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VERIFICATION OF MH-MODEL IN VARIOUS UNDERGROUND FLOW PROBLEMS *

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Abstract. The contribution describes the results of the analysis of the porous media flow model by applying it on basic problems. The model is based on mixed-hybrid FEM with prismatic elements.
The first tests confirm the correctness of numerical algorithm. The next tests test the whole model. For steady saturated flow (linear problem), the model gives correct results if the discretisation mesh is chosen appropriately, i.e. elements have only slightly slanted bases. For non-linear problems of steady and unsteady unsaturated flow, the model gives correct results under the condition, that the parameters of the model are suitable and fineness of dicretisation is chosen according to the retention curve. The determined extent of usability of the model covers the majority of practical flow problems.

Key words. porous media flow, finite element method, numerical models

AMS subject classifications. 76S05, 76M10, 65C20

1. Introduction. This paper deals with a model of porous media flow in several levels. The numerical algorithms for MH-FEM are tested, the linear problems of steady flow in saturated media are solved and the non-linear problems of both steady and unsteady flow with phreatic surface are solved.

The mathematical model of steady saturated flow and its computer implementation were developed during the years 1996-1998 in cooperation of TU Liberec, DIAMO Stráž pod Ralskem and FJFI ČVUT Praha. Later, the model was extended for unsteady flow and phreatic surface, which means that we consider a domain with fully saturated part, transient layer, and unsaturated part.

The models were successfully used for solution of practical problems and the solution was compared with direct measurement. In spite of that, the presented tests are useful for the following reasons: We need to find the extent of usability of the model, for example for using the model for solving another physical problems; next, some standardisation of the model is needed for integration of the model into a compact software package for solving porous medial flow problems.

2. Description of the model. We will present only short summary of equations describing the model. For detailed theoretical derivation and description of implementation see contribution "Mathematical Modelling of the Underground Transport of Contaminants" in these Proceedings. The basic governing equations are Darcy Law and continuity equation

$$\mathbf{u} = K(\nabla p + \nabla z)\ , \qquad \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = q\ , \tag{1}$$

where $\mathbf{u}$ is specific discharge, p is pressure head, q is density of sources, K is permealility, q is density, z is vertical coordinate and t is time. The mixed-hybrid finite element method is used for solution. Discretisation to prismatic elements and linear

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and constant base functions are considered. Calculating the scalar products, we get the linear algebraic system. The matrix of this system is sparse and it is solved by iterative solver.

In the unsaturated part of space domain, additional equations are considered: the relation between negative pressuse and effective water content θ and the expression for relative permeability k_r or relative resistance R_r. Van Genuchten formula and Irmay formula are used:

$$\theta(p) = \frac{1}{(1+\alpha|p|^m)^{\frac{m-1}{m}}} \,, \qquad \frac{1}{R_r(p)} = k_r(\theta) = \left(\frac{\theta-\theta_0}{n-\theta_0}\right)^3 \,, \tag{2}$$

where θ_0 is residual watr content, n is porosity and α and m describe the statistical distribution of size of grains. The modified Darcy Law $\mathbf{u} = -k_r(\theta)K(\nabla p + k_v \nabla z)$ is used, with a calibration constant k_v which is described in section 3.3.1. The non-linear algebraic system has the following form:

$$\begin{array}{rcrcrcl} \mathbf{R}_r(\lambda_n)A\,\mathbf{u}_n & + & B\,\mathbf{p}_n & + & C\,\lambda_n & = & \mathbf{q}_{1,n}(\overline{\mathbf{p}}_n)\,, \\ B^T\mathbf{u}_n & + & \kappa(\overline{\mathbf{p}}_n)V\mathbf{p}_n & & & = & \mathbf{q}_{2,n}(\overline{\mathbf{p}}_n)\,, \\ C^T\mathbf{u}_n & & & + & S\,\lambda_n & = & \mathbf{q}_3\,. \end{array} \tag{3}$$

It differs from the linear case in the non-linear terms $\mathbf{R}_r(\lambda_n)$ (relative resistance) and $\kappa(\overline{\mathbf{p}}_n)$ (water capacity). The compoments of unknown vectors $\mathbf{u}_n$, $\mathbf{p}_n$, λ_n are fluxes through element sides, pressures in centres of elements and pressures in centres of sides.

