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ROOTS OF CONTINUOUS PIECEWISE MONOTONE MAPS OF AN INTERVAL

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1. Introduction

At the Third Czechoslovak Summer School on Dynamical Systems, K. Simon raised the following question.

If a continuous piecewise linear interval map has a continuous square root, must it have a continuous piecewise linear square root?

We shall consider slightly more general problems. Namely, we shall investigate the existence of continuous: piecewise monotone, piecewise strictly monotone, and piecewise linear n-th roots of interval maps which have a continuous n-th root.

Here by an **n-th root** of f we mean a map g such that $f = g^n$ (g^n is the n-th iterate of g). A continuous map $f : I \to J$, where I, J are closed intervals, is **piecewise monotone** (resp. **piecewise strictly monotone**, resp. **piecewise linear**) if there are finitely many points $a_0 < a_1 < \cdots < a_m$ in I such that $I = [a_0, a_m]$ and f is monotone (resp. strictly monotone, resp. affine) on $[a_{i-1}, a_i]$ for $i = 1, \dots, m$. We shall denote the class of these maps by $M(I, J)$ (resp. $S(I, J)$, resp. $L(I, J)$). We shall also denote the class of all continuous maps from I into J by $C(I, J)$. In addition, the class of all closed intervals will be denoted by $\mathcal{I}$, the set of all positive integers by $\mathbb{Z}_+$ and the set of all continuous n-th roots of f by $\sqrt[n]{f}$. For an interval J we shall denote its boundary (i.e. the set of its endpoints) by $\operatorname{bd} J$.

Clearly, any iterate of a map from $M(I, I)$, $S(I, I)$ or $L(I, I)$ has to belong to the same class. Therefore if we look for piecewise monotone (resp. piecewise strictly monotone, resp. piecewise linear) roots of f then we can restrict our attention to piecewise monotone (resp. piecewise strictly monotone, resp. piecewise linear) f. We obtain the following results.

Theorem A. *Let $I \in \mathcal{I}$, $n \in \mathbb{Z}_+$ and let $f : I \to I$ be a continuous piecewise monotone map. Assume that $\sqrt[n]{f} \neq \emptyset$. Then:*

(a) *There is a piecewise monotone map in $\sqrt[n]{f}$.*
(b) *If $f(I) = I$ then every map in $\sqrt[n]{f}$ is piecewise monotone.*

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(c) *If f is piecewise strictly monotone then every map in $\sqrt[n]{f}$ is piecewise strictly monotone.*

Theorem B. *For each $I \in \mathcal{I}$ and $n \in \mathbb{Z}_+$ with $n \geq 2$, there exists a continuous piecewise linear map $f : I \to I$ such that $\sqrt[n]{f} \neq \emptyset$, but no element of $\sqrt[n]{f}$ is piecewise linear.*

On our way to prove Theorem B, we shall investigate roots of horseshoe maps. By a **lap** of a piecewise strictly monotone map we mean a maximal interval of monotonicity of this map. If $I \in \mathcal{I}$ then a map $f \in S(I, I)$ is called a **horseshoe map**, if it has more than one lap and each lap is mapped by f onto the whole I. We shall call a horseshoe map **strict** if it has no homtervals (intervals on which all iterates of the map are monotone). The **type** of a horseshoe map f will be a pair (m, σ), where m is the number of laps of f and σ indicates whether f is increasing (then $\sigma = +$) or decreasing (then $\sigma = -$) on the leftmost lap.

Our main theorem on the roots of strict horseshoe maps is the following.

Theorem C. *Let $I \in \mathcal{I}$, $n \in \mathbb{Z}_+$ and let $f : I \to I$ be a strict horseshoe map of type (m, σ).*

(1) *If $\sqrt[n]{m}$ is not an integer, then $\sqrt[n]{f} = \emptyset$.*
(2) *If $\sqrt[n]{m}$ is an integer, but $m(n+1)$ is odd and $\sigma = -$, then $\sqrt[n]{f} = \emptyset$.*
(3) *If $\sqrt[n]{m}$ is an integer and $m(n+1)$ is even, then $\sqrt[n]{f}$ has a unique element. This element is a strict horseshoe map of type $(\sqrt[n]{m}, \sigma)$.*
(4) *If $\sqrt[n]{m}$ is an integer, $m(n+1)$ is odd and $\sigma = +$, then $\sqrt[n]{f}$ has exactly two elements. Both of them are strict horseshoe maps; one of type $(\sqrt[n]{m}, +)$, the other one of type $(\sqrt[n]{m}, -)$.*

Notice that exactly one of the cases (1)-(4) has to occur.

