Proceedings of ALGORITMY 2000
Conference on Scientific Computing, pp. 245–253

MIC(0) DD PRECONDITIONING OF FEM ELASTICITY PROBLEMS ON NON-STRUCTURED MESHES*

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Abstract. Two preconditioning techniques for PCG iterative solution of $2D$ elasticity problems are described and studied. Linear and quadratic Lagrangian finite elements are applied for discretization of the elliptic system of partial differential equations. Non-structured meshes, including a local refinement, are used for triangulation of the computational domain. The obtained coupled stiffness matrix has a two-by-two block structure corresponding to a simple separable displacement ordering of the unknowns. Then, two displacement decomposition (DD) algorithms are considered whereas $MIC(0)$ incomplete factorization is used to approximate the related diagonal blocks. A set of numerical experiments are presented to illustrate the PCG convergence rate of the related $MIC(0)$ DD preconditioners.

Key words. PCG, $MIC(0)$, displacement decomposition, FEM

AMS subject classifications. 65F10, 65N30

1. Introduction. The performance of two displacement decomposition (DD) preconditioners that exploit modified incomplete Cholesky factorization $MIC(0)$ is studied in the case of finite element (FEM) matrices arising from the discretization of the two-dimensional ($2D$) equations of elasticity. There is a lot of work dealing with preconditioning iterative solution methods for the Navier equations of elasticity. Here we will briefly comment on some of the approaches used. Multigrid, multilevel, domain decomposition, and patched local refinement are directly applicable in a very general setting to the coupled stiffness matrix. Another example of this first approach is the block-ILU factorization based on block-size reduction ([6], see also in [7, 8]).
In this paper we focus our attention on a different preconditioning strategy based on a first step block-diagonal approximation of the stiffness matrix written in a separable displacement two-by-two block form. In an earlier paper, Axelsson and Gustafsson [3] have implemented modified point-ILU factorization to this problem. As the coupled system does not lead to an M-matrix, they construct their preconditioners based on the point-ILU factorization of the displacement decoupled (displacement decomposition) block-diagonal part of the original matrix. This approach is based on the second Korn's inequality, providing good convergence of the resulting preconditioned conjugate gradient (PCG) algorithm for elasticity problems with moderate Poisson ratio $\nu \in [0, \frac{1}{2})$ (relatively far from the incompressible limit $\nu = \frac{1}{2}$).
The displacement decomposition remains up to now one of the most robust approaches for preconditioning of FEM elasticity systems (see also, e.g., [5, 13]). The most often studied case is when the mesh is (topo)logically equivalent to uniform one in the unit squire/cube (see, e.g., [3, 5, 10, 6, 12, 10]), or more generally, when a patched local refinement is additionally applied [7]. The problem considered here is in a general setting. The computational domain is arbitrary where the triangulation is

*This work was supported partially by Grant I801/98.

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automatically generated including subdomains with local refinement. Discretization by linear and quadratic Lagrangian triangle finite elements is considered. As the diagonal blocks of the displacement decomposed matrix are not M-matrices in the case of quadratic elements, the $MIC(0)$ preconditioner is applied to the related linear FEM stiffness matrices defined on an auxiliary globally refined mesh.
The aim of the presented numerical tests is to analyze the performance of the studied algorithms. Our consideration is focussed on the scalability of the PCG convergence rate where the mutually interacted impacts of the higher order FEM, the non-structured grids including local refinement, the different kinds of singularities of the solution and the coefficients jumps are compared.
The reminder of the paper is organized as follows: In section 2 we survey some major results about the Navier equations of elasticity ant their FEM approximation. The needed background results related to the DD $MIC(0)$ preconditioner are presented in section 3. Section 4 contains a rich set of numerical experiments organized in two subsections. The first of them is dedicated to the asymptotic behaviour of the studied iterative solvers based on a model test problem. The second one contains real-life benchmark problems on non-structured meshes including local mesh refinement. Short concluding remarks are given in the last section.

