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INVERSE METHOD FOR DETERMINING TEMPERATURE-DEPENDENT THERMAL PROPERTIES OF POROUS MEDIA

JOZEFA LUKOVIČOVÁ, JOZEF ZÁMEČNÍK

Abstract. Determining the temperature dependent thermal conductivity and heat capacity based on the solution of the nonlinear inverse problem of the parameter estimation by using the last square-method is presented. A set of temperature measurements at a single sensor location inside the heat conducting body is required. The solution of the corresponding direct problem is obtained using a time marching boundary element method. In the application of the inverse method is employed one-dimensional heat conduction through the furnace slag -based concrete.

Keywords: thermal conductivity, heat capacity, inverse problem, building materials

1. Introduction. Many theoretical and experimental methods for measuring the thermal properties are developed in the literature. Most of these methods only enable measurements at constant temperature, which makes the determination of the temperature-dependent thermal properties very time-consuming. This paper deals with the method for determining temperature-dependent thermal conductivity and heat capacity, which is based on the solution of the inverse problem of the identification of thermal parameters of a heat conducting body. One can be counsidered as a reasonable alternative to the clasical methods for measuring thermal properties, because one provides to determine thermal properties as functions of temperature for wide temperature range.

The problem of parameter identification is solved by nonlinear least-square method. The solution of this inverse problem requires a finite set of temperature measurements taken inside the body and assumes that the thermal properties belong to set polynomials. Significant contributions made in the field of inverse heat conduction problems are published by Beck et al. [1] and of parameter identification by Cannon [2] and Ingham [3]. The effectiveness of the inverse problem's solution is substantially dependent on the numerical realisation of the direct problem's solution. Here the boundary element method (BEM) is used. A review of BEM can be found in Brebbia [4].

Parameter estimation problem is fairly difficult, since the described equations are nonlinear with respect to the unknown parameters. Therefore experiment is arranged where the specimen is designed in the simplest shape and the thermal conduction is chosen in a simple way, in order to make determining of the unknown coefficients as easy as possible.

Department of Physics, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovak Republic

2. Formulation of the problem. We consider the one-dimensional, nonlinear heat conduction problem in slab geometry. The dimensionless mathematical formulation of this problem can be expressed as

$$C(T)\frac{\partial T(x,t)}{\partial t} = F_0 \frac{\partial}{\partial x}(k(T)\frac{\partial T(x,t)}{\partial x}) \qquad (x,t)\in(0,1)\times(0,1] \tag{1a}$$

$$-k(T)\frac{\partial T(x,t)}{\partial x} = q_0(t) \quad \text{at } x=0, \quad t\in(0,1] \tag{1b}$$

$$T(x,t) = T_1(t) \quad \text{at } x=1, \quad t\in(0,1] \tag{1c}$$

$$T(x,t) = T_0(x) \quad \text{for } t=0, \quad x\in[0,1] \tag{1d}$$

where T is the temperature, $q = k(T)\frac{\partial T(x,t)}{\partial x}$ is the heat flux, $k(T)$ is the thermal conductivity, $C(T)$ is the heat capacity per unit volume, $q_0(t)$, $T_1(t)$, $T_0(x)$ are known functions. The distance, time, temperature, heat capacity and thermal conductivity are dimensionless with respect to L (length of the slab), t_e (final time of interest during which a specific practical heat conduction experiment is performed), T_r, C_r, k_r (reference values), respectively. The Fourier number $F_0 = (k_r t_e)/(C_r L^2)$.

For the inverse problem, the thermal properties $k(T)$ and $C(T)$ are regarded as being unknown, but everything else in equation (1) is known. To determine $k(T)$ and $C(T)$ from boundary and initial data we need additional temperature measurements $T^{(m)}(t)$ at an arbitrary spatial position $x = d \in (0,1)$, which are recorded in time, namely

$$T(x,t) = T^{(m)}(t) \quad \text{at} \quad x=d, \qquad t\in(0,1] \tag{2}$$

3. Solution of the direct problem. The first step of inverse analysis is to develop the corresponding direct solution for the problem (1). A boundary element method is employed for solution of the nonlinear system (1). By using the Kirchhoff transformation

