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STATIONARY SOLUTIONS, BLOW UP AND CONVERGENCE TO STATIONARY SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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1. Introduction

Consider the problem

$$
(1.1)\qquad \begin{cases} u_t = \triangle u - au^p & x \in \Omega,\ t > 0 \\ \dfrac{\partial u}{\partial n} = u^q & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_o(x) \geq 0 & x \in \overline{\Omega}, \end{cases}
$$

with $p, q > 1$, $a > 0$, Ω – bounded domain in $\mathbb{R}^N$, $u_o \not\equiv 0$.

If $a = 0$ then it follows from [**F**] that any solution blows up in finite time. The starting point of our investigations was the question whether the damping term in the equation can prevent blow up if $a > 0$.

For $N = 1$ we give the following complete answer:

(i) If $p < 2q - 1$ or $p = 2q - 1$, $a < q$ then there are initial data for which blow up occurs.

(ii) If $p > 2q - 1$ or $p = 2q - 1$, $a > q$ then any solution exists globally and stays uniformly bounded.

(iii) If $p = 2q - 1$, $a = q$ then any solution exists globally but it is not uniformly bounded. More precisely, any solution tends pointwise (as $t \to \infty$) to the unique function v which satisfies

$$
\begin{aligned} v_{xx} - qv^{2q-1} &= 0 \qquad \text{in } \Omega \\ v &= \infty \qquad \text{on } \partial\Omega. \end{aligned}
$$

For $N > 1$ and Ω a ball we also show that (i), (ii) hold. For general domains the answer is far from being complete. We show global existence and boundedness only for

$$
q < \frac{N+1}{N-1}, \quad p > \frac{N - q(N-2)}{N + 1 - q(N-1)}(q+1) - 1
$$

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and blow up of solutions starting from initial functions with negative energy for $p \le q$. For $p \le q$ and q subcritical ($q < N/(N-2)$ if $N > 2$) we give also another sufficient condition for blow up. Namely, u blows up provided $u_o \ge v$, $u_o \not\equiv v$, v is any positive stationary solution. Positive stationary solutions exist if $p < q$ or $p = q$, $a > a_\Omega := \dfrac{|\partial\Omega|}{|\Omega|}$. If $p = q$, q is subcritical and $a < a_\Omega$ then any solution blows up.

If Ω is a ball and p, q, a are as in (i) then we prove blow up of solutions which emanate from radial subsolutions that are sufficiently large on $\partial\Omega$.

For $N = 1$ and p, q, a as in (i), a sufficient condition for blow up is that u_o lies above an arbitrary maximal stationary solution. If $q \le p \le 2q - 1$ then we shall see below that for any interval Ω there exists $a_o = a_o(\Omega, p, q) > 0$ such that for $a < a_o$ the maximal stationary solution is 0, which means that any solution blows up.

For $N = 1$ we also show that for suitable initial functions blow up occurs only on the boundary of the interval Ω.

Since we are interested in all possible types of behavior of solutions, we are led to the question if there are global unbounded solutions for p, q, a as in (i). For $N = 1$ or $p \le q$, q subcritical, the answer is no. Therefore, there are only two possibilities in this case: blow up in finite time or global existence and boundedness. The latter possibility means that the ω–limit set is nonempty and consists of stationary solutions.

Let us now give a sketch of our results concerning the stationary solutions. For $N = 1$ ($\Omega = (-l, l)$) our description of the set of (positive) stationary solutions is almost complete.

Denote the set of positive stationary solutions by E and the subset of symmetric positive stationary solutions by E_s. For fixed $l > 0$ we distinguish five cases:

(i) If $p > 2q - 1$ then
   $\operatorname{card} E = 1,\ E = E_s$ for any $a > 0$.

(ii) If $p = 2q - 1$ then
   $E = \emptyset$ for $0 < a \le q$,
   $\operatorname{card} E = 1,\ E = E_s$ for $a > q$.

(iii) If $q < p < 2q - 1$ then there are $0 < a_o < a_1$ such that
   $E = \emptyset$ for $0 < a < a_o$,
   $\operatorname{card} E = 1,\ E = E_s$ for $a = a_o$,
   $\operatorname{card} E = 2,\ E = E_s$ for $a_o < a \le a_1$,
   $\operatorname{card} E \ge 4,\ \operatorname{card} E$ is even, $\operatorname{card} E_s = 2$ for $a > a_1$.
  If, in addition, $p \le 4$ or $p > 4$, $q \ge p - 1 - \frac{1}{p-2}$, then
   $\operatorname{card} E = 4$ for $a > a_1$.

(iv) If $p = q$ then there is an $a_1 > 0$ such that
   $E = \emptyset$ for $0 < a \le 1/l$,
   $\operatorname{card} E = 1,\ E = E_s$ for $1/l < a \le a_1$,
   $\operatorname{card} E = 3,\ \operatorname{card} E_s = 1$ for $a > a_1$.

(v) If $p < q$ then there is an $a_1 > 0$ such that
   $\operatorname{card} E = 1,\ E = E_s$ for $0 < a \le a_1$,

$$\operatorname{card} E = 3, \ \operatorname{card} E_s = 1 \text{ for } a > a_1 .$$

Our results are summarized in the following bifurcation diagrams:

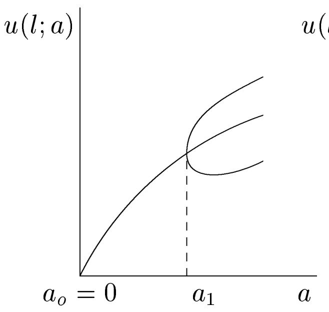


Fig.1: $p < q$

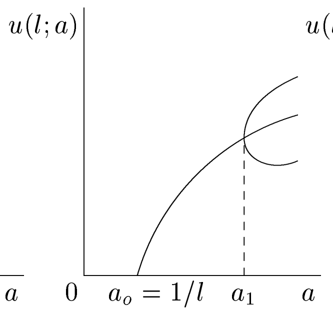


Fig.2: $p = q$

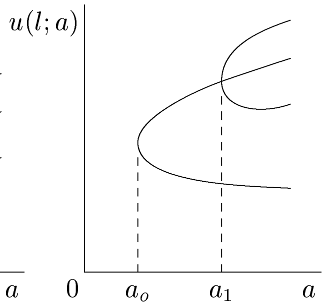


Fig.3: $2q-1 > p > q$

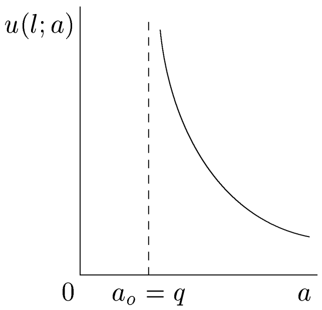


Fig.4: $p = 2q - 1$

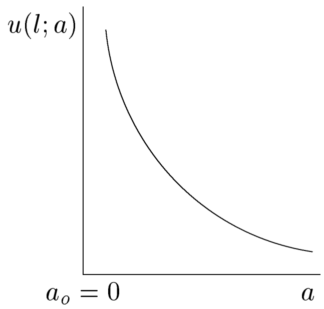


Fig.5: $p > 2q - 1$

In higher space dimension we have also some existence, nonexistence and multiplicity results for the stationary problem on general domains and more precise results for the radially symmetric problem on a ball. These results confirm that several facts indicated in Figures 1–5 hold also for $N > 1$. See Theorems 2.1, 2.2 for more details.

We mentioned above that for $N = 1$ a sufficient condition for blow up is that u_o lies above an arbitrary maximal stationary solution. This leads to the question how are the stationary solutions ordered. We show that for $N = 1$ any positive stationary solution is maximal except for the case $q < p < 2q - 1$, $a > a_o$, when there is a $v \in E_s$ such that $v < w$ for any $w \in E$, $w \neq v$. Any $w \in E$, $w \neq v$ is maximal.

To give a description of the local semiflow generated by the problem (1.1) we determine the stability properties of stationary solutions. For $N = 1$ we show that positive stationary solutions which do not correspond to $a = a_o$ or $a = a_1$ are hyperbolic, i.e. zero is not an eigenvalue of the linearization (if $q < p < 2q - 1$ then also the smaller solution corresponding to $a = a_1$ is hyperbolic). Then we compute the Morse indices of the hyperbolic stationary solutions. This will be used to draw the picture of the flow, more precisely, to find orbits which connect the stationary solutions.

For $N = 1$, $p < q$, $a > a_1$ the flow is depicted in the following figure.

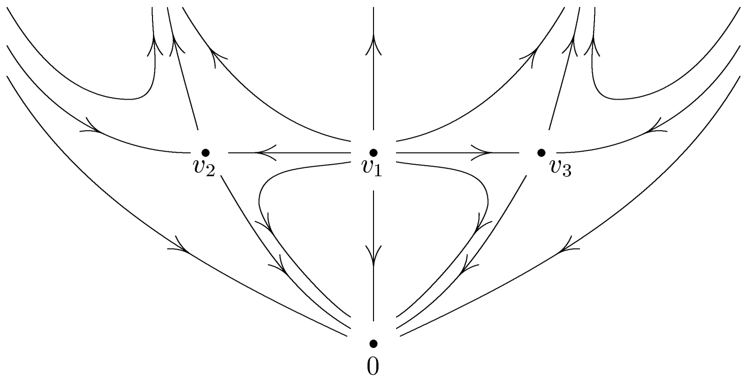


Figure 6. The flow for $N = 1$, $p < q$, $a > a_1$.

In Figure 6, the function v_1 is the symmetric positive stationary solution, v_2 and v_3 are nonsymmetric stationary solutions. The zero solution is stable, the unstable manifolds of v_2, v_3 are one–dimensional, the unstable manifold of v_1 is two–dimensional. Any positive stationary solution is connected by an orbit to 0, v_1 is connected to v_2 and v_3.

Moreover, if $N = 1$, $p < q$, then for any u_o there is a $\lambda_o > 0$ such that the solution $u(t, \lambda u_o)$ starting from λu_o tends to 0 in $W^{1,2}(\Omega)$ as $t \to \infty$ if $\lambda < \lambda_o$; $u(t, \lambda_o u_o)$ tends to a positive stationary solution; while $u(t, \lambda u_o)$ blows up in finite time if $\lambda > \lambda_o$.

A weaker result is proved in a more general situation. Denote the set of initial nonnegative data for which the solutions exist globally by G. Then G is star–shaped with respect to zero and closed in $C^+ := \{v \in W^{1,2}(\Omega)\,;\, v \geq 0 \text{ a.e. }\}$ provided $N > 1$, $p < q < (N + 1)/(N - 1)$ or $p = q < \min(2, (N + 2)/N)$.

The paper is organized as follows. Section 2 contains results on the N–dimensional stationary problem. The bifurcation diagrams for the 1–dimensional stationary problem are established in Section 3. In Section 3 also the Morse indices of the stationary solutions for $N = 1$ are computed. In Section 4 we give sufficient conditions for blow up and global existence. In Section 5 we establish the connecting orbits and study the behavior of $u(t, \lambda u_o)$, $\lambda > 0$.

2. Stationary solutions for $N \geq 1$

Throughout this section[1] we shall suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with the smooth boundary $\partial\Omega$, $a > 0$ and $p, q > 1$ are subcritical, i.e. $p < \dfrac{N+2}{N-2}$ and $q < \dfrac{N}{N-2}$ if $N > 2$. Then we have the compact imbedding of the Sobolev space $W^{1,2}(\Omega)$ into $L^{p+1}(\Omega)$ and the trace operator $\mathrm{Tr} : W^{1,2}(\Omega) \to L^{q+1}(\partial\Omega)$ is also compact.