3. The results of performed tests.

3.1. Testing of base functions and local matrices. The first test should verify the correctness of the algorithm for generating the base functions from $\mathbf{RT}^0$ space, which is described in [4]. The computed values are compared with values calculated analytically. Several tests were performed to cover all the properties and situations, which can influence the results: the shape of element, bases declination and element size. The tests are specified in [7]. These tests have proved correctness and reliability of algorithm; magnitude of errors equals to accuracy of computer.

The next basic algorithm of the MH-FEM model is the computation of local matrices. In our case, this is an analytical integration of quadratic functions on the element volume. Three different methods were compared: Zienkiewicz [8], Ciarlet [1] and the method derived at TU Liberec (see [5]). The difference corresponds to accuracy of computer aritmetics.

Globally, we can indicate the tests as successful. The parts of the program work correctly and we can now test the whole program.

3.2. Testing of 1D problems, influence of element shape. In this section, we consider steady flow in saturated porous media. This is a linear problem and we can solve it analytically for 1D case. The tests are performed for flow in horizontal and vertical direction. We will call the first case “canal” and the second case “column”. The mesh “canal” is shown on figure 4. Here, Dirichlet conditions $p = p_{d1}$ and $p = p_{d2}$ are prescribed on the sides b-h and homogeneous Neumann condition is prescribed on the sides l-b and l-h. Dimensions are l = 10m, b = h = 1m, permeability K is a unitary tensor and values of pressure p_{d1} = 110m and p_{d2} = 10m. The mesh “column” is exactly the same as “canal” but placed vertically. Dirichlet conditions on the top and bottom side ($p_{d1} = 200$m and $p_{d2} = 100$m respectively) and homogeneous Neumann condition on the vertical sides are prescribed.

Relative error was calculated as a difference between the results of the model and the values of analytical solution. Distribution of the relative error in the places of the mesh was examined. The piezometric head is computed in the centres of sides and centres of elements. The errors in vertical sides are in order of 10^{-9} and in horizontal sides and centres of elements they are 10^{-5} (this difference is interesting). The maximum value of errors of flux through sides is $6.8 \cdot 10^{-10}$. All values correspond to accuracy of solver and `double` aritmetics. The parameters and the results of the problem "column" are analogical with the problem "canal". This test is successful, the accuracy for 1D linear problems is sufficient.

Next, the influence of "non-standard" elements in the mesh is examined. Particularly, we consider elements with non-parallel bases and pyramidal and simplex elements. In the tests, the same 1D problem is solved and the meshes are derived from the mesh in previous test.

The global shape of the domain was kept. The mesh has now 2 layers to be possible to change the shapes of elements. There were used prismatic elements with one non-horizontal base and degenerated elements – pyramid and simplex. Various meshes were constructed using only some of them and using various ordering. For example see the right part of figure 4. The following set of tests was preformed:

- Comparison among mesh types: The errors of piezometric head in elements and sides and of flux values were examined. The accuracy of results on reference regular double-layer mesh (with only non-slanted sides) corresponds to the results on single-layer mesh. If some deformated elements are in the mesh, the error grows to the order of 10^{-2}, which is insufficient for simple linear problem.
- The same comparison for "flat" meshes (x,y coordinates multiplied by 100): The results are very similar to the previous.
- Observing of dependence of error on bases declination: The graphs of relative error values can be seen on figures 1, 2, 3. The error rises with rising of the angle of the base. We obtain sufficient accuracy for value of angle less than 1 degree. It can be seen that the results of two variants of the mesh "column" differ. In the first case, only the error of piezometric head rises, while in the second case the errors of both piezometric head and flux rise. It shows that geometrical properties of the mesh influence the results.

3.3. Testing of phreatic surface problems. Now we consider a more complex model derived from the model of steady saturated flow. The solved equations are non-linear and the solution is performed by an iteration cycle. The iterations are stopped when the maximum difference in location of phreatic surface between two steps does not exceed given value (accuracy).

An analysis of the results was performed in a quite different way. Since the correct solution is not known, we review the results by comparison with physical knowledge and by comparison of values computed in different ways. Because of non-linearity and chosen accuracy of iteration cycle 0.5m, the error in the order of tenths of percent is still acceptable.