$$u(T) = \int_0^T k(T)\,dT \tag{3}$$

and denoting $u(x,t) = u(T(x,t))$, $(x,t)\in(0,1)\times[0,1]$, equations (1) in the new variable u can be written as

$$\frac{\partial u(x,t)}{\partial t} = a(T(x,t))\frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t)\in(0,1)\times(0,1] \tag{4a}$$

$$-\frac{\partial u(x,t)}{\partial x} = u(q_0(t)) \quad \text{at} \quad x=0, \quad t \in (0,1] \tag{4b}$$

$$u(x,t) = u(T_1(t)) \quad \text{at} \quad x=1, \quad t \in (0,1] \tag{4c}$$

$$u(x,t) = u(T_0(x)) \quad \text{for} \quad t=0, \quad x \in [0,1] \tag{4d}$$

where
$$a(T) = F_0 \frac{k(T)}{C(T)} \tag{5}$$

is the dimensionless thermal diffusivity.

According to weighted residuals method, the residue of (4a) weighted with fundamental solution u^* and integrated over the domain produces zero

$$\iint_{t_e 1} (a(T)\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t})u^* dx d\tau = 0 \tag{6}$$

If the thermal diffusivity a is constant, then for (4a) a fundamental solution is available, see for example Beck [1]

$$u^*(x,t;\xi,\tau) = \frac{1}{(4\pi a(t-\tau))^{\frac{1}{2}}} \exp(-\frac{(x-\xi)^2}{4a(t-\tau)}) H(t-\tau)$$

where H is the Heaviside function and ξ and τ are generic space and time variables, respectively. In the case of non-constant thermal diffusity the use of the fundamental solution (6) is accompanied by a time marching technique in which $a(T)$ is assumed constant at the beginning of each time step. Therefore, starting from the initial time $t_0 = 0$, over each time element $[t_{i-1}, t_i]$, the value of a_i is taken as the mean average

$$a_i = \overline{a(T)} = \int_0^1 a(T(x,t_{i-1}))dx \tag{7}$$

Therefore, over each time step nonlinear partial differential equation is linearised. Applying the Gaussian reciprocity theorem, using the fundamental solution u^* and approximation (7), equation (6) is transformed into following integral equation for each time step $[t_{i-1}, t_i]$, see Brebbia.[4]

$$\eta(x)u(x,t) = \int_{t_{i-1}}^{t_i} a_i u'(0,\tau)u^*(x,t;0,\tau)d\tau + \int_{t_{i-1}}^{t_i} a_i u'(1,\tau)u^*(x,t;1,\tau)d\tau \; -$$

$$\int_{t_{i-1}}^{t_i} a_i u(0,\tau)u^{*\prime}(x,t;0,\tau)d\tau - \int_{t_{i-1}}^{t_i} a_i u(1,\tau)u^{*\prime}(x,t;1,\tau)d\tau +$$

$$\int_0^1 u(y,t_{i-1})u^*(x,t;y,t_{i-1})dy\,, \qquad t\in[t_{i-1},t_i]\,, \quad x,y\in(0,1) \tag{8}$$

where primes denote differentiation with respect to x and $\eta(x)$ is a coefficient which is equal to 1 for $x\in(0,1)$ and 0.5 if $x\in\{0,1\}$.

Assuming that the temperature and the heat flux are constant over each time step, the constant BEM approximation of integral equation (8) and of equations (4b) and (4c), can be written in the form

$$\eta(x)u(x,\tilde{t}_i) = u^{©}(0,\tilde{t}_i)\int_{i-1}^{i} a_i u^*(x,\tilde{t}_i;0,\tau)d\tau + u^{©}(1,\tilde{t}_i)\int_{t_{i-1}}^{t_i} a_i u^*(x,\tilde{t}_i;1,\tau)d\tau\ -$$

$$u(0,\tilde{t}_i)\int_{t_i}^{t_i} a_i u^{*©}(x,\tilde{t}_i;0,\tau)d\tau - u(1,\tilde{t}_i)\int_{t_{i-1}}^{t_i} a_i u^{*©}(x,\tilde{t}_i;1,\tau)d\tau +$$