We shall look for (weak) solutions of the problem

$$
\begin{cases} \triangle u = a\,|u|^{p-1}u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = |u|^{q-1}u & \text{on } \partial\Omega \end{cases} \tag{2.1}
$$

By standard L^p regularity theory (see e.g. [**A1,** Theorem 3.2]) we get that any solution of (2.1) is in $C^1(\overline{\Omega}) \cap C^2(\Omega)$. Moreover, the maximum principle (see [**GT,** Theorem 3.5, Lemma 3.4]) implies that any nonnegative solution $u \not\equiv 0$ of (2.1) is positive in $\overline{\Omega}$. In what follows, by $|\partial\Omega|$ we denote the (N-1)–dimensional measure of $\partial\Omega$, by $|\Omega|$ we mean the N–dimensional measure of Ω. Finally, we put $a_\Omega = \dfrac{|\partial\Omega|}{|\Omega|}$ and $c_\Omega = |\Omega|^{-1/2}$.

The main result of this section are the following two theorems.

Theorem 2.1.

(i) *Let $p \leq q$ and let $a > a_o$, where $a_o := 0$ if $p < q$ and $a_o := a_\Omega$ if $p = q$. Then there exists a positive solution of* (2.1). *The zero solution is stable, any positive solution is unstable (both from above and from below) in $W^{1,2}(\Omega)$ in the Lyapunov sense. The graphs of any two positive solutions intersect.*

(ii) *Let $p = q$ and $a < a_\Omega$. Then* (2.1) *does not have positive solutions. The zero solution is unstable.*

(iii) *Let $p > q$. Then the zero solution is unstable and there exists $a_o \in [0, \infty)$ such that* (2.1) *has a positive stable solution for $a > a_o$ and* (2.1) *does not have positive solutions for $0 < a < a_o$.*

(iv) *Let $q < p < 2q-1$ and put $\zeta = \left(\dfrac{p+1}{2}\right)^{(p-1)/(q-1)}$. If $\tilde{a} > 0$ is sufficiently large, then there exists $a \in (\tilde{a}, \tilde{a}\zeta)$ such that* (2.1) *has at least two positive solutions.*

[1] except of Remark 2.6 where supercritical p, q are considered

(v) [2] *Let* $q < \dfrac{N+1}{N-1}$ *and* $p+1 > (q+1)q^*$, *where* $q^* = \dfrac{N-q(N-2)}{N+1-q(N-1)}$. *Then* $p > 2q-1$ *and* $a_o = 0$, *i.e.* (2.1) *has a positive stable solution for any* $a > 0$.

Theorem 2.2. *Let* Ω *be a ball in* $\mathbb{R}^N$.

(i) *If* $p < q$ *or* $p = q$ *and* $a > a_\Omega$, *then there exists a positive symmetric solution of* (2.1). *This solution is unique among positive symmetric functions.*
(ii) *If* $q < p < 2q-1$ *then there exists* $a_o^s > 0$ *such that* (2.1) *has a symmetric positive solution iff* $a \geq a_o^s$. *If* $a > a_o^s$, *then* (2.1) *has et least* 2 *symmetric positive solutions.*
(iii) *Let* $p = 2q-1$. *If* $a > q$ *then* (2.1) *has a symmetric positive stable solution. If* $a \leq q$ *then* (2.1) *does not have symmetric positive solutions.*
(iv) *If* $p > 2q-1$ *then there exists a symmetric positive stable solution of* (2.1) *for any* $a > 0$.

We shall use the variational formulation of (2.1), i.e. we shall look for critical points of the C^2 functional

$$\Phi : X \to \mathbb{R} : u \mapsto \mathcal{I}(u) + a\mathcal{P}(u) - \mathcal{Q}(u),$$

where

$$X = W^{1,2}(\Omega) \text{ is endowed with the scalar product}$$
$$\langle u, v\rangle = \int_\Omega \nabla u\,\nabla v\,dx + \int_\Omega uv\,dx,$$
$$\mathcal{I}(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx,$$
$$\mathcal{P}(u) = \frac{1}{p+1}\int_\Omega |u(x)|^{p+1} dx \text{ and}$$
$$\mathcal{Q}(u) = \frac{1}{q+1}\int_{\partial\Omega} |u(x)|^{q+1} dS.$$

Hence $\|u\|^2 := \langle u,u\rangle = 2\mathcal{I}(u) + 2\mathcal{K}(u)$, where $\mathcal{K}(u) = \frac{1}{2}\int_\Omega u(x)^2 dx$. By F, P, Q and K we denote the Fréchet derivatives of Φ, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{K}$, respectively. Notice that K, P and Q are compact C^1 operators in X and the problem (2.1) is equivalent to the problem

$$F(u) = 0, \tag{2.2}$$

where $F = F_a : X \to X : u \mapsto u - K(u) + aP(u) - Q(u)$.

If u is an isolated solution of (2.2), we shall denote by $d(u)$ or $d_a(u)$ the local Leray–Schauder degree of F at u with respect to 0, i.e. $d(u) = \deg(F, 0, B_\varepsilon(u))$ for ε sufficiently small (where $B_\varepsilon(u) = \{v \in X\,;\, \|v-u\| \leq \varepsilon\}$).

[2] cf. also Remark 2.5(i)

If C is a closed convex set in X, we denote by P^C the orthogonal projection in X onto C and we put $F^C(u) = u - P^C\big(K(u) - aP(u) + Q(u)\big)$ i.e. the solutions of $F^C(u) = 0$ correspond to the solutions of the variational inequality

$$u \in C: \qquad \langle F(u), \varphi - u\rangle \geq 0 \qquad \text{for any } \varphi \in C \tag{2.3}$$

which are the critical points of Φ with respect to C. If $C = C^+ := \{u \in X\,;\ u \geq 0 \text{ a.e.}\}$, then we write briefly F^+ instead of F^{C^+} and we denote by $d^+(u)$ the local Leray–Schauder degree of F^+ at u with respect to 0. We call $\underline{u}$ a **subsolution** of (2.2) if $\Phi'(\underline{u})\varphi \leq 0$ for any $\varphi \in C^+$. Analogously we define a supersolution of (2.2).

Following [**H2**], we call an operator $T : X \to X$ E–regular, if there exists a finite sequence $\{E_i\}_{i=0}^{n+1}$ of real Banach spaces such that $E = E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_n \hookrightarrow E_{n+1} = X$ and T induces continuous operators $T_i \in C(E_i, E_{i-1})$ for $i = 1, \dots, n+1$. The L^p regularity for (2.1) implies that the operators K, P and Q are $W^{1,r}(\Omega)$–regular for any $r \geq 2$. Moreover, one can easily prove the following Lemma (cf. [**H2,** Lemma 2]).

Lemma 2.1. *Let $T_j : X \to X$ be E–regular operators for $j = 1, \dots, m$ and let the corresponding E_i spaces in the definition of E–regularity be independent of j. Let $\{\alpha_k^{(j)}\}_{k=1}^{\infty}$ be a sequence of real numbers converging to $\alpha^{(j)}$ for $j = 1, \dots, m$. Let $v_k \in X$, $v_k \to v \in E$ in X and let $v_k = \sum_{j=1}^{m} \alpha_k^{(j)} T_j(v_k)$. Then $v_k \in E$ and $v_k \to v$ in E.*

In the following two lemmas we study solutions which are close to zero.

Lemma 2.2. *Let $p, q > 1$ be fixed, $\infty > A \geq a_k \geq 0$ $(k = 1, 2, \dots)$, $F_{a_k}(u_k) = 0$, $0 \neq \|u_k\| \to 0$. Then one of the following assertions is true*

(i) *$p < q$, $a_k \to 0$, $a_k > 0$ for k large enough.*
(ii) *$p = q$, $a_k \to a_\Omega$.*

Moreover, $\dfrac{|u_k|}{\|u_k\|} \to c_\Omega$ in $X \cap C(\overline{\Omega})$.

Proof. Putting $v_k = \dfrac{|u_k|}{\|u_k\|}$ we may suppose that v_k converges weakly in X to some element $v \in X$ (otherwise we choose a suitable subsequence). Dividing the equation $F_{a_k}(u_k) = 0$ by $\|u_k\|$ we obtain

$$v_k = Kv_k - a_k P(v_k)\|u_k\|^{p-1} + Q(v_k)\|u_k\|^{q-1}. \tag{2.4}$$

Passing to the limit in (2.4) and using the compactness of K we get $v_k \to v = Kv$ (strong convergence), $\|v\| = 1$, which implies $v \equiv \pm c_\Omega$. Lemma 2.1 and the imbedding $W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $r > N$ imply $v_k \to v$ in $C(\overline{\Omega})$, hence $v_k > 0$ (or $v_k < 0$) for k large enough. Without loss of generality we may suppose

$v_k > 0$. Integrating the equation $\triangle u_k = a_k u_k^p$ over Ω and multiplying the resulting equation by $\|u_k\|^{-q}$ we get

$$a_k \int_\Omega v_k^p dx \, \|u_k\|^{p-q} = \int_{\partial\Omega} v_k^q dS \to |\partial\Omega| \, |\Omega|^{-q/2}, \tag{2.5}$$

which implies $p \le q$, $a_k > 0$, $a_k \to 0$ if $p < q$ and $a_k \to a_\Omega$ if $p = q$. □

Remark 2.1. By Theorem 2.1(ii) it will follow that $a_k \ge a_\Omega$ for k large enough in the case of Lemma 2.2(ii). If Ω is a ball, then using (2.5) one can even prove $a_k > a_\Omega$, since $\triangle(u_k^p) > 0$.

Lemma 2.3.

(i) *If $p < q$ and $a > 0$ or if $p = q$ and $a > a_\Omega$, then $u = 0$ is a strict local minimum of Φ, $d(0) = 1$.*

(ii) *If $a = 0$ or $p = q$ and $0 \le a < a_\Omega$ or if $p > q$ and $a \ge 0$, then $d(0) = -1$.*

Proof. (i) We shall argue by contradiction. Suppose there exist $0 \ne u_k \to 0$ (in X) such that $\Phi(u_k) \le 0$. Since Φ is bounded on bounded sets and weakly lower semicontinuous, there exists $0 \ne u_k$ such that $\Phi(u_k) = \min_{\|v\| \le 1/k} \Phi(v) \le 0$. Hence, there exists a Lagrange multiplier $\lambda_k \ge 0$ such that $F(u_k) + \lambda_k u_k = 0$, i.e.

$$u_k = \frac{1}{1+\lambda_k}\big(Ku_k - aP(u_k) + Q(u_k)\big). \tag{2.6}$$

We may suppose that $v_k := \dfrac{u_k}{\|u_k\|} \rightharpoonup v$, $\dfrac{1}{1+\lambda_k} \to \mu \in [0,1]$. Dividing (2.6) by $\|u_k\|$ and passing to the limit we get $v_k \to v = \mu K v$, $\|v\| = 1$, which yields $\mu = 1$, $|v| \equiv c_\Omega$. By Lemma 2.1 we get $v_k \to v$ in $C(\overline{\Omega})$. Now

$$\frac{\Phi(u_k)}{\|u_k\|^{p+1}} \ge a\mathcal{P}(v_k) - \mathcal{Q}(v_k) \, \|u_k\|^{q-p}, \tag{2.7}$$

where the right-hand side converges to $a\mathcal{P}(v)$ for $q > p$ or to $a\mathcal{P}(v) - \mathcal{Q}(v)$ for $p = q$. Since in both cases the limit is positive, we have a contradiction. Hence $u = 0$ is a (strict) local minimizer for Φ and by [**A2**] $d(0) = 1$.