3.3.1. Problem of steady flow. We consider a 2D problem, which models the steady phreatic surface when only a gradient of piezometric head is prescribed. Domain is vertically placed layer with dimensions $500 \times 50 \times 200$m discretised with prismatic elements with right-triangle bases, 50×50m, 10m height.

Homogeneous Neumann conditions on the large vertical sides and the bottom side are prescribed. Dirichlet conditions on smaller vertical sides are prescribed: 140m and 30m (these are the positions of phreatic surface). Neumann condition on the upper side is prescribed, which expresses precipitation (standard value 600mm per year – 2m^3 per day in 1 element).

The model computes correctly the shape of phreatic surface and the flow in saturated zone. In the unsaturated zone, only vertical flow is expected, but the results are different (see figure 5 left). The calibration constant k_v in the term with ∇z is used to repair this defect caused by nonregularity of precipitation flow – rain is falling only in short periods and in the rest time, the ground is dry. The computations were performed for values $k_v = 1$, 0.5, 0.3, 0.1 and precipitation $10\text{m}^3/\text{d}$, $2\text{m}^3/\text{d}$ and $0.05\text{m}^3/\text{d}$. The criteria for correctness are horizontal isolines of piezometric head and values of pressure not less than -10m in unsaturated zone.

Results: The smaller is k_v and larger is rain flow (up to about $20\text{m}^3/\text{d}$), the better are results. The value $k_v = 0.1$ is applicable for all reasonable values of precipitation. The precipitation flow must not exceed about $50\text{m}^3/\text{d}$ – this causes unstability.

3.3.2. Problem of unsteady flow. The domain and the mesh are the same as in problem "column" in the case of saturated flow. The dimensions of bases are 50×50m, the height of column is 200m. All sides are impermeable except of the bottom, where a Dirichlet condition is prescribed: 20m in the beginning, 150m after that. We expect the surface would rise from the initial value 20m to the limit value 150m.

Computations were performed for following set of values:

retention curve (m, α):	(3,0.002)	(3,0.001)	(3,0.03)	
element height (dz):	20m	10m	5m	2m
time step (dt):	10d	3d	1d	0.5d

Necessary requirements on the results are: regular shape of time dependence graph and independence of results on selected discretisation. These were satisfied for suitable combinations of the values presented above.

For more steep retention curve, the smaller dz is needed. Also dt must be selected in dependence to dz: in one time step, the surface must not pass through larger number of elements, it means that the value of dt is bounded from above (the smaller dz is, the smaller dt must be). Values in larger extent are presented in the report [2]. For typical cases of correct and incorrect results see figure 6.

4. Conclusion. The tests checked the model in several levels. The implementation of numerical algorithms is correct. The model as a whole has some restrictions of usability. In the case of saturated flow, the limitation of angles between sides of element was observed. In the case of unsaturated flow the parameters and discretisation must be suitably selected. The reason is in the physical model and in non-linearity. The accuracy of the model is fully sufficient for common hydrogeological applications.

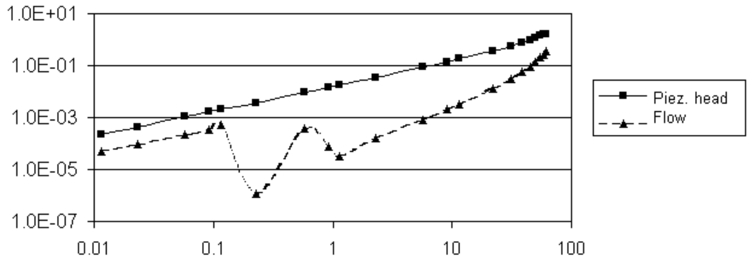

FIG. 1. *Dependence of relative error (vertical axis) on base slant in degrees (horizontal axis) for the modified mesh "canal".*

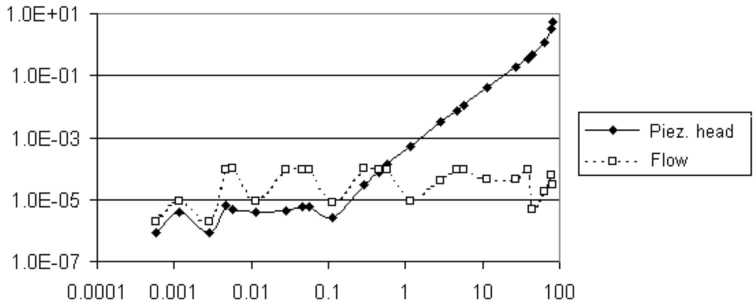

FIG. 2. *Dependence of relative error on base slant in degrees for the first variant of modified mesh"column" (four elements have non-parallel bases).*