$$\sum_{j=1}^{N} u(\tilde{y}_j,\tilde{t}_{i-1})\int_{y_{j-1}}^{y_j} u(x,\tilde{t}_i,y,t_{i-1})dy \quad , \ x,y\in(0,1) \tag{9a}$$

$$-u'(0,\tilde{t}_i) = u(q_0(t_i))\,, \qquad t\in[t_{i-1},t_i] \tag{9b}$$

$$u(1,\tilde{t}_i) = u(T_1(t_i))\,, \qquad t\in[t_{i-1},t_i] \tag{9c}$$

where $\tilde{t}_i = (t_{i-1}+t_i)/2$ is midpoint of the element $|t_{i-1},t_i|$ and $\tilde{y}_j = (y_{j-1}+y_j)/2$ for $j=1$ to N, are the midpoints of the elements $|y_{j-1},y_j|$ which are used to discretise the segment $[0,1]$ into N elements. The integrals in (9) are calculated analytically.

The solution of the equations (9) provides the transformed temperature function $u(x,t)$ at any point inside the layer $[0,1]\times[t_{i-1},t_i]$. Once the values of $u(x,t)$ are obtained, the temperature $T(x,t)$ is calculated by inverting the transformation (3), namely

$$T(x,t) = u^{-1}(u(x,t))\,, \qquad (x,t)\in[0,1]\times[t_{i-1}.t_i] \tag{10}$$

The values of $u(\tilde{y}_j,t_i)$ for $j=1$ to N need to be calculated in order to provide the ‘initial’ condition at the time t_i and to proceed to the next time step $[t_i,t_{i+1}]$. The corresponding values of the temperature, $T(\tilde{y}_j,t_i)$ for $j=1$ to N, are required in order to calculate the new constant value of the thermal diffusiviy a_{i+1}, given by (7) at the time t_i. Based on this time marching technique the BEM provides the values of u and C at any point in the solution domain, and in particular, the values of the temperature at $x=d$, $T^{(c)}(t)$ are calculated.

4. Inverse problem. For the inverse problem, the thermal property *k(T)* and *C(T)* are regarded as being unknown, but everything else in equations (1) is known. In addition temperature readings $T^{(m)}(t)$ taken at arbitrary spatial position $x = d \in (0,1)$ are considered available. The k(T) and C(T) are obtained as the solution of the minimization problem of the least-squares norm $\left\| T^{(c)} - T^{(m)} \right\|^2$, where $T^{(c)} = T(x,t;C,k)$ is solution of (9) for given $k(T)$ and $C(T)$ (initial guesses of $k(T)$ and $C(T)$) at $x = d$. Because, in practice, only a finite set of time measurements may be available (at some discrete times, $t_i^{©}$), namely

$$T(d,t_i^{©}) = T^{(m)}(t_i^{©}) = T_i^{(m)}, \quad i = 1 \text{ to } M, \tag{11}$$

in order to achieve a unique solution problem, the unknown functions *k(T)* and *C(T)* are parameterised by assuming that the *k(T)* and *C(T)* are taken as a set of polynomials

$$C(T) = \sum_{n=1}^{R} C_n T^{n-1} \text{ and } \quad k(T) = \sum_{n=1}^{R} k_n T^{n-1} \tag{12}$$

and the least-squares norm in discretised form is

$$S(\underline{C},\underline{k}) = \sum_{i=1}^{M} [T_i^{(m)} - T_i^{(c)}(\underline{C},\underline{k})]^2 \tag{13}$$

where $\underline{C} = (C_n)$ and $\underline{k} = (k_n)$, for $n = 1 \text{ to } R$, are the unknown vectors of the thermal conductivity and heat capacity, and $T_i^{(c)}(\underline{C},\underline{k})$ is the calculated value of the temperature at $t = t_i'$, for $i = 1$ to M , obtained from the BEM solution of the direct problem (9), by using the estimated values of the $(\underline{C},\underline{k})$. Determining of the unknown parameters k_n and C_n is based on minimizing of the nonlinear least-squares norm (13) using the Newton -Raphson method.

5. Results. Numerical experiments for numerical stability, accuracy, sensitivity to the experimental errors and estimation of number of temperature measurements are examined by using simulated exact and inexact measurements in [5].