We prove Theorem A in Section 2, Theorem C in Section 3 and Theorem B in Section 4.

2. Piecewise monotone and piecewise strictly monotone roots

We start by proving an auxiliary lemma.

Lemma 1. *Let $I, J \in \mathcal{I}$ and let $\varphi, \psi \in C(J, I)$ be maps such that*

(i) *ψ is monotone,*
(ii) *$\varphi|_{\mathrm{bd}\, J} = \psi|_{\mathrm{bd}\, J}$,*
(iii) *if $x \in J$ and $\varphi(x) \neq \psi(x)$ then φ is constant in some neighborhood of x.*

Then $\varphi = \psi$.

Proof. Suppose that $\varphi \neq \psi$. Then there is some $x \in J$ such that $\varphi(x) \neq \psi(x)$. Let $[c, d]$ be the maximal interval containing x on which φ is constant. If $\varphi(c) \neq \psi(c)$ then by (iii), φ is constant in some neighborhood of c. By the maximality of $[c, d]$, this means that c is the left endpoint of J, but this contradicts (ii). Therefore we have $\varphi(c) = \psi(c)$. Analogously, $\varphi(d) = \psi(d)$. Hence, $\psi(c) = \varphi(c) = \varphi(x) =$

$\varphi(d) = \psi(d)$. Thus, by (i), ψ is constant on $[c, d]$, and we get $\varphi(x) = \psi(x)$ – a contradiction. $\square$

Now we can prove a basic lemma, showing what happens if a composition of two maps is monotone. We shall use the notation $\langle x, y\rangle$ for the closed interval whose endpoints are x and y (regardless of whether $x < y$, $x > y$ or $x = y$ – in which case $\langle x, y\rangle = \{x\}$).

Lemma 2. *Let $I, J, K \in \mathcal{I}$ and let $F \in C(J, K)$, $G \in C(K, I)$. Assume that $G \circ F$ is monotone. Then:*

(a) *G is monotone on $F(J)$.*
(b) *There exists a monotone map $H \in C(J, K)$ such that $G \circ H = G \circ F$ and $H|_{\mathrm{bd}\, J} = F|_{\mathrm{bd}\, J}$.*

Proof.
(a) Suppose that G is not monotone on $F(J)$. Then there exist $x < y < z$ in $F(J)$ such that $G(y) \notin \langle G(x), G(z)\rangle$. There are $u < v$ in J such that $F(\{u, v\}) = \{x, z\}$. Then there exists $w \in (u, v)$ such that $F(w) = y$. We get $u < w < v$ and $G \circ F(w) \notin \langle G \circ F(u), G \circ F(v)\rangle$ – a contradiction.
(b) Let $F(\mathrm{bd}\, J) = \{c, d\}$ with $c \le d$. Define a map $H_1 \in C(J, K)$ by

$$H_1(x) = \begin{cases} c & \text{if } F(x) < c, \\ d & \text{if } F(x) > d, \\ F(x) & \text{otherwise.} \end{cases}$$

Then apply the "pouring water" construction of [**ALMS**]. That is, define a map $H \in C(J, K)$ by

$$H(x) = \min(\max\{H_1(t) : t \in J,\, t \le x\}, \max\{H_1(t) : t \in J,\, t \ge x\}).$$

(To get the graph of H, we pour water to the graph of H_1, until it is full; see Figure 1.) Clearly, $H|_{\mathrm{bd}\, J} = F|_{\mathrm{bd}\, J}$ and H is monotone. Moreover, if $H(x) \ne F(x)$ at some x then H is constant in some neighborhood of x. Therefore the hypotheses of Lemma 1 are satisfied for $\varphi = G \circ H$ and $\psi = G \circ F$. Hence, we get $\varphi = \psi$. $\square$

Remark 3. In the proof of Lemma 2 (b), either we get $H = F$ or H is constant on some interval.