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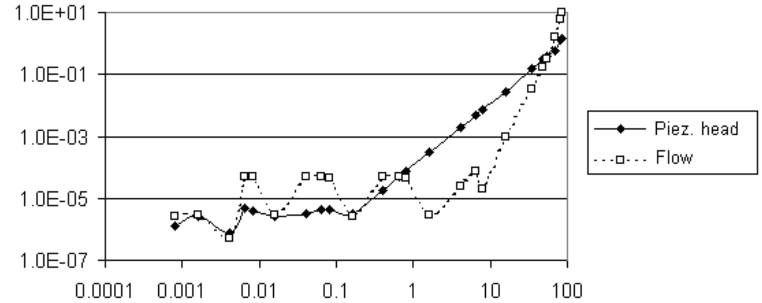

FIG. 3. *Dependence of relative error on base slant in degrees for the second variant of modified mesh "column" (two elements have non-parallel bases).*

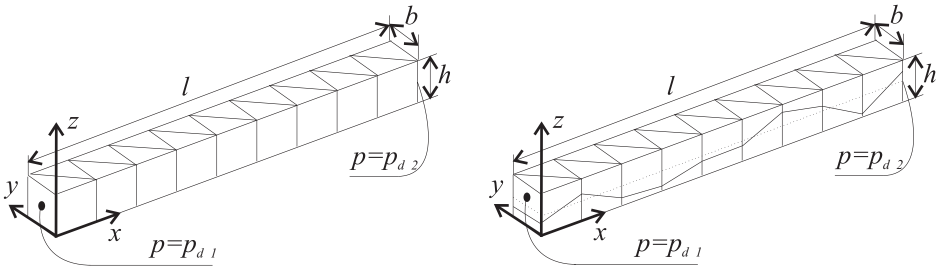

FIG. 4. *The mesh "canal". The basic type in the left and the modified double-layer version with slanted bases of elements in the right. The reference double-layer mesh is drawn with dotted line.*

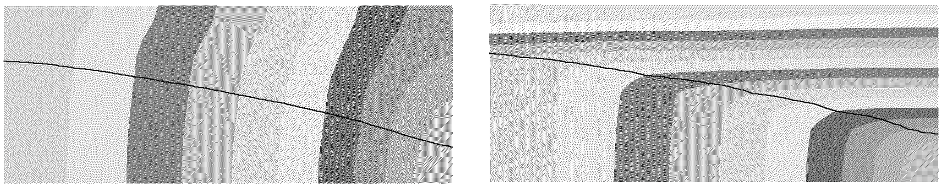

FIG. 5. *Results of the steady flow problem. Isolines of piezometric head with a step 10m and a line of water surface are displayed. The model without without calibration in the left and the model with k_v=0.1 in the right.*

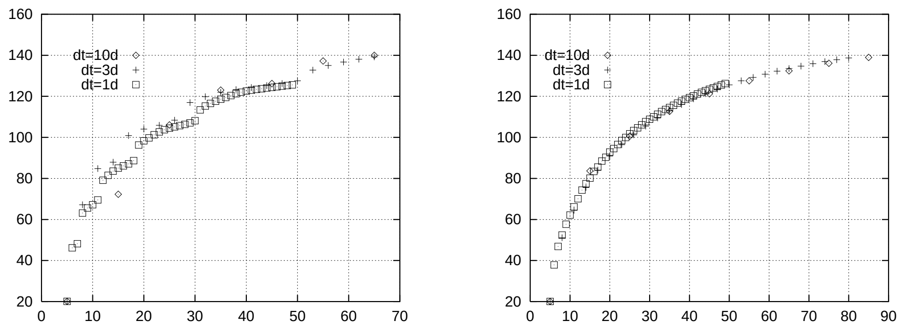

FIG. 6. *Results of the unsteady flow problem - dependence of water height on time for various values of time step. Parameters* $m = 3$, $\alpha = 0.002$ *and dz=20m in the left and dz=5m in the right were used.*