2. FEM approximation of the Navier equations of elasticity. Let us consider the weak formulation of the linear elasticity problem in the form: find $\underline{u} \in V_E = [H^1_E(\Omega)]^2 = \{\underline{v} \in [H^1(\Omega)]^2 : \underline{v}_{|G_D} = \underline{u}_S\}$ such that

$$\int_\Omega [2\mu\varepsilon(\underline{u}) : \varepsilon(\underline{v}) + \lambda \; div \; \underline{u} \; div \; \underline{v}] \, d\Omega = \int_\Omega \underline{f}^t \underline{v} d\Omega + \int_{G_N} \underline{g}^t_S \underline{v} dG, \tag{1}$$

$\forall \underline{v} \in V_0 = [H^1_0(\Omega)]^2 = \{v \in [H^1(\Omega)]^2 : \underline{v}_{|G_D} = 0\}$, with the positive parameters λ and μ of Lamé, the symmetric strains

$$\varepsilon(\underline{u}) := \frac{1}{2}(\nabla \underline{u} + (\nabla \underline{u})^t),$$

the volume forces $\underline{f}$, and the boundary tractions $\underline{g}$. Usually one introduces Young's modulus $E > 0$ and Poisson's ratio $\nu \in [0, 1)$. The relations between these and the material parameters λ, μ are

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

Assume that Ω is a polygonal domain, $\mathcal{T}$ is a a triangulation of Ω, and let for simplicity E and ν be piecewise constants with respect to the elements of $\mathcal{T}$. Now we denote by $V_E^{(h,k)} \subset V_E$ and $V_0^{(h,k)} \subset V_0$ the FEM spaces of conforming piecewise linear and pieswise quadratic ($k \in \{1, 2\}$) functions with nodal Lagrangian basis corresponding to the triangulation $\mathcal{T}$ with a mesh parameter h. Applying FEM discretization to (1) we get the linear algebraic system

$$K^{(h,k)} \underline{u}_h = \underline{f}_h \tag{2}$$

where $K^{(h,k)}$ is the related symmetric and positive definite stiffness matrix and $\underline{u}_h$ is the vector of nodal unknowns. Let us assume that $K^{(h,k)}$ is written in a two-by-two block form

$$K^{(h,k)} = \begin{pmatrix} K_{11}^{(h,k)} & K_{12}^{(h,k)} \\ K_{21}^{(h,k)} & K_{22}^{(h,k)} \end{pmatrix}, \tag{3}$$

where the block structure corresponds to a separable displacement component ordering of the vector of nodal unknowns. The aim of our study is to develop robust preconditioning solution methods for the system (3) suitable for large-scale problems. For a block-approximation of $K^{(h,k)}$ we use two displacement decomposition approaches, see e.g. [3, 5], referred as separable displacement preconditioner $C_{SDC}^{(h,k)}$ and isotropic preconditioner $C_{ISO}^{(h,k)}$ defined as follows

$$C_{SDC}^{(h,k)} = \begin{pmatrix} K_{11}^{(h,k)} & \\ & K_{22}^{(h,k)} \end{pmatrix} \tag{4}$$

and

$$C_{ISO}^{(h,k)} = \begin{pmatrix} \bar{K}^{(h,k)} & \\ & \bar{K}^{(h,k)} \end{pmatrix} \tag{5}$$

where $\bar{K}^{(h,k)} = (K_{11}^{(h,k)} + K_{22}^{(h,k)})/2$. Such preconditioning strategies of the coupled matrix K are theoretically motivated by the second Korn's inequality. More precisely, the next lemma holds.

LEMMA 2.1. *The displacement decomposition preconditioners (4, 5) are of optimal order, that is,*

$$cond((C_{SDC}^{(h,k)})^{-1}K^{(h,k)}) = O(\frac{1}{1-2\nu_{max}}), \qquad cond((C_{ISO}^{(h,k)})^{-1}K^{(h,k)}) = O(\frac{1}{1-2\nu_{max}})$$

where $\nu_{max} = \max_\Omega \nu$.

The above estimates straightforwardly follow from second Korn's [5, 11] inequality for the continuous elasticity problem, and from the inclusion $V_0^{(h,k)} \subset V_0$. The constants in the estimates depend on the Poisson's ratio.