(ii) Using the homotopies $H^t(u) = F_{ta}(u)$, $t \in [0,1]$, and $H_\alpha(u) = u - Q(u) - (1+\alpha)Ku$, $\alpha \in [0, \alpha_o]$, we obtain $d(0) = \deg(H_\alpha, 0, B_\varepsilon(u)) = -1$, since the operator $H'_\alpha(0)$ is regular and has exactly one negative eigenvalue for $\alpha > 0$ small. We have to verify $H^t(u) \ne 0$ and $H_\alpha(u) \ne 0$ for $\|u\| = \varepsilon$ small and $\alpha \ge 0$ small. The condition $H^t(u) \ne 0$ and $H_0(u) \ne 0$ follows from Lemma 2.2. Hence suppose $H_{\alpha_k}(u_k) = 0$ for $0 \ne u_k \to 0$ and $\alpha_k > 0$, $\alpha_k \to 0$. Putting $v_k = \dfrac{u_k}{\|u_k\|}$ we get similarly as in Lemma 2.2 $v_k \to v \equiv \pm c_\Omega$ in $X \cap C(\overline{\Omega})$ and we may assume $v_k > 0$ for k large. Then $\triangle u_k = -\alpha_k u_k < 0$, $\dfrac{\partial u_k}{\partial n} = u_k^q > 0$, which yields a contradiction. □

Remark 2.2 It can be shown that in the situation of Lemma 2.3(ii) the critical point $u = 0$ is of mountain–pass type in the sense of [**H1**].

Lemma 2.4. *Let $0 \le \underline{u} \le \overline{u} \le M < \infty$, where $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution of* (2.2), *respectively. Then there exists a solution u of* (2.2) *with $\underline{u} \le u \le \overline{u}$.*

Moreover, if $\underline{u}, \overline{u} \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ are not local minimizers of Φ with respect to $C := \{v \in X\,;\, \underline{u} \le v \le \overline{u}\}$ and $\underline{u} < \overline{u}$ in $\overline{\Omega}$, then there exists a solution u lying strictly between $\underline{u}$ and $\overline{u}$ and being a local minimizer of Φ.

Proof. In the first part of the proof we shall proceed similarly as in [**St,** Theorem I.2.4.]. The set C is convex and (weakly) closed and $\Phi : C \to \mathbb{R}$ is lower bounded and weakly lower semicontinuous, hence there exists $u \in C$ such that $\Phi(u) = \min_{v \in C} \Phi(v)$. Consequently, u solves (2.3).

Choose $\varphi \in C^1(\overline{\Omega})$, $\varepsilon > 0$ and put

$$v_\varepsilon = \min\{\overline{u}, \max\{\underline{u}, u + \varepsilon\varphi\}\} = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon \in C,$$

where $\varphi^\varepsilon = \max\{0, u + \varepsilon\varphi - \overline{u}\} \ge 0$ and $\varphi_\varepsilon = -\min\{0, u + \varepsilon\varphi - \underline{u}\} \ge 0$. We have $0 \le \langle \Phi'(u), v_\varepsilon - u\rangle = \varepsilon\langle \Phi'(u), \varphi\rangle - \langle \Phi'(u), \varphi^\varepsilon\rangle + \langle \Phi'(u), \varphi_\varepsilon\rangle$, so that

$$\langle \Phi'(u), \varphi\rangle \ge \frac{1}{\varepsilon}\Big(\langle \Phi'(u), \varphi^\varepsilon\rangle - \langle \Phi'(u), \varphi_\varepsilon\rangle\Big). \tag{2.8}$$

Since $\overline{u}$ is a supersolution, we have

$$\begin{aligned}
\langle \Phi'(u), \varphi^\varepsilon\rangle &\ge \langle \Phi'(u) - \Phi'(\overline{u}), \varphi^\varepsilon\rangle \\
&= \int_{\Omega^\varepsilon} \big(\nabla(u - \overline{u})\,\nabla(u + \varepsilon\varphi - \overline{u}) + a(u^p - \overline{u}^p)(u + \varepsilon\varphi - \overline{u})\big)\,dx \\
&\qquad - \int_{\Gamma^\varepsilon} (u^q - \overline{u}^q)(u + \varepsilon\varphi - \overline{u})\,dS \\
&\ge \varepsilon \int_{\Omega^\varepsilon} \big(\nabla(u - \overline{u})\,\nabla\varphi + a(u^p - \overline{u}^p)\varphi\big)\,dx - \varepsilon \int_{\Gamma^\varepsilon} |u^q - \overline{u}^q|\,|\varphi|\,dS,
\end{aligned}$$

where Ω^ε or Γ^ε are the sets of all $x \in \Omega$ or $x \in \partial\Omega$, for which $u(x) + \varepsilon\varphi(x) \ge \overline{u}(x) > u(x)$, respectively. Since $|\Omega^\varepsilon| \to 0$ and $|\Gamma^\varepsilon| \to 0$ as $\varepsilon \to 0$, we get $\langle \Phi'(u), \varphi^\varepsilon\rangle \ge o(\varepsilon)$. Analogously we get $\langle \Phi'(u), \varphi_\varepsilon\rangle \le o(\varepsilon)$, hence (2.8) implies $\langle \Phi'(u), \varphi\rangle \ge 0$ for all $\varphi \in C^1(\overline{\Omega})$, so that $\Phi'(u) = 0$.

Suppose now the additional assumptions on $\underline{u}$ and $\overline{u}$ and let u be as above. Then $\triangle\overline{u} \le a\overline{u}^p$, $\dfrac{\partial\overline{u}}{\partial n} \ge \overline{u}^q$. Putting $w = \overline{u} - u$ one obtains $w \not\equiv 0$, $w \ge 0$, $\triangle w \le a(\overline{u}^p - u^p) \le cw$, where $c = apM^{p-1}$. By [**GT,** Theorem 3.5] $w > 0$ in Ω. If $w(x_o) = 0$ for some $x_o \in \partial\Omega$, then [**GT,** Lemma 3.4] implies $\dfrac{\partial w}{\partial n}(x_o) < 0$. However, $\dfrac{\partial w}{\partial n}(x_o) \ge \overline{u}^q(x_o) - u^q(x_o) \ge 0$, a contradiction. Hence, $w = \overline{u} - u > 0$ in $\overline{\Omega}$. Similarly one gets also $u - \underline{u} > 0$ in $\overline{\Omega}$.

Now suppose that u is not local minimum of Φ. Similarly as in the proof of Lemma 2.3 we find $u_k \to u$ such that $\Phi(u) > \Phi(u_k) = \min_{\|v-u\|\le 1/k} \Phi(v)$, $F(u_k) + \lambda_k(u_k - u) = 0$ for some $\lambda_k \geq 0$. The last equation is equivalent to

$$u_k = \frac{1}{1+\lambda_k}(K - aP + Q)(u_k) + \frac{\lambda_k}{1+\lambda_k}u,$$

which together with Lemma 2.1 implies $u_k \to u$ in $C(\overline{\Omega})$. However, this is a contradiction with $\Phi(u_k) < \Phi(u) = \min_{\underline{u}\le v\le \overline{u}} \Phi(v)$. □

Lemma 2.5. *Any solution of the variational inequality* (2.3) *with* $C = C^+$ *solves also the problem* (2.2).

Proof. Proof is based on the same arguments as the first part of the proof of Lemma 2.4. Choosing $\varphi \in C^1(\overline{\Omega})$ and putting $v_\varepsilon = \max\{0, u + \varepsilon\varphi\}$ one gets

$$0 \le \frac{1}{\varepsilon}\langle \Phi'(u), v_\varepsilon - u\rangle = \langle \Phi'(u), \varphi\rangle + o(1),$$

hence $\Phi'(u) = 0$. □

Lemma 2.6. *If* $u \in C^+$ *is an isolated solution of* (2.2)*, then the degree* $d^+(u)$ *is well defined. If* $N = 1$ *and* $u \neq 0$*, then* $d^+(u) = d(u)$*. Moreover, except for the case* $p = q$*,* $a = a_\Omega$*, we have (for any* N*)*

(i) $d^+(0) = 1$ *if* $d(0) = 1$,
(ii) $d^+(0) = 0$ *if* $d(0) = -1$.

Proof. If $u \in C^+$ is an isolated solution of (2.2) then u is an isolated solution of (2.3) by Lemma 2.5. If $N = 1$, then u lies in the interior of $C^+ \subset X$, hence $F^+ = F$ in a neighbourhood of u.

If $p < q$ and $a > 0$ or $p = q$ and $a > a_\Omega$, then 0 is a strict local minimum of Φ by Lemma 2.3, hence it is a (strict) local minimum of Φ with respect to C^+. Now [**Q2**] implies $d^+(0) = 1$. Now it is sufficient to show $d^+(0) = 0$ for $a = 0$, since then (ii) follows from the homotopy invariance property of the degree. Hence suppose $a = 0$. Then we may use the homotopies $H^t_\alpha(u) = u - P^+\big((1+\alpha)Ku + tQ(u)\big)$, $\alpha \in [0, \alpha_o]$, $t \in [0,1]$, to derive $d^+(0) = \deg(H^1_{\alpha_o}, 0, B_\varepsilon(0)) = \deg(H^0_{\alpha_0}, 0, B_\varepsilon(0)) = 0$, where the last equality follows from [**Q1,** Theorem 2(i)]. The admissibility of H^t_α follows from the fact the the solutions of $H^t_\alpha(u) = 0$ correspond to the solutions of the inequality $u \ge 0$, $-\Delta u \ge \alpha u$, $\dfrac{\partial u}{\partial n} \ge tu^q$. □

Proof of Theorem 2.1(i). Suppose $p < q$ and $a > 0$ or $p = q$ and $a > a_\Omega$. Then 0 is a strict local minimum of Φ by Lemma 2.3(i). Choosing $u > 0$ such that $a\mathcal{P}(u) < \mathcal{Q}(u)$ we simply get $\Phi(tu) < 0$ for $t > 0$ sufficiently large. Put $\Phi^+(u) = \Phi(u)$ for $u \in C^+$, $\Phi^+(u) = +\infty$ for $u \notin C^+$. We show that the functional Φ^+ fulfils the Palais–Smale condition introduced by Szulkin [**Sz**], hence by the corresponding mountain–pass theorem [**Sz,** Theorem 3.2] there exists a nontrivial solution u of the variational inequality (2.3) with $C = C^+$. By Lemma 2.5 u is a positive solution of (2.2).

Thus suppose $u_k \in C^+$, $\varepsilon_k \downarrow 0$, $\Phi(u_k) \to d$ and

$$\langle \Phi'(u_k), v - u_k \rangle \geq -\varepsilon_k \|v - u_k\| \quad \text{for any } v \in C^+. \tag{2.9}$$

Put $w_k := P^+(u_k - F(u_k))$, then

$$\langle u_k - F(u_k) - w_k, w_k - v \rangle \geq 0 \quad \text{for any } v \in C^+. \tag{2.10}$$

To prove the relative compactness of the sequence $\{u_k\}$ it is sufficient to show its boundedness, since then $\{w_k\}$ is relatively compact and putting $v = w_k$ in (2.9), $v = u_k$ in (2.10) and adding the resulting inequalities one simply gets $\|u_k - w_k\| \leq \varepsilon_k$. Now using (2.9) with $v = 2u_k$ we get for k sufficiently large

(2.11)

$$(q+1)(d+1)+\varepsilon_k\|u_k\| \geq (q+1)\Phi(u_k)-\langle\Phi'(u_k),u_k\rangle = (q-1)\mathcal{I}(u_k)+a(q-p)\mathcal{P}(u_k).$$

If $q > p$, then the right-hand side in (2.11) can be estimated below by $c\|u_k\|^2$ for some $c > 0$, hence the assertion follows. Let $p = q$ and suppose $\|u_k\| \to \infty$. Using the decomposition $u_k = c_k + u_k^\perp$, where $\int_\Omega u_k^\perp dx = 0$ and c_k is constant, (2.11) and [**N,** Theorem 7.1] yield $\|u_k^\perp\| \leq M\mathcal{I}(u_k^\perp) = o(c_k)$ for some $M > 0$, which implies $u_k/\|u_k\| \to c_\Omega$. Therefore,

$$\frac{\Phi(u_k)}{\|u_k\|^p} \to a\mathcal{P}(c_\Omega) - \mathcal{Q}(c_\Omega) = \frac{c_\Omega^{p+1}}{p+1}(a|\Omega| - |\partial\Omega|) > 0,$$

which gives a contradiction with the assumption $\Phi(u_k) \to d$.