We can restate Theorem A in the following form, more convenient for the proof (notice that $g(I) = I$ is equivalent to $g^n(I) = I$).

Theorem 4. *Let $I \in \mathcal{I}$, $n \in \mathbb{Z}_+$ and $g \in C(I, I)$. Assume that $g^n \in M(I, I)$. Then:*

(a) *There exists $h \in M(I, I)$ such that $h^n = g^n$.*
(b) *If $g(I) = I$ then $g \in M(I, I)$.*
(c) *If $g^n \in S(I, I)$ then $g \in S(I, I)$.*

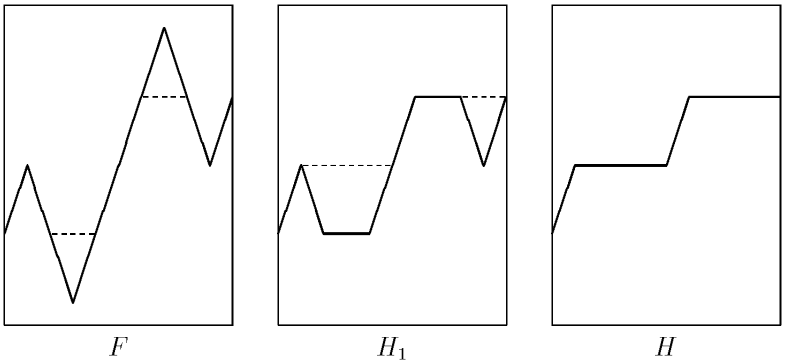

Figure 1. Consecutive steps in the construction of the map H.

Proof. We start by proving that

($\star$) g is piecewise monotone on $g^{n-1}(I)$.

Let J be an interval of monotonicity of g^n. Apply Lemma 2 (a) to $K = g^{n-1}(J)$, $F = g^{n-1}|_J$ and $G = g|_K$. We obtain that g is monotone on $g^{n-1}(J)$. (If $g^{n-1}(J)$ consists of one point, then formally we cannot use Lemma 2, but clearly g is also monotone on $g^{n-1}(J)$.) Since $g^{n-1}(I)$ is a finite union of $g^{n-1}(J)$ over intervals J of the type considered, ($\star$) follows.

Now, to prove (a), notice that $g^{n-1}(I) \subset g^{n-2}(I) \subset \cdots \subset g(I) \subset I$. We shall construct h inductively on these intervals. We shall do it in such a way that h^{n-k} will also coincide with g^{n-k} on $g^k(I)$ and h will coincide with g on $\operatorname{bd}(g^k(I))$.

In view of ($\star$), we can set $h|_{g^{n-1}(I)} = g|_{g^{n-1}(I)}$. Assume now that $1 \le k \le n-1$, that h is already defined on $g^k(I)$, and that h^{n-k} coincides with g^{n-k} on $g^k(I)$ and h coincides with g on $\operatorname{bd}(g^k(I))$. We are going to define h on $g^{k-1}(I)$. The set $\operatorname{cl}(g^{k-1}(I) \setminus g^k(I))$ consists of 0,1 or 2 closed intervals. Assume that J is the intersection of such an interval with an interval of monotonicity of g^n. Then we apply Lemma 2 (b) to $K = g(J)$, $F = g|_J$ and $G = g^{n-k}|_K$. We obtain a continuous monotone map $H : J \to g(J)$ such that $g^{n-k} \circ H = g^{n-k+1}|_J$ and $H|_{\operatorname{bd} J} = g|_{\operatorname{bd} J}$. We set $h = H$ on J. Since $g(J) \subset g(g^{k-1}(I)) = g^k(I)$, we have $h^{n-k}|_{g(J)} = g^{n-k}|_{g(J)}$, so $h^{n-k+1}|_J = (g^{n-k}|_{g(J)}) \circ H = g^{n-k+1}|_J$. Therefore, if we proceed as above on each possible J then we get a piecewise monotone h on $g^{k-1}(I)$, such that h^{n-k+1} coincides with g^{n-k+1} on $g^{k-1}(I) \setminus g^k(I)$. However, if $x \in g^k(I)$ then $h^{n-k}(x) = g^{n-k}(x)$ and $g^{n-k}(x) \in g^n(I) \subset g^{n-1}(I)$, so $g(g^{n-k}(x)) = h(g^{n-k}(x))$. Therefore h^{n-k+1} coincides with g^{n-k+1} also on $g^k(I)$, and consequently h^{n-k+1} coincides with g^{n-k+1} on $g^{k-1}(I)$. From the conditions $H|_{\operatorname{bd} J} = g|_{\operatorname{bd} J}$ and $h|_{\operatorname{bd}(g^k(I))} = g|_{\operatorname{bd}(g^k(I))}$ it follows that h is continuous on the whole $g^{k-1}(I)$ and that h coincides with g on $\operatorname{bd}(g^{k-1}(I))$. This completes the induction step.