The experiments were performed on the furnace-slag based concrete with bulk density $1230\,\text{kg.m}^{-3}$. The measured sample with an alloy geometry of length $L = 0.2\,\text{m}$, with reference thermal conductivity $k_r = 1\,\text{W/mK}$, heat capacity $C_r = 10^5\ \text{J/m}^3\text{K}$, temperature $T_0 = 20\,°C$, $T_r = T_{max} - T_0 = 60\ ^{o}\text{C}$ is subjected to a heat transfer experiment with $q_0(t) = 2.3 + 4T$ and $T_1(t) = T_0 = 20/60$ over a period of time $t_e = 3.6 \times 10^3 \text{s}$. The heat flux $q_0(t)$ is known with a prescribed rate of $\Delta t^* = t_e \Delta t$, temperature $T^{(m)}$ is recorded with a sampling rate of $\Delta t^{**} = t_e / M$ at $d = 0.5$. The Fourier number $F_0 = 0.9$.

The inverse determination of the heat capacity and the thermal conductivity starting initially with the guesses:

$C(T) = C_1 + C_2 T + C_3 T^2$ and $k(T) = k_1 + k_2 T + k_3 T^2$

where $C_2 = 0.75$, $C_3 = 1.25$, $k_2 = 4$, $k_3 = 4.75$, in order to obtain a unique solution, $k(T)$ and $C(T)$ are fixed at one point and we set $C(0) = C_1 = 1$ and $k(0) = k_1 = 3.4$.

For the time step $\Delta t = 0.01$ and number of measurements $M = 60$, numerical dimensionless results are obtained as follows:

$C(T) = 1 + 0.203T + 0.6015T^2$ and $k(T) = 3.4 + 1.1821T + 3.4521T^2$

Values of temperature dependence of thermal conductivity of furnace-slag based concrete obtained by the inverse method and two other standard methods [6] are showed on Fig. 1.

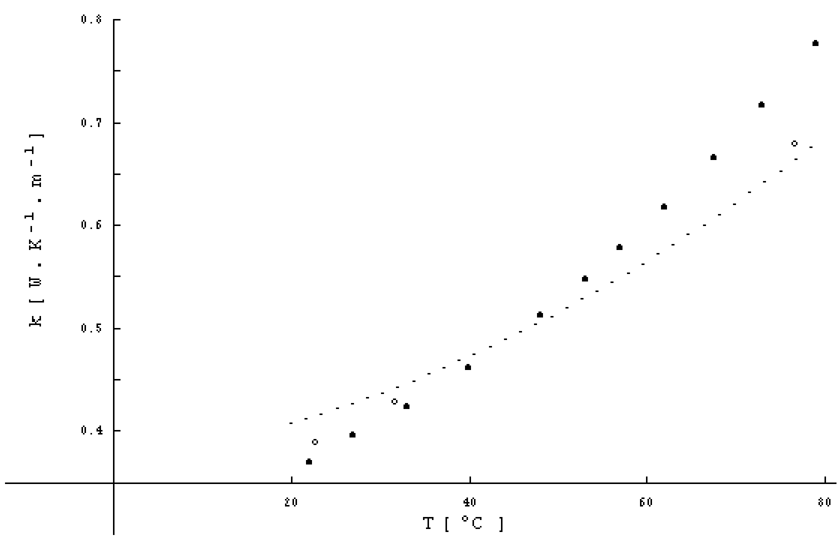

FIG.1. *Thermal conductivity vs. temperature obtained by the methods:*
· · · · · · *inverse*
○○○○○ *standard steady state*
●●●●● *standard Van –Rinsum*

6. Conclusion. Inverse heat conduction was applied to the simultaneous determining of the temperature dependence of furnace-slag based concrete thermal conductivity and thermal capacity in the interval $(20-80)\,°C$. The comparison of the temperature dependent thermal conductivity obtained by this inverse method with well-established steady state method and Van-Rinson method shows that results are coincident.

This numerical method allows large time periods between practical sampling measurements and reasonably large tolerance error in measurement. The method is suited for practical calculation of thermal properties of porous materials and one for analyse of heat conduction measurement problems.

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