To see that any positive solution u is unstable (both from above and from below) notice that

$$\Phi''(u)(u,u) = q\langle\Phi'(u),u\rangle + (1-q)2\mathcal{I}(u) + a(p-q)(p+1)\mathcal{P}(u) < 0 \tag{2.12}$$

and suppose e.g. that u is stable from above. Choosing $\varepsilon > 0$ we may find $\delta > 0$ such that the solution u_δ of (1.1) starting from $(1+\delta)u$ fulfils $\|u_\delta(t) - u\| < \varepsilon$ for any $t > 0$. Moreover, choosing δ sufficiently small we have $\Phi\big(u_\delta(0)\big) < \Phi(u)$ and due to the compactness and monotonicity of the flow (see Proposition 5.1) we get $u_\delta(t) \to \overline{u_\delta}$ as $t \to +\infty$, where $\overline{u_\delta}$ is a stationary solution fulfilling $\|\overline{u_\delta} - u\| \leq \varepsilon$, $\overline{u_\delta} \geq u$ and $\Phi(\overline{u_\delta}) < \Phi(u)$; the last inequality follows from the fact that the function $\Phi\big(u_\delta(\cdot)\big)$ is nonincreasing. The maximum principle implies $\overline{u_\delta} > u$ in $\overline{\Omega}$ and Lemma 2.4 together with (2.12) (used both for u and for $\overline{u_\delta}$) yield a contradiction. The last argument shows also the nonexistence of two positive solutions u_1, u_2 with $u_1 \leq u_2$. □

Remarks 2.3. Let us briefly mention some other possibilities how to prove Theorem 2.1(i).

(i) One can use the standard mountain–pass theorem for the functional Φ to get a critical point u which is either a local minimum or of mountain–pass type (see [**H1,** Theorem]). If u changes sign in Ω, one gets similarly as in (2.12) $\Phi''(u)(w,w) < 0$ for any $0 \neq w \in \operatorname{span}\{u^+, u^-\}$ (where $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = -\min\{u(x), 0\}$) and using this information it is not difficult to show that u is neither local minimum nor of mountain–pass type.

(ii) If one is able to prove suitable apriori estimates for the positive solutions of (2.2), then one can use the degree theory: if $\|u\| < R$ for any solution u of (2.2) with $0 \le a \le A$, then

$$0 = d_0^+(0) = \deg(F_0^+, 0, B_R(0)) = \deg(F_A^+, 0, B_R(0)) \neq d_A^+(0) = 1,$$

hence there exists a nontrivial solution for $a = A$.
The apriori estimates can be easily found e.g. for symmetric solutions on a ball (see the proof of Theorem 2.2). For a general domain we have the following assertion: *Let $p < q$ and let $q < \dfrac{N-1}{N-2}$ if $N > 2$. Then for any $A > 0$ there exists $R > 0$ such that any positive solution u of* (2.2) *with $0 \le a \le A$ fulfils $\|u\| < R$. Moreover, the solutions tend to zero if $a \to 0+$.*

Proof. Denote by $\|\cdot\|_r$ or $|||\cdot|||_r$ the norm in $L^r(\Omega)$ or $L^r(\partial\Omega)$, respectively. By R we denote various constants, which may vary from step to step.
We have $\|u\|^2 \le R(\mathcal{I}(u) + \mathcal{Q}(u)) + \eta$ for any $u \in X$, where $\eta > 0$ and $R = R(\eta)$. If u is a solution, then obviously $2\mathcal{I}(u) \le (q+1)\mathcal{Q}(u)$. Choosing $\varepsilon > 0$ such that the trace operator $\operatorname{Tr} : X \to L^r(\partial\Omega)$, where $r = q\dfrac{2-\varepsilon}{1-\varepsilon}$, is continuos, we obtain using Hölder inequality

$$-\eta + \|u\|^2 \le R\mathcal{Q}(u) \le R|||u|||_r^{2-\varepsilon}|||u|||_q^{q-1+\varepsilon} \le R\|u\|^{2-\varepsilon}|||u|||_q^{q-1+\varepsilon},$$

hence

$$\|u\|^\varepsilon \le \eta' + R|||u|||_q^{q-1+\varepsilon}, \tag{2.13}$$

where $\eta' \to 0$ as $\eta \to 0$. Now $\triangle(u^p) \ge 0$, hence $||u^p||_1 \le R|||u^p|||_1$, where R does not depend on u. Using this inequality, Hölder inequality and the equation $\triangle u = au^p$ integrated over Ω, we obtain

$$||u||_p^q \le R|||u|||_p^q \le R|||u|||_q^q = Ra||u||_p^p,$$

hence $||u||_p \le Ra^{1/(q-p)}$ and $|||u|||_q \le Ra^{1/(q-p)}$. Now (2.13) implies $\|u\| \le R$ and $\|u\| \to 0$ if $a \to 0+$. $\square$

Let us also note that using the degree theory and Lemmas 2.3, 2.6 one can easily prove (without apriori estimates) the following assertion:

$$\begin{gathered}(\forall\varepsilon > 0)(\exists\delta > 0)(\forall\eta \in (0,\delta))(\exists a \in (a_o, a_o + \varepsilon))(\exists u \in X)\\ u \text{ is a positive solution of (2.1) and } \|u\| = \eta.\end{gathered} \tag{2.14}$$

(iii) In Section 4 we show that under the assumptions of Theorem 2.1(i) there exists a positive bounded initial condition u_o, for which the solution of the parabolic problem (1.1) blows up in a finite time, and that any global solution of (1.1) with bounded initial condition is globally bounded. Since zero is a stable stationary solution, we may use Theorem 5.1 to show the existence of $\alpha \in (0,1)$ such that the solution with the initial condition αu_o tends to a positive stationary solution as time tends to infinity. However, this dynamical proof of the existence of stationary solution has (similarly as in the case (ii)) one disadvantage: we have to impose some additional assumptions on p and q (see Theorem 5.1).

Proof of Theorem 2.1(ii). Let $p = q$, $a < a_\Omega$, and suppose there exists a positive solution u of (2.2). Choose $\tilde{a} \in (a, a_\Omega)$. Then u is a supersolution for the operator $F_{\tilde{a}}$, 0 is a solution of $F_{\tilde{a}}(v) = 0$ and neither u nor 0 is a minimizer of $\Phi = \Phi_{\tilde{a}}$ with respect to $C = \{v \in X\,;\, 0 \le v \le u\}$. By Lemma 2.4 the equation $F_{\tilde{a}}(v) = 0$ has a solution $\tilde{u} \in C$, which is a local minimizer of $\Phi_{\tilde{a}}$. However, this a contradiction with the estimate (2.12). □

Proof of Theorem 2.1(iii). Let $p > q$, $a > 0$. If there exists a positive solution u of (2.2) and $\tilde{a} > a$, then similarly as in the proof of Theorem 2.1(ii) we get a positive solution $\tilde{u}$ of $F_{\tilde{a}}(v) = 0$, which is a local minimizer of $\Phi_{\tilde{a}}$ and fulfils $0 < \tilde{u} < u$ in $\overline{\Omega}$. Hence to prove the assertion (iii), it is sufficient to prove the existence of a positive solution for some $a > 0$.

Choose $\tilde{p} \in (1, q)$, $\tilde{a} > 0$ and let $\tilde{u}$ be a positive solution of (2.1) with p and a replaced by $\tilde{p}$ and $\tilde{a}$, respectively (its existence follows from Theorem 2.1(i)). It is easily seen that $\tilde{u}$ is a supersolution for our problem if a is sufficiently large, since then $a\tilde{u}^p > \tilde{a}\tilde{u}^{\tilde{p}}$. Hence Lemma 2.4 yields the assertion. □

Proof of Theorem 2.1(iv). Choose $b > 0$ and put

$$\Lambda_b(u) = \mathcal{I}(u) + b\mathcal{P}(u), \qquad M = \{u \in X\,;\, \mathcal{Q}(u) = 1\}.$$

Due to the compactness of the trace operator $\mathrm{Tr} : X \to L^{q+1}(\partial\Omega)$, the set M is weakly closed. The C^1 functional $\Lambda_b : X \to \mathbb{R}$ is convex and coercive, hence there exists $u_b \in M$ such that $\Lambda_b(u_b) = \inf_{u \in M} \Lambda_b(u)$. We may suppose $0 \not\equiv u_b \ge 0$ (otherwise we put $\tilde{u}_b = |u_b|$). The minimizer u_b fulfils the equation

$$\Lambda_b'(u_b) = \nu_b \mathcal{Q}'(u_b),$$

where

$$\nu_b = \frac{\langle \Lambda_b'(u_b), u_b\rangle}{\langle \mathcal{Q}'(u_b), u_b\rangle} = \frac{2\mathcal{I}(u_b) + b(p+1)\mathcal{P}(u_b)}{q+1} > 0 \tag{2.15}$$

is the corresponding Lagrange multiplier. Putting $t_b = \nu_b^{1/(q-1)}$ and $u = t_b u_b$ one can easily show that u is a positive solution of (2.2) with

$$a = \frac{b}{t_b^{p-1}} = b\nu_b^{-(p-1)/(q-1)} =: f(b), \tag{2.16}$$

where the function f depends not only on b but also on u_b.

It is easily seen that the function $g : b \mapsto \Lambda_b(u_b)$ is continuous (and does not depend on u_b, of course). Moreover, (2.15) implies

$$2g(b) \le (q+1)\nu_b \le (p+1)g(b), \tag{2.17}$$

so that (2.16) yields the estimate

$$\Big(\frac{q+1}{2}\Big)^{(p-1)/(q-1)} h(b) > f(b) > \Big(\frac{q+1}{p+1}\Big)^{(p-1)/(q-1)} h(b), \tag{2.18}$$

where $h(b) := b\big(g(b)\big)^{-(p-1)/(q-1)}$ is continuous. Now (2.18) and the continuity of h will imply our assertion if we show $\lim_{b\to+\infty} f(b) = +\infty$ and $\|u\| = \|t_b u_b\| \to \infty$ for the corresponding solutions, since the solutions that we found in the proof of (iii) were bounded (in L^∞ and, consequently, in X). Hence, suppose $b \to +\infty$. If we put $v_b(x) = d \max\{0, 1 - \sqrt{b}\,\mathrm{dist}(x, \partial\Omega)\}$, where $d = \Big(\frac{q+1}{|\partial\Omega|}\Big)^{1/(q+1)}$, we have $\mathcal{Q}(v_b) = 1$, hence

$$g(b) \le \Lambda_b(v_b) \le c\sqrt{b}, \tag{2.19}$$

where c is some constant independent of b. This implies

$$h(b) \ge b(c\sqrt{b})^{-(p-1)/(q-1)} = \tilde{c}\, b^{(2q-p-1)/(2q-2)} \to \infty,$$

hence by (2.18) also $f(b) \to \infty$. Now (2.15), (2.17) and (2.19) imply $\mathcal{P}(u_b) \le c/\sqrt{b}$, so that $u_b \to 0$ in $L^2(\Omega)$. Now choose $\xi < 1$ such that the trace operator $\mathrm{Tr} : W^{\xi,2}(\Omega) \to L^{q+1}(\partial\Omega)$ is continuous. Using an interpolation inequality we obtain

$$1 = \mathcal{Q}(u_b) \le c\|u_b\|_{\xi,2}^{q+1} \le c\|u_b\|^{\xi(q+1)} \|u_b\|_2^{(1-\xi)(q+1)}, \tag{2.20}$$

where $\|\cdot\|_{\xi,2}$ and $\|\cdot\|_2$ is the norm in $W^{\xi,2}(\Omega)$ and $L^2(\Omega)$, respectively. Since $\|u_b\|_2 \to 0$, (2.20) implies $\|u_b\| \to \infty$. $\square$

Remark 2.4. If we could choose u_b such that $f(b)$ became continuous, then this would imply in the case of Theorem 2.1(iv) the existence of two positive solutions for any a large. If one could prove Palais-Smale condition in this case, this would also lead to the proof of two positive solutions for a large. Another way how to prove this existence is to prove corresponding apriori estimates and to use the degree theory – this will be done for the symmetric solutions on the ball.