When we arrive with the induction at $k = 0$ then we obtain $h \in M(I, I)$ such that $h^n = g^n$. This proves (a).

(b) follows immediately from ($\star$).

To prove (c), notice that in the proof of (a), whenever we use Lemma 2 (b), then by Remark 3, we either get $h|_J = g|_J$ or h is constant on some interval. If g^n is piecewise strictly monotone then, since $h^n = g^n$, the map h^n is also piecewise strictly monotone. Therefore h cannot be constant on any interval, and we get $h|_J = g|_J$ on all J. Therefore $h = g$, so $g \in M(I, I)$. Since g cannot be constant on any interval, we get $g \in S(I, I)$. □

3. Roots of horseshoe maps

The first step in the investigation of roots of horseshoe maps is the following simple theorem.

Theorem 5. *Let $I \in \mathcal{I}$, $n \in \mathbb{Z}_+$ and let $f \in S(I, I)$ be a horseshoe map. Then any element of $\sqrt[n]{f}$ is a horseshoe map.*

Proof. Suppose that $g \in \sqrt[n]{f}$. By Theorem A, $g \in S(I, I)$. If g is not a horseshoe map then there is a lap J of g such that $g(J) \neq I$ (since f has more than one lap, clearly g has more than one lap). Since $f(I) = I$, we have also $g(I) = I$, so $g^{n-1}(I) = I$. Therefore there exists $x \in I$ such that $g^i(x) \in \operatorname{int} J_i$ for some laps J_i of g for $i = 0, 1, \dots, n-1$ and $J_{n-1} = J$. Then $K = \bigcap_{i=0}^{n-1} g^{-i}(J_i)$ is a lap of $f = g^n$. Since $K \subset g^{-(n-1)}(J)$, we get $f(K) \subset g(J)$. Hence, $f(K) \neq I$ – a contradiction. □

The following lemma is very simple and we leave its proof to the reader.

Lemma 6. *Let g be a horseshoe map of type (k, τ) and let $n \in \mathbb{Z}_+$. Then g^n is a horseshoe map of type (k^n, σ), where $\sigma = +$ whenever $\tau = +$ and also when $\tau = -$, k is odd and n is even; otherwise $\sigma = -$.*

Now we turn to the investigation of strict horseshoe maps. Clearly, if $n \in \mathbb{Z}_+$ then F is strict if and only if F^n is strict.

We say that two maps $F \in C(I, I)$ and $G \in C(J, J)$ are **topologically conjugate** if there exists a homeomorphism $H : I \to J$ such that $H \circ F = G \circ H$. The next lemma follows immediately from the kneading theory (see e.g. [**MT**]).

Lemma 7. *Any two strict horseshoe maps of the same type are topologically conjugate via an orientation preserving homeomorphism.*

Clearly, any map topologically conjugate to a horseshoe map is a horseshoe map. Moreover, if the conjugacy preserves orientation then these two horseshoe maps have the same type. Also, if one of them is strict, so is the other one.

It turns out that a strict horseshoe map cannot have more than one n-th root which is a strict horseshoe map of a given type.

