t, \bar{u}, \bar{v}) \right| \leq \frac{te^{-2t^2+1}}{20(1+t^2)} |u - \bar{u}| + \frac{te^{-2t^2+1}}{20(1+t^2)} |v - \bar{v}|.
\]

If we put \( q_2(t) = \tilde{q}_2(t) = \frac{te^{-2t^2+1}}{20(1+t^2)} \), then we obtain
\[
q_2^* = \tilde{q}_2^* = \int_0^{+\infty} \left(1 + t^2\right)^{-\frac{1}{2}} \frac{t \exp(-2t^2+1)}{20(1+t^2)} dt = \frac{e}{280} < \infty.
\]

Hence, the condition (H3) is satisfied. Moreover, we have
\[
\frac{\beta(q_1^* + \bar{q}_1^*)}{\Gamma(\delta_1)} + \frac{\beta(q_2^* + \tilde{q}_2^*)}{\Gamma(\delta_2)} \approx 6.6579 \times 10^{-2} < 1,
\]
the condition (24) is satisfied. It follows from Theorem 5 that the boundary value problem (28) has a unique solution \((u, v) \in E\).

5 Conclusion

In this work, the existence and uniqueness of solution for the (SFDEs) with initial conditions comprising the Erdélyi-Kober fractional derivatives have been discussed in a special Banach space. For our discussion, we have used the Leray-Schauder nonlinear alternative fixed point Theorem, as well as the Banach contraction principle. Finally, several indispensable examples are presented to illustrate the usefulness of our main results. Future work will be directed toward the Caputo version of the Erdélyi-Kober fractional differential equation and fractional coupled systems.

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References