In the proof of Theorem 2.1(v) we will need the following lemma from [**FK**].

Lemma 2.7. *Let q, q^* be as in Theorem* 2.1(v)*, let $\varepsilon > 0$ and $r > q^*$. Then there exists a constant $c = c(\varepsilon, r)$ such that*

$$\int_{\partial\Omega} |u|^{q+1} dS \le \varepsilon\|u\|^2 + c\Big(\int_\Omega |u|^{q+1} dx\Big)^r \tag{2.21}$$

for any $u \in X$.

Proof. Proof is based on the continuity of the trace operator $\mathrm{Tr} : W^{\theta z, q+1}(\Omega) \to L^{q+1}(\partial\Omega)$, on an interpolation inequality and the continuity of the imbedding $X \hookrightarrow W^{z,q+1}(\Omega)$ for suitable $z, \theta \in (0,1)$. A detailed proof can be found in [**FK**].□

Proof of Theorem 2.1(v). Let $a > 0$ be fixed. Our assumptions imply $p + 1 > (q+1)r$ for suitable $r > q^*$. Choosing $\varepsilon > 0$ and using Lemma 2.7 and Hölder inequality we obtain for any $u \in X$ (and suitable $c > 0$ varying from step to step)

$$\begin{aligned}\langle Q(u), u\rangle &= \int_{\partial\Omega} |u|^{q+1} dS \\ &\le \varepsilon \int_\Omega |\nabla u|^2 dx + c\Big(\int_\Omega |u|^{p+1} dx\Big)^{\frac{2}{p+1}} + c\Big(\int_\Omega |u|^{p+1} dx\Big)^{\frac{r(q+1)}{p+1}} \\ &\le \varepsilon \int_\Omega |\nabla u|^2 dx + \varepsilon a \int_\Omega |u|^{p+1} dx + c \\ &= \varepsilon\langle u - Ku + aP(u), u\rangle + c\end{aligned}$$

which implies a uniform apriori bound for the solutions $t \in [0,1]$, $u \in C^+$ of the inequality

$$\langle u - Ku + aP(u) - tQ(u), v - u\rangle \ge 0 \qquad \forall v \in C^+. \tag{2.22}$$

Consequently, denoting $H_t(u) = u - P^+\big(Ku - aP(u) + tQ(u)\big)$ we get

$$\deg(F^+, 0, B_c(0)) = \deg(H_1, 0, B_c(0)) = \deg(H_0, 0, B_c(0)) = 1, \tag{2.23}$$

where the last equality follows from [**Q2,** Corollary 1], since the functional $\Lambda_a(u) = \mathcal{I}(u) + a\mathcal{P}(u)$ corresponding to H_0 is coercive. On the other hand, Lemma 2.6 yields

$$\deg(F^+, 0, B_\varepsilon(0)) = d^+(0) = 0. \tag{2.24}$$

The existence of a positive solution follows from (2.23), (2.24) and Lemma 2.5. □

Remarks 2.5.

(i) According to the results for Ω being a ball, the condition on p, q in Theorem 2.1(v) does not seem to be optimal. In fact, a finer apriori estimate can lead to weaker assumptions. Suppose e.g. that p, q fulfil the following assumptions: $q < \dfrac{N+1}{N-1}$, $p \ge q + 1$ and $p + 1 + \dfrac{p-1}{p+1} > (q+1)q^*$ (so that p, q need not fulfil the condition from Theorem 2.1(v)). We show that this condition is also sufficient for the apriori bound and, consequently, also for the existence.

Let u be a solution of (2.22), i.e. it solves the problem $\triangle u = au^p$, $\dfrac{\partial u}{\partial n} = tu^q$. Choosing a test function $\varphi_d(x) = \min\{1, \frac{1}{d}\operatorname{dist}(x, \partial\Omega)\}$ for $d > 0$ small and putting $\Omega_d = \{x \in \Omega\,;\, \varphi_d(x) = 1\}$ we get

$$a \int_{\Omega_d} u^p dx \le a \int_\Omega u^p \varphi\, dx = -\int_\Omega \nabla u \nabla\varphi\, dx \le \|u\|\,\|\varphi\| \le \frac{c}{\sqrt{d}}\|u\| \tag{2.25}$$

and using Hölder inequality we obtain

$$\int_{\Omega\setminus\Omega_d} u^p dx \le c\Big(\int_\Omega u^{p+1}dx\Big)^{\frac{p}{p+1}} d^{\frac{1}{p+1}}. \tag{2.26}$$

Choosing $d = \big(\int_\Omega u^{p+1}dx\big)^{-\nu}$, where $\nu = (p-1)/(p+3)$, and using (2.25) and (2.26) in Lemma 2.7 we get the desired apriori estimate for u.
Similar improvements can be made also for $p < q+1$.

(ii) In order to prove Theorem 2.1(v) one can use also the function $f(b)$ introduced in the proof of Theorem 2.1(iv) and show $\liminf\limits_{b\to\infty} f(b) = 0$. However, this leads to estimates which are close to those already used in the proof of Theorem 2.1(v).

(iii) The investigation of the function $f(b)$ gives an information for the existence of solutions also in other cases; however, in these cases other methods turned out to be more powerfull. Nevertheless, the likely behaviour of f (indicated in the figures below) gives us a good insight on the stationary solutions. To support the figures below, let us only mention that it is easy to show that $f(b) \to \infty$ if $p > q$, $b \to 0$, or if $p < q$, $b \to \infty$. In both cases one can use a simple estimate $\nu_b \le cb$.

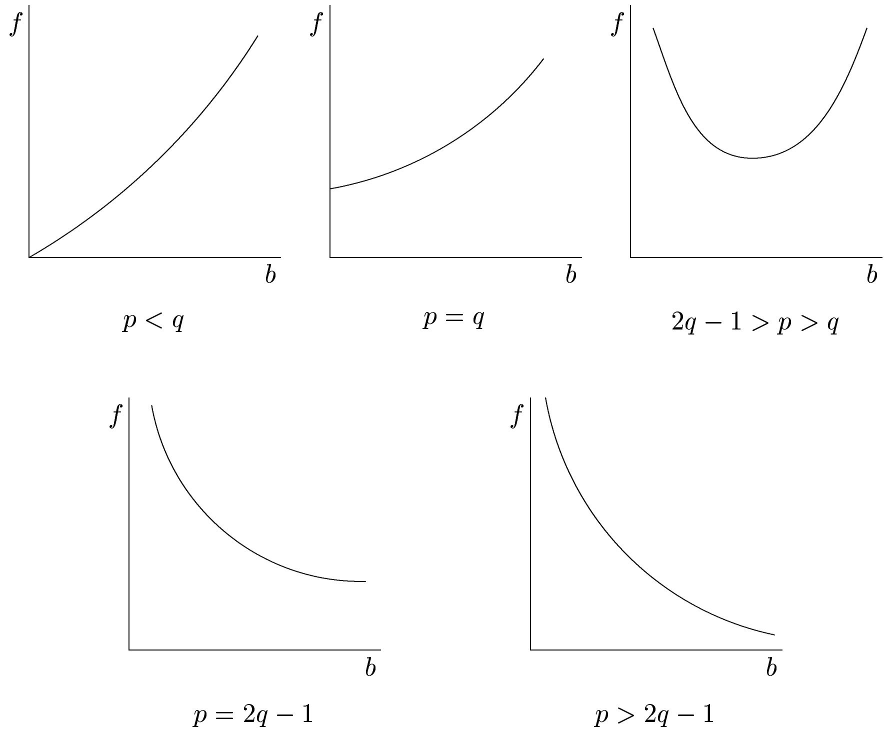


Figure 7. The graphs of f.

Proof of Theorem 2.2(i). Let $\Omega = B_R(0)$. The existence of a positive symmetric solution to (2.1) follows by the same way as in Theorem 2.1(i); we have only to restrict ourselves to the space X_s of all radially symmetric functions in $X = W^{1,2}(\Omega)$. Hence it suffices to prove the uniqueness. Denote $r = |x|$. Any positive symmetric solution of (2.1) fulfils the O.D.E.

$$u_{rr} + \frac{N-1}{r}u_r = au^p, \qquad r \in (0,R) \tag{2.27}$$

together with the boundary conditions

$$u_r(0) = 0, \quad u_r(R) = u^q(R). \tag{2.28}$$

If u_1, u_2 are two different positive symmetric solutions, then the uniqueness of the solution of the initial problem for (2.27) implies $u_1(0) \neq u_2(0)$. Hence we may suppose $u_1(0) < u_2(0)$. Since $w := u_2 - u_1$ fulfils

$$w_{rr} + \frac{N-1}{r}w_r = a(u_2^p - u_1^p), \qquad w_r(0) = 0, \quad w(0) > 0,$$

it is easily seen that $w(r) > 0$ for any $r \in [0,R]$, so that $u_2 > u_1$ in $\overline{\Omega}$. By (2.12) neither u_1 nor u_2 is a local minimum of Φ with respect to $C := \{u\,;\, u_1 \leq u \leq u_2\}$, hence Lemma 2.4 implies the existence of a local minimizer of Φ between u_1 and u_2, which contradicts (2.12). □

Proof of Theorem 2.2(ii). Let $\Omega = B_R(0)$. Considering only the space X_s of symmetric functions we get similarly as in the proof of Theorem 2.1(iii) the existence of $a_o^s \geq 0$ such that the problem (2.1) has a stable symmetric positive solution if $a > a_o^s$ and (2.1) does not have symmetric positive solution if $a < a_o^s$.

To show the rest of the assertion we need some apriori estimates for symmetric positive solutions. Hence suppose that u is such solution. Multiplying (2.27) by u_r and integrating resulting equation over $(0,R)$ we get using (2.28)

$$\begin{aligned}\frac{1}{2}u^{2q}(R) = \frac{1}{2}u_r^2(R) &\leq \frac{1}{2}u_r^2(R) + \int_0^R \frac{N-1}{r}u_r^2(r)\,dr \\ &= \frac{a}{p+1}\big(u(R)^{p+1} - u(0)^{p+1}\big) < \frac{a}{p+1}u(R)^{p+1}\end{aligned}$$

which implies

$$u(R)^{2q-p-1} < \frac{2a}{p+1}. \tag{2.29}$$

Moreover, (2.27) implies $u_{rr} > 0$ whenever $u_r \leq 0$, hence $u_r \geq 0$ and (2.29) yields an apriori bound for u, which is independent of $a \in [0,A]$ for any $A < \infty$ fixed.

Denoting by d^{s+} the local degree corresponding to F^+/X_s and using apriori estimates (2.29) we obtain for $R > 0$ sufficiently large

$$\deg\big(F_a^+/X_s, 0, B_R(0)\big) = \deg\big(F_0^+/X_s, 0, B_R(0)\big) = d_0^{s+}(0) = 0 = d_a^{s+}(0), \tag{2.30}$$

where the last two equalities follow analogously as the corresponding equality in Lemma 2.6(ii).