REMARK 1. *In the model case of pure displacement problem of homogeneous material the exact estimates read as flows [3]:*

$$cond((C_{SDC}^{(h,k)})^{-1}K^{(h,k)}) < \frac{2}{1-\tilde{\nu}}, \qquad cond((C_{ISO}^{(h,k)})^{-1}K^{(h,k)}) < \frac{3+\tilde{\nu}}{1-\tilde{\nu}},$$

where $\tilde{\nu} = \frac{\nu}{1-\nu} \in [0,1)$. *Some interesting generalizations of these estimates are presented in the more recent paper [10].*

3. DD MIC(0) preconditioning. In this section we first recall some facts about the modified incomplete Cholesky factorization MIC(0) preconditioning algorithm. Our presentation at this point follows those from [5]; see also [13] for an alternative approach. Let us rewrite the real $N \times N$ symmetric matrix $A = (a_{ij})$ in the form

$$A = D - L - L^t \tag{6}$$

where D is the diagonal and $(-L)$ is the strictly lower triangular part of A. Then we consider the following approximate factorization of A:

$$C_{MIC(0)} = (X-L)X^{-1}(X-L)^t$$

where $X = diag(x_1, \cdots, x_N)$ is a diagonal matrix determined by the condition of equal rowsums, i.e.,

$$C_{MIC(0)}\underline{e} = A\underline{e}, \qquad \underline{e} = (1, \cdots, 1)^t \in \mathcal{R}^N$$

For the purpose of preconditioning, we are interested in the case when $X > 0$ and thus $C_{MIC(0)}$ is positive definite. If this holds, we speak about *stable* $MIC(0)$ factorization. Concerning stability of the $MIC(0)$ factorization, we have the following theorem.

THEOREM 3.1. *Let $A = (a_{ij})$ be a symmetric real $N \times N$ matrix and let $A = D - L - L^t$ be the splitting (6) of A. Let us assume that*

$$\begin{array}{rcl} L & \geq & 0 \\ A\underline{e} & \geq & 0 \\ A\underline{e} + L^t\underline{e} & > & 0 \qquad \underline{e} = (1, \cdots, 1)^t \in \mathcal{R}^N, \end{array}$$

i.e. that A is a weakly diagonally dominant matrix with nonpositive offdiagonal entries and that $A + L^t = D - L$ is strictly diagonally dominant.
Then the relation

$$x_i = a_{ii} - \sum_{k=1}^{i-1} \frac{a_{ik}}{x_k} \sum_{j=k+1}^{N} a_{kj}$$

gives the positive values x_i and the diagonal matrix $X = diag(x_1, \cdots, x_N)$ defines stable $MIC(0)$ factorization of A.
The conditions of the theorem are obviously not always satisfied for FEM stiffness matrices. Now we introduce the displacement decomposition preconditioners for the case of linear FEM ($k = 1$) in the form

$$C_{SDC\ MIC(0)}(K^{(h,1)}) = \begin{pmatrix} C_{MIC(0)}(K_{11}^{(h,1)}) & \\ & C_{MIC(0)}(K_{22}^{(h,1)}) \end{pmatrix}, \tag{7}$$

$$C_{SDC\ ISO(0)}(K^{(h,1)}) = \begin{pmatrix} C_{MIC(0)}(\bar{K}^{(h,1)}) & \\ & C_{MIC(0)}(\bar{K}^{(h,1)}) \end{pmatrix}. \tag{8}$$

Here $C_{MIC(0)}(A)$ stands for the $MIC(0)$ preconditioner of A. A local elementwise analysis of the $MIC(0)$ stability is applicable in this case. Since the blocks $K_{11}^{(h,1)}$ and $K_{22}^{(h,1)}$ correspond to elliptic bilinear forms they can be assembled by element stiffness matrices written in a general form (see for more details in [1, 4])

$$A_T^{(h,1)} = r_T \begin{pmatrix} b_T + c_T & -c_T & -b_T \\ -c_T & a_T + c_T & -a_T \\ -b_T & -a_T & a_T + b_T \end{pmatrix} \tag{9}$$

where $a_T = \cot\alpha_T$, $b_T = \cot\beta_T$, $c_T = \cot\gamma_T$, $(\alpha_T, \beta_T, \gamma_T)$ can be viewed as angles of some triangle, and r_T is a constant, all generally depending on both the shape of $T \in \mathcal{T}$ and on the Poisson's ratio ν. The next lemma follows from (9).

LEMMA 3.2. *If the corresponding angles $(\alpha_T, \beta_T, \gamma_T)$ satisfy the condition*

$$\max_{e \in \mathcal{T}} \ \max\{\alpha_T, \beta_T, \gamma_T\} \leq \frac{\pi}{2} \tag{10}$$

then $MIC(0)$ factorization of the blocks $K_{11}^{(h,1)}$, $K_{22}^{(h,1)}$ and $\bar{K}^{(h,1)}$ is stable.
The condition (10) is easily controlled in the case of moderate Poisson's ratio by a proper choice of the minimal angle of the triangulation which is one of the options of the used general purpose mesh generator.