Lemma 8. *Let $F, G \in S(I, I)$ be strict horseshoe maps of the same type and let $n \in \mathbb{Z}_+$. Assume that $F^n = G^n$. Then $F = G$.*

Proof. Let $[a_0, a_1], [a_1, a_2], \dots, [a_{m-1}, a_m]$ be the laps of F and let $[b_0, b_1]$, $[b_1, b_2], \dots, [b_{m^n-1}, b_{m^n}]$ be the laps of F^n. We know that the set $\{b_0, b_1, \dots, b_{m^n}\}$ is equal to $\bigcup_{i=0}^{n-1} F^{-i}(\{a_0, a_1, \dots, a_m\})$. Hence, there is a map $\varphi : \{0, 1, \dots, m^n\} \to \{0, 1, \dots, m^n\}$ such that $F(b_i) = b_{\varphi(i)}$ for $i = 0, 1, \dots, m^n$. In view of Lemma 7, this map remains the same if we replace F by G. Thus, F and G coincide on the set $\bigcup_{i=0}^{n-1} F^{-i}(\{a_0, a_1, \dots, a_m\})$. Since $F^{kn} = G^{kn}$ for $k \in \mathbb{Z}_+$, this remains true if we replace n by kn. Therefore F and G coincide on the set $R = \bigcup_{i=0}^{\infty} F^{-i}(\{a_0, a_1, \dots, a_m\})$. Since F is strict, the set R is dense in I. Consequently, F and G coincide on the whole I. □

We need the existence of strict horseshoe maps of all types. For this, we introduce **linear horseshoe maps**, that is those horseshoe maps which are affine on each lap. Note that a horseshoe map which belongs to $L(I, I)$ need not be a linear horseshoe map, since the former need only be piecewise linear on each of its laps. Figure 2 gives an example of a horseshoe map which is piecewise linear, but which is not a linear horseshoe map.

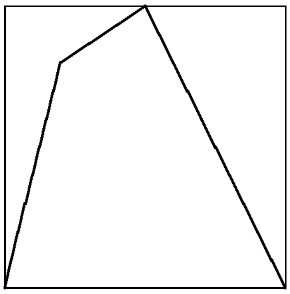

Figure 2. This map is not a linear horseshoe map.

Clearly, for each (m, σ) where $m \geq 2$ is an integer and $\sigma \in \{+, -\}$, there exists a linear horseshoe map of type (m, σ).

Lemma 9. *Any linear horseshoe map is strict.*

Proof. Let F be a linear horseshoe map. Since it has more than one lap, it is expanding on each lap. This means that there is a constant $\alpha > 1$ such that if K is an interval on which F is monotone then the length of $F(K)$ is at least α times larger than the length of K. Therefore if F has a homterval J then the lengths of $F^n(J)$ tend to infinity as $n \to \infty$ – a contradiction. □

The last ingredient of the proof of Theorem C is the following lemma.

Lemma 10. *Let $m, k \geq 2$ and $n \geq 1$ be integers and let $\sigma, \tau \in \{+, -\}$. Assume that $m = k^n$ and one of the following conditions holds.*

(i) $\sigma = \tau = +$,
(ii) $\sigma = +,\ \tau = -$ *and* $m(n+1)$ *is odd,*
(iii) $\sigma = \tau = -$ *and* $m(n+1)$ *is even.*

Then for any strict horseshoe map f of type (m,σ) there exists a strict horseshoe map of type (k,τ) in $\sqrt[n]{f}$.

Proof. Let $f \in S(I,I)$ be a strict horseshoe map of type (m,σ). There exists a linear horseshoe map $G \in L(I,I)$ of type (k,τ). By Lemma 6, G^n is a linear horseshoe map of type (m,σ). By Lemma 9, G^n is strict, and by Lemma 7, G^n is conjugate to f via an orientation preserving homeomorphism H (that is, $H \circ G^n = f \circ H$). Set $g = H \circ G \circ H^{-1}$. Then g is a strict horseshoe map of type (k,τ). We have $g^n = H \circ G^n \circ H^{-1} = f$. □

Now Theorem C follows from Theorem 5 and Lemmas 6, 8 and 10.

4. Piecewise linear roots of linear horseshoe maps

If $F \in L(I,I)$ then we shall call the maximal intervals on which F is affine the **linear laps** of F.

When we want to prove a counterpart to Theorem 5 for linear horseshoe maps, then we cannot just repeat the proof of Theorem 5 with small modifications. The reason is that laps are the maximal intervals on which the map is one-to-one. If we take a longer interval then some points get glued together by the map and its further applications cannot "unglue" them. With linear laps (which can be shorter than the laps of the same map), the situation is different. A longer interval can be "bent" by the map and "straightened" by its further applications. Nevertheless, we can prove the following theorem.