Now if $a > a_o^s$, then we have a positive symmetric solution u_1 which is a local minimizer of Φ in X_s (cf. the proof of Theorem 2.1(iii)), hence [**Q2**] implies $d^{s+}(u_1) = 1$. If this were the only positive symmetric solution, (2.30) would imply

$$0 = \deg\bigl(F_a^+/X_s, 0, B_R(0)\bigr) = d_a^{s+}(u_1) + d_a^{s+}(0) = 1,$$

a contradiction. Hence there exist at least two symmetric positive solutions for $a > a_o^s$.

Now we show the existence of a positive symmetric solution for $a = a_o^s$ and this will also imply $a_o^s > 0$, since the equation $F_0(u) = 0$ does not have positive solutions. Thus let u_n be positive symmetric solutions of (2.1) with $a = a_n \downarrow a_o^s$. Then

$$u_n = Ku_n - a_n P(u_n) + Q(u_n) \tag{2.31}$$

and the boundedness of u_n implies that we may suppose $u_n \rightharpoonup u$ (weak convergence). Now (2.31) implies

$$u_n \to u = Ku - a_o^s P(u) + Q(u),$$

hence u is a nonnegative symmetric solution for $a = a_o^s$. It is now sufficient to notice that $u \neq 0$ by Lemma 2.2. □

Proof of Theorem 2.2(iii), (iv). If $p > 2q - 1$ or $p = 2q - 1$ and $a > q$, then the proof of Theorem 4.1 yields a positive symmetric supersolution to our problem, hence the existence follows from Lemma 2.4 (used for the space X_s). If $p = 2q - 1$, $a \le q$ and u were a positive symmetric solution, then (2.29) yields a simple contradiction. □

Remark 2.6. If $p > 1$ or $q > 1$ is not subcritical, then one can still expect similar results as in Theorems 2.1, 2.2. More precisely,

(i) *if $p > q$, then there exists $a_o \in [0, \infty)$ such that* (2.1) *has a classical positive solution for $a > a_o$ and* (2.1) *does not have classical positive solutions for $0 < a < a_o$. If Ω is a ball and $p > 2q - 1$, then $a_o = 0$.*

(ii) *If Ω is a ball, $p \le q$ and $a > a_o$ (where a_o is defined in Theorem* 2.1(i)*), then* (2.1) *has a classical positive symmetric solution. If Ω is a ball and $q < p < 2q - 1$, then the conclusions of Theorem* 2.2(ii) *are true.*

Proof. (i) Let $p > q > 1$, let u be a classical positive solution of (2.1) and let $\tilde{a} > a$. Then u is a supersolution of (2.1) in which a is replaced by $\tilde{a}$ and the nonlinearities v^p and v^q are suitably modified for $v > \max u$ (so that the corresponding functional is well defined and differentiable). An obvious modification of Lemma 2.4 implies now the existence of a solution $\tilde{u}$ for the problem (2.1) with a replaced by $\tilde{a}$. Hence the existence of $a_o \in [0, \infty]$ follows.

To see that $a_o < \infty$, choose subcritical $\tilde{p}, \tilde{q} > 1$ such that $\tilde{q} < \min(\tilde{p}, q)$. If $\tilde{a} > 0$ is large enough, we have a positive solution $\tilde{u}$ of (2.1) with p, q and a replaced by $\tilde{p}, \tilde{q}$ and $\tilde{a}$, respectively. The proof of Theorem 2.1(iii) shows that we may suppose $0 < \tilde{u} < 1$ in $\overline{\Omega}$, hence $\tilde{u}^{\tilde{q}} > \tilde{u}^q$. Moreover, choosing $a > 0$ large enough we have $a\tilde{u}^p > \tilde{a}\tilde{u}^{\tilde{p}}$, so that $\tilde{u}$ is a supersolution for the problem (2.1) (with the nonlinearities v^p, v^q modified for $v > 1$), which implies the existence of a solution for a large.

If Ω is a ball and $p > 2q - 1$, we may use the supersolution from Theorem 4.1.

(ii) Replacing the nonlinearities u^p and u^q by $m(u) = u \min(u, C)^{p-1}$ and $n(u) = u^{1+\varepsilon} \min(u, C)^{q-1-\varepsilon}$, respectively (where $\varepsilon > 0$ is small and $C > 0$ is large) we obtain similarly as in (2.29) the following apriori bound for the positive symmetric solutions of the modified problem:

$$\frac{n^2\big(u(R)\big)}{M\big(u(R)\big)} < 2a, \tag{2.32}$$

where $M(u) = \int_o^u m(v)\, dv$. If $u(R) > C$, then (2.32) yields

$$2a > \frac{u(R)^{2+2\varepsilon} C^{2q-2-2\varepsilon}}{\frac{u(R)^2 - C^2}{2} C^{p-1} + \frac{C^{p+1}}{p+1}} > \frac{u(R)^{2+2\varepsilon} C^{2q-2-2\varepsilon}}{u(R)^2 C^{p-1}} > C^{2q-p-1},$$

which is a contradiction for C large. Consequently, any positive symmetric solution of the modified problem is a solution of our original problem for C large enough.

The existence of a positive symmetric solution for the modified problem for $p \le q$ and $a > a_o$ follows from the mountain pass theorem similarly as in Theorems 2.1(i), 2.2(i) or from the degree theory (see Remark 2.3(ii)). The existence of a_o^s (as in Theorem 2.2(ii)) for $q < p < 2q - 1$ follows from an obvious modification of the proof of Theorem 2.2(ii). $\square$

Finally let us note, that if Ω is a general domain in $\mathbb{R}^N$ and $p \le q$, then one can easily show that (2.14) is true also for supercritical p, q.

3. Stationary solutions for $N = 1$

Consider the O.D.E.

$$u_{xx} = au^p \qquad \text{for } \; x > 0, \tag{3.1}$$

with the initial conditions

$$u(0) = m > 0, \quad u_x(0) = 0. \tag{3.2}$$

We are looking for $L > 0$ such that

$$u_x(L) = u^q(L). \tag{3.3}$$

This will provide a symmetric solution to

$$\tag{3.4} \begin{cases} u_{xx} = au^p & \text{on } (-l,l), \\ \dfrac{\partial u}{\partial n} = u^q & \text{at } -l,l, \end{cases}$$

with $l = L$. If for given m there are two values L_1, L_2 such that (3.3) is satisfied, then by shift and reflection we obtain a pair of nonsymmetric solutions u_1, u_2 to the problem (3.4) with $l = (L_1 + L_2)/2$, $u_1(x) = u_2(-x)$.

Multiplying (3.1) by u_x and integrating we see that

$$\tag{3.4a} \frac{1}{2}u_x^2 - \frac{a}{p+1}u^{p+1} = \text{ const } = -\frac{a}{p+1}m^{p+1}.$$

Note that $u_{xx} \geq 0$, hence u_x is nondecreasing and since $u_x(0) = 0$ we have that $u_x \geq 0$. Therefore

$$\tag{3.4b} u_x = \sqrt{\frac{2a}{p+1}}\sqrt{u^{p+1} - m^{p+1}}$$

and integrating this equation we obtain

$$\tag{3.5} \int_m^{u(x)} \frac{dv}{\sqrt{v^{p+1} - m^{p+1}}} = \sqrt{\frac{2a}{p+1}}\,x.$$

For m given, the solvability of (3.1)–(3.3) is equivalent to finding L such that

$$\int_m^{u(L)} \frac{dv}{\sqrt{v^{p+1} - m^{p+1}}} = \sqrt{\frac{2a}{p+1}}L,$$
$$u^q(L) = \sqrt{\frac{2a}{p+1}}\sqrt{u^{p+1}(L) - m^{p+1}}.$$

The last equation may be written in the form

$$\frac{p+1}{2a}u^{2q}(L) - u^{p+1}(L) + m^{p+1} = 0.$$

If we now denote by $R(m)$ a root of the equation

$$\tag{3.6} \frac{p+1}{2a}x^{2q} - x^{p+1} + m^{p+1} = 0$$

and assume that $R(m) > m$, then (3.5) gives us a solution to (3.4) on the interval $\bigl(-L(m), L(m)\bigr)$ with

$$L(m) = \sqrt{\frac{p+1}{2a}}\int_m^{R(m)} \frac{dv}{\sqrt{v^{p+1} - m^{p+1}}}.$$

Setting $V = \dfrac{v}{m}$ we get

$$L(m) = \sqrt{\frac{p+1}{2a}}\, m^{-(p-1)/2} \int_1^{\frac{R(m)}{m}} \frac{dV}{\sqrt{V^{p+1}-1}}. \tag{3.7}$$

Theorem 3.1. *Assume that $p > 2q-1$. Then for any l the problem (3.4) has a unique nontrivial solution. This solution is symmetric.*

Proof. Consider the function

$$\mathcal{F}(x) = \frac{p+1}{2a} x^{2q} - x^{p+1} + m^{p+1}. \tag{3.8}$$

One has

$$\mathcal{F}'(x) = (p+1)\Big(\frac{q}{a} x^{2q-1} - x^p\Big).$$

Hence $\mathcal{F}'$ vanishes only for

$$x = \Big(\frac{a}{q}\Big)^{1/(2q-p-1)}. \tag{3.9}$$

Thus $\mathcal{F}$ is increasing up to this value and decreasing next. Hence (3.6) has only one root

$$R(m) \geq \Big(\frac{a}{q}\Big)^{1/(2q-p-1)},$$

in particular

$$\lim_{m\to 0} \frac{R(m)}{m} = +\infty. \tag{3.10}$$

Since

$$0 < \int_1^{+\infty} \frac{dV}{\sqrt{V^{p+1}-1}} < +\infty, \tag{3.11}$$

we deduce from (3.7), (3.10) that

$$\lim_{m\to 0} L(m) = +\infty. \tag{3.12}$$

Combining (3.7), (3.11) we have also

$$\lim_{m\to \infty} L(m) = 0 \tag{3.13}$$

and the range of L is $(0,+\infty)$. Now we show that L is a decreasing function. Indeed, from (3.7) we have

$$
\begin{aligned}
L'(m) = &- \frac{p-1}{2}\sqrt{\frac{p+1}{2a}}m^{-(p+1)/2}\int_1^{\frac{R(m)}{m}} \frac{dV}{\sqrt{V^{p+1}-1}} \\
&+ \sqrt{\frac{p+1}{2a}}m^{-(p-1)/2}\frac{1}{\sqrt{\left(\frac{R(m)}{m}\right)^{p+1}-1}}\left(\frac{R(m)}{m}\right)'.
\end{aligned} \tag{3.14}
$$

But since $R(m)$ is the only root to (3.6), it follows from the implicit function theorem that R is differentiable and by differentiation one gets

$$
R'(m) = \frac{m^p}{R(m)^p - \frac{q}{a}R(m)^{2q-1}}. \tag{3.15}
$$

It follows that

$$
\begin{aligned}
\left(\frac{R(m)}{m}\right)' &= -\frac{1}{m^2}R(m) + \frac{1}{m}R'(m) = -\frac{1}{m^2}\left(R(m) + \frac{m^{p+1}}{\frac{q}{a}R(m)^{2q-1} - R(m)^p}\right) \\
&= -\frac{1}{m^2}\left(\frac{q}{a}R(m)^{2q-1} - R(m)^p\right)^{-1}\left(\frac{q}{a}R(m)^{2q} - R(m)^{p+1} + m^{p+1}\right) \\
&< -\frac{1}{m^2}\left(\frac{q}{a}R(m)^{2q-1} - R(m)^p\right)^{-1}\left(\frac{p+1}{2a}R(m)^{2q} - R(m)^{p+1} + m^{p+1}\right) \\
&= 0,
\end{aligned}
$$

the last inequality follows from the fact, that

$$
\frac{q}{a}R(m)^{2q-1} - R(m)^p = \mathcal{F}'\big(R(m)\big) < 0.
$$

Recalling (3.14) we obtain that

$$
L'(m) < 0. \tag{3.16}
$$

(3.12), (3.13) and (3.16) yield the assertion. □

Theorem 3.2. *Assume that $p = 2q - 1$.*

(i) *If $a \le q$ then the problem (3.4) cannot have nontrivial solutions.*
(ii) *If $a > q$ then for any l the problem (3.4) has a unique nontrivial solution. This solution is symmetric.*