In the case of quadratic FEM discretization the diagonal blocks do not belong to the class of M-matrices even for the model problem $-\Delta u = f$. To get a stable $MIC(0)$ factorization, in this case, we first globally refine the triangulation $\mathcal{T}$ by connecting the midpoints of the triangles sides nd denote by $K_{11}^{(h/2,1)}$, $K_{22}^{(h/2,1)}$ and $\bar{K}^{(h/2,1)}$ the related blocks corresponding to linear FEM implemented on the refined mesh. Then the $MIC(0)$ displacement decomposition preconditioners are defined as follows:

$$C_{SDC\ MIC(0)}(K^{(h,2)}) = \begin{pmatrix} C_{MIC(0)}(K_{11}^{(h/2,1)}) & \\ & C_{MIC(0)}(K_{22}^{(h/2,1)}) \end{pmatrix} \tag{11}$$

$$C_{SDC\ ISO(0)}(K^{(h,2)}) = \begin{pmatrix} C_{MIC(0)}(\bar{K}^{(h/2,1)}) & \\ & C_{MIC(0)}(\bar{K}^{(h/2,1)}) \end{pmatrix}. \tag{12}$$

The next lemma provides the theoretical background of the above preconditioning approach.

LEMMA 3.3. *The following spectral equivalence estimates hold*

$$cond((K_{ii}^{(h/2,1)})^{-1} K_{ii}^{(h,2)}) = O(1), \qquad i \in \{1,2\}.$$

The proof of the lemma can be done by a local element by element analysis similarly as for the model elliptic problem presented, e.g., in [1].

REMARK 2. *The numerical tests discussed in the last section are performed using the perturbed version of $MIC(0)$ algorithm, where the incomplete factorization is applied to the matrix $\tilde{A} = A + \tilde{D}$. The diagonal perturbation $\tilde{D} = \tilde{D}(\xi) = diag(\tilde{d}_1, \ldots \tilde{d}_N)$ is defined as follows:*

$$\tilde{d}_i = \begin{cases} \xi a_{ii} & if \quad a_{ii} \geq 2w_i \\ \xi^{1/2} a_{ii} & if \quad a_{ii} < 2w_i \end{cases}$$

where $0 < \xi < 1$ is a constant of order $O(h^2)$ and $w_i = \sum_{j>i} -a_{ij}$.

REMARK 3. *It is well known that the convergence of the incomplete factorization preconditioning algorithms strongly depends on the ordering of the unknowns. The results presented in this paper correspond to a coordinatewise ordering of the nodes.*

4. Numerical experiments. Two types of numerical experiments are reported in this section. In both cases, the number of iterations is used as a measure for the robustness of the iterative solvers.

4.1. Model problem. Consider the model pure displacement ($G_D = \partial\Omega$) problem with homogeneous material ($\lambda = 1$, and $\mu = 1.5$), and right hand side corresponding to the given solution $u_1 = x^2 + \cos(x+y)$, $u_2 = x^3 + y^4 + \sin(y-x)$. A uniform mesh with n mesh points along each of the coordinate directions is used where the discretization parameter $n \in \{4, 8, 16, 32, 64\}$ is varied. In Table 1 are presented number of iterations for five methods: the well known Gauss-Seidel ($G-S$), Steepest Decent (SD), and Conjugate Gradient (CG), and the studied in this paper two displacement decomposition $MIC(0)$ PCG methods. The stopping criteria here is $||\underline{u}_h^{(k_{it})} - \underline{u}_h||_0 \leq 10^{-10}||\underline{u}_h||_0$. The advantages of $SDC\ MIC(0)$ and $ISO\ MIC(0)$ are well expressed even for relatively small discrete problems. For the model problem $SDC\ MIC(0)$ converges a little bit faster, but as will be shown in the next subsection, this is not a general conclusion. The asymptotic of the number of iterations of both $DD\ MIC(0)$ preconditioners is $O(\sqrt{n})$.