Theorem 11. *Let $I \in \mathcal{I}$, $n \in \mathbb{Z}_+$ and let $f \in L(I,I)$ be a linear horseshoe map. Then any element of $\sqrt[n]{f} \cap L(I,I)$ is a linear horseshoe map.*

Proof. Suppose that $g \in \sqrt[n]{f} \cap L(I,I)$. Let A be the set of those points of I which belong to the interior of some lap of g but not to the interior of any linear lap of g. The set A is finite; if it is empty then by Theorem 5, g is a linear horseshoe map.

Suppose that $x \in I$ is a point for which $g^{n-1}(x) \in A$. The points $x, g(x), \dots, g^{n-2}(x)$ belong to the interiors of some laps of g (otherwise, by Theorem 5, $g^{n-1}(x)$ would be an endpoint of I). Therefore x belongs to the interior of some lap of f and hence to the interior of some linear lap of f. Consequently, one of the points $x, g(x), \dots, g^{n-2}(x)$ has to belong also to A.

This shows that if $(x_i)_{i=0}^{\infty}$ is a sequence of points of I such that $g(x_i) = x_{i-1}$ for all $i > 0$ and $x_0 \in A$ then for infinitely many i's, the points x_i belong to A. Since by Theorem 5, g is a horseshoe map, for every $y \in I$ we can find a sequence $(y_i)_{i=0}^{\infty}$ such that $g(y_i) = y_{i-1}$ for all $i > 0$, all points y_i are distinct, and $y_0 = y$. Consequently, if $y \in A$ then A is infinite. Therefore A must be empty. This completes the proof. □

We turn now to the proof of Theorem B. Let $I \in \mathcal{I}$ and let β be the length of I. For any integer $m > 2$, $\sigma \in \{+,-\}$ and a vector $P = (p_1, \dots, p_m)$ with $p_i > 0$ and $\sum_{i=1}^{m} p_i = \beta$ there exists a unique linear horseshoe map $F_{P,\sigma} \in L(I,I)$ of type (m,σ) with laps of length $p_1, \dots, p_m$ (from left to right). Let $n \in \mathbb{Z}_+$ and suppose that $g \in \sqrt[n]{F_{P,\sigma}} \cap L(I,I)$. By Theorem 11, g is a linear horseshoe map. Therefore $g = F_{Q,\tau}$ for some $\tau \in \{+,-\}$ and a vector $Q = (q_1, \dots, q_k)$ with $q_i > 0$ and $\sum_{i=1}^{k} q_i = \beta$. By Lemma 6, $k^n = m$ and one of the conditions (i) - (iii) of Lemma 10 holds. Moreover, there are additional conditions on the vectors P and Q. Namely, it is clear that there are polynomials (in fact, monomials) $W_1, \dots, W_m$ of degree n in k variables such that $p_i = W_i(Q)$ for $i = 1, \dots, m$. The monomials W_i depend only on k, τ and n.

It is worth mentioning that with some work one can obtain explicit formulas for the monomials W_i (at least in some cases). For instance, if $k = 2$ and $\tau = +$ then $W_i(x,y) = x^{n-\alpha_i-1}y^{\alpha_i+1}$, where for $i \geq 2$, α_i is the number of switches (from 1 to 0 and from 0 to 1) in the binary expansion of $i-1$, and $\alpha_1 = -1$.

Now Theorem B follows easily. For a given integer $n \geq 2$ we choose $m \geq 2$ and $\sigma \in \{+,-\}$ such that one of the conditions (3) or (4) of Theorem C is satisfied. If $I \in \mathcal{I}$ and $f \in L(I,I)$ is a linear horseshoe map of type (m,σ) then by Theorem C, $\sqrt[n]{f} \neq \emptyset$. The set $\sqrt[n]{f} \cap L(I,I)$ is parametrized by the vectors P from an open $(m-1)$-dimensional simplex S. However, those elements of $\sqrt[n]{f} \cap L(I,I)$ which have piecewise linear n-th root, correspond to the parameters $P = (W_1(Q), \dots, W_m(Q))$ for Q from an open $(\sqrt[n]{m}-1)$-dimensional simplex. Therefore they form an $(\sqrt[n]{m}-1)$-dimensional algebraic subset of S. Since $\sqrt[n]{m} - 1 < m - 1$, this proves Theorem B.

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