Proof. (i) The boundary value $u(L)$ must be a solution to (3.6). But (3.6) reduces to

$$
m^{p+1} = x^{p+1}\left(1 - \frac{q}{a}\right) \le 0.
$$

(ii) In this case (3.6) has a unique root $R(m)$ which is given by the explicit formula

$$R(m) = m\Big(1 - \frac{q}{a}\Big)^{-\frac{1}{2q}}.$$

Hence $\Big(\dfrac{R(m)}{m}\Big)' = 0$ and it is easily seen from (3.14) that $L'(m) < 0$. (3.7) immediately yields (3.12) and (3.13). □

Next we turn to the case $p < 2q - 1$. Considering $\mathcal{F}$ given by (3.8) we see that $\mathcal{F}$ has an absolute minimum given by (3.9). So, in order for (3.6) to have a root we need

$$\mathcal{F}\Big(\Big(\frac{a}{q}\Big)^{\frac{1}{2q-p-1}}\Big) \le 0$$

which reads also

$$m \le c_{(a)} := a^{\frac{1}{2q-p-1}} c(p,q), \tag{3.17}$$

where

$$c(p,q)^{p+1} = \Big(\frac{1}{q}\Big)^{\frac{1}{2q-p-1}} \Big(\frac{2q-p-1}{2q}\Big)^{\frac{1}{p+1}}.$$

Then for m satisfying (3.17), the graph of $\mathcal{F}$ looks like

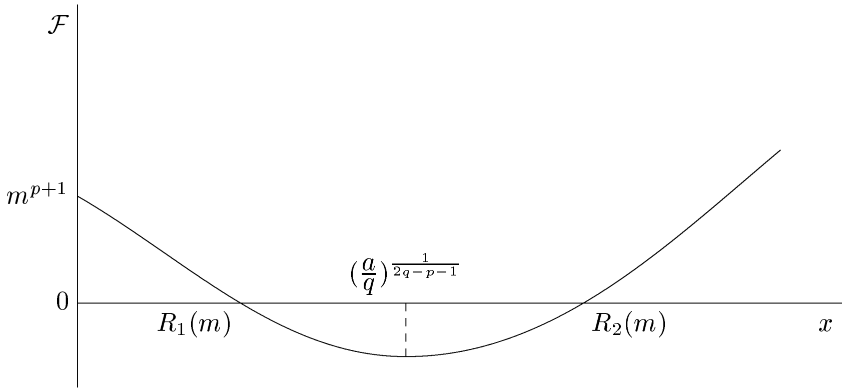


Figure 8. The graph of $\mathcal{F}$.

and (3.6) has two roots $R_1(m), R_2(m)$ which are equal to $\Big(\dfrac{a}{q}\Big)^{\frac{1}{2q-p-1}}$ when $m = c_{(a)}$. Note that if m satisfies (3.17) then

$$m \le \Big(\frac{a}{q}\Big)^{\frac{1}{2q-p-1}}.$$

Since $\mathcal{F}(m) \geq 0$, one has

$$m \leq R_1(m) \leq R_2(m).$$

Let us now study the two curves

$$L_i(m) = \sqrt{\frac{p+1}{2a}}\, m^{-(p-1)/2} \int_1^{\frac{R_i(m)}{m}} \frac{dV}{\sqrt{V^{p+1}-1}} \qquad i=1,2 \tag{3.18}$$

on the interval $(0, c_{(a)})$.

Lemma 3.1. *Assume that $p < 2q-1$. Then we have*

$$L_1(m) \leq L_2(m), \tag{3.19}$$

and $L_2(m)$ is decreasing for $m \in (0, c_{(a)})$. Moreover,

$$\lim_{m \to c_{(a)}} L_i(m) = \sqrt{\frac{p+1}{2a}}\, c_{(a)}^{-\frac{p-1}{2}} \int_1^{d(p,q)} \frac{dV}{\sqrt{V^{p+1}-1}} =: L_{(a)}, \tag{3.20}$$

where $d(p,q) = \dfrac{1}{c(p,q)} \Big(\dfrac{1}{q}\Big)^{\frac{1}{2q-p-1}}$.

Proof. (3.19) and (3.20) are obvious. In order to show that $L_2' < 0$ it is sufficient to prove that $\Big(\dfrac{R_2(m)}{m}\Big)' < 0$ (see (3.14)). From (3.15) we get

$$\Big(\frac{R_i(m)}{m}\Big)' = \frac{1}{m^2}\Big(\frac{\frac{q}{a}R_i(m)^{2q} - R_i(m)^{p+1} + m^{p+1}}{R_i(m)^p - \frac{q}{a}R_i(m)^{2q-1}}\Big).$$

According to (3.6), the last equality implies that

$$\Big(\frac{R_i(m)}{m}\Big)' = \frac{1}{am^2}\Big(q - \frac{p+1}{2}\Big)\frac{R_i(m)^{2q}}{R_i(m)^p - \frac{q}{a}R_i(m)^{2q-1}}. \tag{3.21}$$

Since $R_2(m) \geq \Big(\dfrac{a}{q}\Big)^{\frac{1}{2q-p-1}}$, $R_2(m)$ is in the region where

$$\frac{1}{p+1}\mathcal{F}'(x) = \frac{q}{a}x^{2q-1} - x^p > 0.$$

Hence, the right hand side of (3.21) is negative for $i = 2$. $\square$

Lemma 3.2. *Assume that $p \leq q$. Then $L_1(m)$ is increasing.*

Proof. To prove that $L_1' > 0$ means (see (3.14)) to prove that

$$m^{-\frac{p-1}{2}} \frac{\big(\frac{R_1(m)}{m}\big)'}{\sqrt{\big(\frac{R_1(m)}{m}\big)^{p+1} - 1}} > \frac{p-1}{2} m^{-\frac{p+1}{2}} \int_1^{\frac{R_1(m)}{m}} \frac{dV}{\sqrt{V^{p+1}-1}}.$$

The last inequality is equivalent to the following one:
(3.22)

$$\Psi_L(m) := m\Big(\frac{R_1(m)}{m}\Big)' > \frac{p-1}{2}\sqrt{\Big(\frac{R_1(m)}{m}\Big)^{p+1}-1}\int_1^{\frac{R_1(m)}{m}}\frac{dV}{\sqrt{V^{p+1}-1}} =: \Psi_R(m).$$

Using (3.21) we get

$$\Psi_L(m) = \frac{1}{a}\frac{R_1(m)}{m}R_1(m)^{2q-p-1}\Big(q-\frac{p+1}{2}\Big)\frac{1}{1-\frac{q}{a}R_1(m)^{2q-p-1}}.$$

But

$$0 < 1-\frac{q}{a}R_1(m)^{2q-p-1} < 1-\frac{p+1}{2a}R_1(m)^{2q-p-1} = \Big(\frac{R_1(m)}{m}\Big)^{-(p+1)},$$

the last equality follows from (3.6). Hence

$$\Psi_L(m) > \frac{1}{a}\Big(q-\frac{p+1}{2}\Big)\Big(\frac{R_1(m)}{m}\Big)^{p+2}R_1(m)^{2q-p-1}. \tag{3.23}$$

On the other hand,

$$\begin{aligned}\Psi_R(m) &< \frac{p-1}{2}\sqrt{\Big(\frac{R_1(m)}{m}\Big)^{p+1}-1}\int_1^{\frac{R_1(m)}{m}}\frac{V^p dV}{\sqrt{V^{p+1}-1}}\\ &= \frac{p-1}{p+1}\Big(\Big(\frac{R_1(m)}{m}\Big)^{p+1}-1\Big).\end{aligned}$$

According to (3.6) we have

$$\frac{p-1}{p+1}\Big(\Big(\frac{R_1(m)}{m}\Big)^{p+1}-1\Big) = \frac{p-1}{2a}\Big(\frac{R_1(m)}{m}\Big)^{p+1}R_1(m)^{2q-p-1}.$$

Our assumption on p, q implies now that

$$\Psi_R(m) < \frac{1}{a}\Big(q-\frac{p+1}{2}\Big)\Big(\frac{R_1(m)}{m}\Big)^{p+1}R_1(m)^{2q-p-1}.$$

Recalling the inequality $R_1(m) \ge m$, we obtain

$$\Psi_R(m) < \frac{1}{a}\Big(q-\frac{p+1}{2}\Big)\Big(\frac{R_1(m)}{m}\Big)^{p+2}R_1(m)^{2q-p-1}. \tag{3.24}$$

(3.23) and (3.24) yield (3.22). □

Lemma 3.3. *Assume that $p < 2q-1$. Then*

$$\lim_{m\to 0} L_2(m) = +\infty, \tag{3.25}$$

$$\lim_{m\to 0}\frac{aL_1(m)}{m^{q-p}} = 1. \tag{3.26}$$

Proof. First remark that if R is a limit point (as $m \to 0$) of $R_1(m)$ or $R_2(m)$ one must have (see (3.6))

$$\frac{p+1}{2a} R^{2q} - R^{p+1} = 0,$$

which means that $R = 0$ or $R = \Big(\frac{2a}{p+1}\Big)^{\frac{1}{2q-p-1}}$. Since

$$R_1(m) \le \Big(\frac{a}{q}\Big)^{\frac{1}{2q-p-1}} < \Big(\frac{2a}{p+1}\Big)^{\frac{1}{2q-p-1}}$$

and

$$\Big(\frac{a}{q}\Big)^{\frac{1}{2q-p-1}} < R_2(m),$$

the only limit point of $R_2(m)$ is $\Big(\frac{2a}{p+1}\Big)^{\frac{1}{2q-p-1}}$ and the only limit point of $R_1(m)$ is 0. Thus we have

$$\lim_{m\to 0} \frac{R_2(m)}{m} = +\infty.$$

One concludes like in (3.12) that (3.25) holds.