TABLE 1
Model problem. Linear FEM. Performance analysis of iterative solvers.

n	$G-S$	SD	CG	SDC	ISO
4	171	387	58	21	24
8	643	1455	96	26	30
16	2391	5393	190	33	39
32	8822	19841	368	44	54
64	32299	¿50000	725	60	73

4.2. Real-life problems. Three real-life engineering problems are considered as benchmarks in this subsection. The computational tests are performed by the developed FEM integrated program system where the general purpose mesh generator $TRIANGLE$ [14] has been incorporated. The numerical tests are presented in a table form where the number of iterations of CG, and PCG with the studied $SDC\ MIC(0)$ and $ISO\ MIC(0)$ preconditioners are listed for both linear and quadratic FEM with a relative stopping criteria in a form $||K\underline{u}_h^{(k_{it})} - \underline{f}_h||_0 < 10^{-8}||\underline{f}_h||_0$.

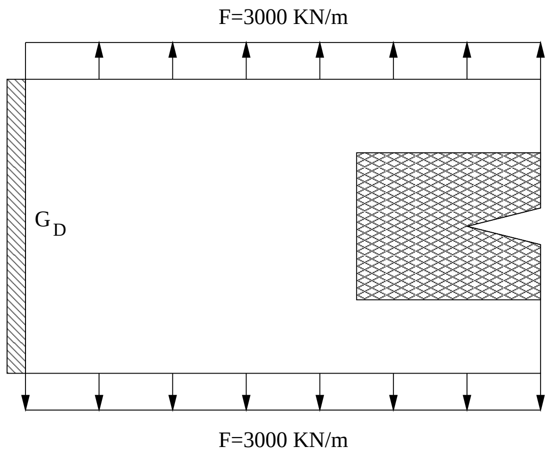

FIG. 1. *Benchmark 1. Concrete body with a crack.*

TABLE 2
Benchmark 1. Concrete body with a crack. Performance analysis of DD PCG solvers.

Mesh	Number of nodes	CG		$SDC\ MIC(0)$		$ISO\ MIC(0)$	
	Linear/Quadratic	1	2	1	2	1	2
1	1609/6242	495	1254	107	215	97	160
2	2784/10833	729	1876	147	302	113	189
3	6221/24492	986	2538	163	415	137	229

Each of the tables contains three rows corresponding to: (1) regular coarser mesh; (2) mesh with local refinement in subdomains (hatched in the figures) of some singularities of the solution; (3) globally refined mesh (2) with a given reduction factor($R_f = 4$ in the first two test problems, and $R_f = 6$ for the last one) with respect of the maximal area of the triangles. Homogeneous Dirichlet boundary conditions are assumed along the the G_D parts of boundaries.

BENCHMARK 1. A rectangular concrete console block of a size 5×8 m with an artificial crack is stretched along its 8 m sides (see Fig.(1)). The material is homogeneous with $E = 2.1 \times 10^{10} Pa$ and $\nu = 0.17$. Local refinement around the crack is applied to get the second mesh. The results presented in Table 2 very well illustrate the convergence properties of the proposed DD preconditioners in a relatively simple local refinement case.

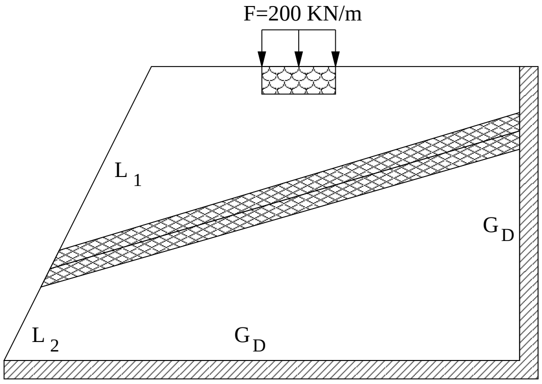

FIG. 2. *Benchmark 2. Two-layer slope problem.*

TABLE 3
Benchmark 2. Two-layer slope problem. Performance analysis of DD PCG solvers.

Mesh	Number of nodes	CG		$SDC\ MIC(0)$		$ISO\ MIC(0)$	
	Linear/Quadratic	1	2	1	2	1	2
1	1594/6224	417	1077	57	95	63	101
2	2186/8571	562	1491	62	107	71	114
3	6248/24826	877	2282	84	149	94	153

BENCHMARK 2. A two-layer soil body illustrating a slope stability computer simulation is considered (see Fig. 2) where the material characteristics are as follows: sandy clay layer L_1 with $E_{L_1} = 8 \times 10^6 Pa$ and $\nu_{L_1} = 0.35$; and clay layer L_2 with $E_{L_2} = 22 \times 10^6 Pa$ and $\nu_{L_2} = 0.18$. The first layer is loaded by the absorbed water (35% of the layer volume). There is also an external vertical load modelling a railway passing over the slope. The local refinement is applied in a strip along the interface boundary between the different layers, and under the seal of the external load. This problem is somehow more complicated as a moderate jump of the coefficients appears. From other point of view the solution is more regular, and as we see in Table 3, the convergence of the iterative methods is even faster then for the first benchmark. Note that the number of the nodes of the finest meshes are almost the same for the first two benchmark problems.

BENCHMARK 3. The interaction between a vertical concrete wall and a weak multilayer soil media is studied (see Fig. 3). The characteristics of the problem are as follows: concrete wall with coefficients $E_C = 3.15 \times 10^{10} Pa$ and $\nu_C = 0.2$; and soil layers with: $E_{L_1} = 5.2 \times 10^6 Pa$ and $\nu_{L_1} = 0.4$; $E_{L_2} = 9.4 \times 10^6 Pa$ and $\nu_{L_2} = 0.35$: $E_{L_3} = 14.0 \times 10^6 Pa$ and $\nu_{L_3} = 0.25$; $E_{L_4} = 21.4 \times 10^6 Pa$ and $\nu_{L_4} = 0.2$. The scheme of the loading is presented in the figure where the external forces are assumed uniformly distributed across the top side of the wall. The zone of local refinement covers the wall.

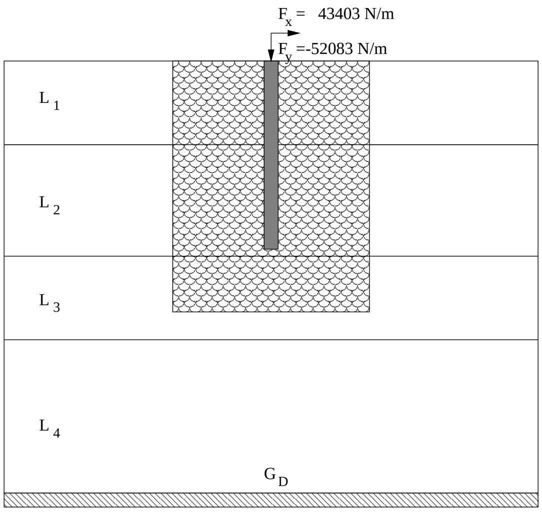

FIG. 3. *Benchmark 3. Foundation wall in multi-layer soil media.*

¿From computational point of view, this problem is the hardest one, due to the strong jump of the elasticity modules between the concrete and the surrounding weak soil. There is also a strongly expressed zone with stresses concentration under the bottom of the wall. The increased number of iterations presented in Table 4 well illustrate the influence of these factors on the convergence rate of the iterative methods.

TABLE 4
Benchmark 3. Foundation wall problem. Performance analysis of DD PCG solvers.

Mesh	Number of nodes	CG		SDC $MIC(0)$		ISO $MIC(0)$	
	Linear/Quadratic	1	2	1	2	1	2
1	1574/6047	5697	26718	269	645	164	300
2	3413/13481	16127	59260	470	1830	221	375
3	9195/36386	26616	95952	1355	10768	295	424

5. Concluding remarks. Two displacenment decomposition preconditioning techniques for PCG iterative solution of $2D$ elasticity problems are considered in this paper where linear and quadratic Lagrangian finite elements are applied for discretization of the Navier system of partial differential equations. The proposed approach is implemented in the developed general purpose FEM package. The modified incomplete Cholesky factorization $MIC(0)$ is used to approximate the obtained diagonal blocks where the stability of the factorization is controlled by the parameters of the incorporated mesh generator $TRIANGLE$. The robustness of the studied algorithms for non-structured meshes, including local refinement in a very general setting, is confirmed by the presented numerical tests. We would like especially to stress the attention of the reader on the efficiency of the algorithms when quadratic finite elements are used. Another important conclusion is that the ISO $MIC(0)$ preconditioner demonstrated some advantages for the real-life problems which is particularly strongly expressed for the most complicated last benchmark problem. We have also observed that the number of the $ISO\ MIC(0)$ iterations have a behaviour very near to $O(N^{5/4})$ for all of the test problems.

Our final conclusion is that the positive experience with the developed $2D\ FEM$ package is a good starting point for the next step research devoted to the more complicated, but definitely more important, $3D$ case.

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