Since $R_1(m) \to 0$ and (3.6) implies that

$$m^{p+1} = R_1(m)^{p+1}\Big(1 - \frac{p+1}{2a} R_1(m)^{2q-p-1}\Big),$$

we have

$$\lim_{m\to 0} \frac{R_1(m)}{m} = 1. \tag{3.27}$$

In the sequel it will be convenient for us to use the following notation: "$f(x) \sim g(x)$ when $x \to x_o$" means that $\lim_{x\to x_o} \frac{f(x)}{g(x)} = 1$. When $h \to 0+$, we have

$$\int_1^{1+h} \frac{dV}{\sqrt{V^{p+1}-1}} = \int_0^h \frac{dV}{\sqrt{(V+1)^{p+1}-1}} \sim \int_0^h \frac{dV}{\sqrt{(p+1)V}} = \frac{2}{\sqrt{p+1}}\sqrt{h}.$$

Using (3.18), (3.27) it follows that

$$L_1(m) \sim \sqrt{\frac{p+1}{2a}}\, m^{-\frac{p-1}{2}} \frac{2}{\sqrt{p+1}} \sqrt{\frac{R_1(m)}{m} - 1} = \sqrt{\frac{2}{am^p}\big(R_1(m) - m\big)}. \tag{3.28}$$

From (3.6) we deduce

$$\frac{p+1}{2a} R_1(m)^{2q-p-1} = 1 - \Big(\frac{m}{R_1(m)}\Big)^{p+1} \sim \Big(1 - \frac{m}{R_1(m)}\Big)(p+1),$$

hence

$$R_1(m) - m \sim \frac{1}{2a} R_1(m)^{2q-p} \sim \frac{1}{2a} m^{2q-p}.$$

Going back to (3.28) we get

$$L_1(m) \sim \sqrt{\frac{2}{a}} m^{-\frac{p}{2}} \frac{1}{\sqrt{2a}} m^{q-\frac{p}{2}} = \frac{1}{a} m^{q-p}$$

and (3.26) is shown. □

Lemma 3.4. *Assume that $p < 2q-1$. Then*

$$\lim_{m \to c_{(a)}} L_1'(m) = +\infty,$$
$$\lim_{m \to c_{(a)}} L_2'(m) = -\infty.$$

Proof. From (3.14), (3.7) we have

$$L_i'(m) = -\frac{p-1}{2m} L_i(m) + G_i(m), \tag{3.29}$$

where

$$G_i(m) := \sqrt{\frac{p+1}{2a}} m^{-\frac{p-1}{2}} \Big(\Big(\frac{R_i(m)}{m} \Big)^{p+1} - 1 \Big)^{-\frac{1}{2}} \Big(\frac{R_i(m)}{m} \Big)'.$$

According to (3.6) we get

$$\Big(\Big(\frac{R_i(m)}{m} \Big)^{p+1} - 1 \Big)^{-\frac{1}{2}} = \sqrt{\frac{2a}{p+1}} R_i(m)^{-q} m^{\frac{p+1}{2}}. \tag{3.30}$$

(3.21) and (3.30) imply that

$$G_i(m) = \frac{1}{am} \Big(q - \frac{p+1}{2} \Big) R_i(m)^{q-p} \Big(1 - \frac{q}{a} R_i(m)^{2q-p-1} \Big)^{-1}. \tag{3.31}$$

The first term on the right hand side of (3.29) tends to a finite limit as $m \to c_{(a)}$ (see (3.20)), while

$$G_1(m) \to +\infty, \quad G_2(m) \to -\infty \qquad \text{as } \; m \to c_{(a)}$$

since

$$R_1(m) \le \Big(\frac{a}{q} \Big)^{\frac{1}{2q-p-1}} \le R_2(m),$$
$$R_i(m) \to \Big(\frac{a}{q} \Big)^{\frac{1}{2q-p-1}} \qquad \text{as } \; m \to c_{(a)}.$$

□

Lemma 3.5. *Assume that $q < p < 2q-1$. Then L_1 has a unique minimum in $(0, c_{(a)})$.*

Proof. It suffices to prove that $L_1''(m) > 0$ at any point m where $L_1'(m) = 0$.
We first rewrite (3.14) in the following form:

$$\sqrt{\frac{2a}{p+1}}m^{\frac{p+1}{2}}L_1'(m) = -\frac{p-1}{2}I(m) + J(m), \tag{3.32}$$

where

$$I(m) := \int_1^{\varrho} \frac{dV}{\sqrt{V^{p+1}-1}}, \qquad J(m) := \frac{m\varrho'}{\sqrt{\varrho^{p+1}-1}}, \qquad \varrho := \frac{R_1(m)}{m}.$$

Differentiating (3.32) and multiplying the result by m, we get

$$\sqrt{\frac{2a}{p+1}}m^{\frac{p+3}{2}}L_1''(m) = \frac{p^2-1}{4}I(m) + J(m)\Big(1-p+\frac{m\varrho''}{\varrho'} - \frac{(p+1)\varrho^p m\varrho'}{2(\varrho^{p+1}-1)}\Big). \tag{3.33}$$

Let us now compute $m\varrho'$, $\dfrac{m\varrho''}{\varrho'}$. From (3.6) we obtain that

$$R_1(m)^{2q-p-1} = \frac{2a}{p+1}\frac{\varrho^{p+1}-1}{\varrho^{p+1}}.$$

(3.15) and the last equality yield

$$\begin{aligned} m\varrho' &= R_1'(m) - \frac{R_1(m)}{m} = \frac{1}{\varrho^p}\frac{1}{1-\frac{q}{a}R_1(m)^{2q-p-1}} - \varrho \\ &= \frac{\varrho(\varrho^{p+1}-1)}{k-\varrho^{p+1}}, \qquad k := \frac{2q}{2q-p-1}. \end{aligned} \tag{3.34}$$

Further $(m\varrho')' = \varrho' + m\varrho''$, hence

$$\frac{m\varrho''}{\varrho'} = -1 + \frac{(p+2)\varrho^{p+1}-1}{k-\varrho^{p+1}} + \frac{(p+1)\varrho^{p+1}(\varrho^{p+1}-1)}{(k-\varrho^{p+1})^2}. \tag{3.35}$$

If $L_1'(m) = 0$, then

$$J(m) = \frac{p-1}{2}I(m). \tag{3.36}$$

Using (3.34)–(3.36) we obtain from (3.33) that

$$\sqrt{\frac{2a}{p+1}}m^{\frac{p+3}{2}}L_1''(m) = J(m)\Big(\frac{1-p}{2} + \frac{(p+3)\sigma-2}{2(k-\sigma)} + \frac{(p+1)\sigma(\sigma-1)}{(k-\sigma)^2}\Big), \ \sigma := \varrho^{p+1}.$$

We will be done if we show that the expression in big brackets is positive. It can be easily seen that this holds if and only if

$$\sigma(3kp - 2p + k) > k^2(p-1) + 2k. \tag{3.37}$$

To prove (3.37) we need the following lower bound for σ:

$$\sigma \geq \frac{p-1}{2p}k \qquad \text{if } \; L_1'(m) \geq 0. \tag{3.38}$$

To derive (3.38) we use successively (3.34), the nonnegativity of L' and an obvious inequality:

$$\frac{\varrho(\varrho^{p+1}-1)}{(k-\varrho^{p+1})\sqrt{\varrho^{p+1}-1}} = \frac{m\varrho'}{\sqrt{\varrho^{p+1}-1}} \geq \frac{p-1}{2}\int_1^{\varrho}\frac{dV}{\sqrt{V^{p+1}-1}} \geq \frac{p-1}{p+1}\frac{\sqrt{\varrho^{p+1}-1}}{\varrho^p}.$$

Now an easy calculation yields (3.38). According to (3.38) it is sufficient to prove that

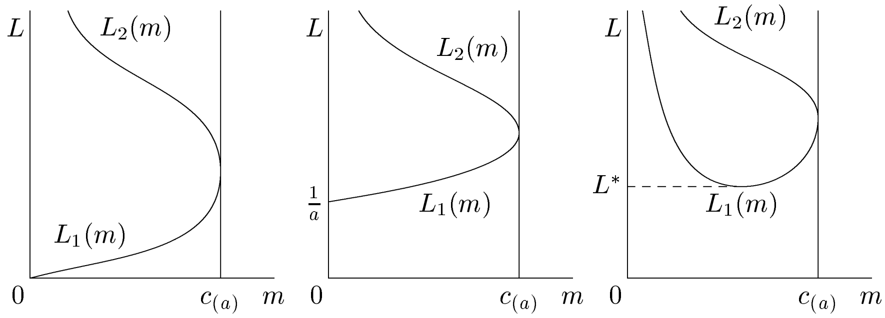
$$\frac{p-1}{2p}\big(2kp - 2p + k(p+1)\big) > k(p-1) + 2.$$

Writing this inequality in the form

$$k(p-1) - (p-1) + k\frac{p^2-1}{2p} > k(p-1) + 2$$

we see that it holds if $k > \dfrac{2p}{p-1}$. But $k = \dfrac{1}{1-\frac{p+1}{2q}} > \dfrac{1}{1-\frac{p+1}{2p}} = \dfrac{2p}{p-1}$, since $q < p$. ☐

The results of Lemmas 3.1–3.5 are summarized in the following figures.



$p < q$ $\qquad$ $p = q$ $\qquad$ $q < p < 2q - 1$

Figure 9. The graphs of $L_i(m)$.

Concerning symmetric solutions to (3.4) we have the following theorem.

Theorem 3.3.

(i) *If $p < q$, then for any $l > 0$ the problem* (3.4) *has a unique positive symmetric solution.*
(ii) *If $p = q$, then for any $l > 1/a$ the problem* (3.4) *has a unique positive symmetric solution, while for $l \le 1/a$ there are no positive solutions.*
(iii) *If $q < p < 2q - 1$, then there is a number L^* (depending on (a, p, q) such that for $l > L^*$ there are exactly two positive symmetric solutions, for $l = L^*$ there is a unique positive symmetric solution and for $l < L^*$ there are no positive solutions.*

Proof. It is an immediate consequence of Lemmas 3.1–3.5. We only remark that the nonexistence results in (ii), (iii) hold also for nonsymmetric solutions (recall the observations at the beginning of this section). □

Now we turn to the study of nonsymmetric solutions. From the fact that for $p < 2q - 1$ and $0 < m < c_{(a)}$ there are two values $L_1(m), L_2(m)$ such that (3.3) holds it follows that there is at least one pair of nonsymmetric solutions for $l = \frac{1}{2}(L_1(m) + L_2(m))$. The following lemma is motivated by the question, whether this pair is unique.

Lemma 3.6. *Assume that $p < 2q - 1$ and either $p \le 4$ or $p > 4$, $q \ge p - 1 - \dfrac{1}{p-2}$. Then*

$$L_1'(m) + L_2'(m) < 0 \qquad \text{for } m \in (0, c_{(a)}). \tag{3.39}$$

Proof. According to (3.29) a sufficient condition for (3.39) is that

$$G_1(m) + G_2(m) \le 0. \tag{3.40}$$

By (3.31) this is equivalent to

$$\frac{\mathcal{F}'\big(R_1(m)\big)}{R_1(m)^q} + \frac{\mathcal{F}'\big(R_2(m)\big)}{R_2(m)^q} \le 0,$$

where $\mathcal{F}$ is defined by (3.8). Setting

$$H(y) := \frac{p+1}{2a} y^{\frac{2q}{q+1}} - y^{\frac{p+1}{q+1}} + m^{p+1},$$

we obtain that $(q+1)H'(y) = x^{-q}\mathcal{F}'(x)$ if $y = x^{q+1}$, hence (3.40) holds if and only if

$$H'(y_1) + H'(y_2) \le 0, \qquad y_i := R_i(m)^{q+1}, \quad i = 1, 2. \tag{3.41}$$

Now we show that if $H'''(y) < 0$ for $y \in [y_1, y_2]$, then (3.41) holds. To do this we first observe that

$$y_o - y_1 \le y_2 - y_o, \tag{3.42}$$

where y_o is the unique point where $H'(y_o) = 0$. Indeed, from Taylor's theorem we have

$$0 = H(y_i) = H(y_o) + \frac{1}{2}H''(\theta_i)(y_i - y_o)^2, \tag{3.43}$$

θ_i lies between y_i and y_o. From (3.43) it is easily seen that

$$y_o - y_1 = \sqrt{\frac{H''(\theta_2)}{H''(\theta_1)}}(y_2 - y_1)$$

and (3.42) follows from the assumption on H'''. Suppose now that $H'(y_2) > -H'(y_1)$. Then

$$H'(y_2 - \eta) > -H'(y_1 + \eta) \qquad \text{for } \ \eta \in [0, y_o - y_1] \tag{3.44}$$

since

$$H''(y_2 - \eta) < H''(y_1 + \eta).$$

But (3.44) leads to

$$H\big(y_2 - (y_o - y_1)\big) < H(y_o)$$

what is a contradiction.

Suppose now that there is a point m such that $L_1'(m) + L_2'(m) \ge 0$ (hence $L_1'(m) \ge 0$). For such m we get using (3.34) and the fact that $\left(\frac{R_2(m)}{m}\right)' < 0$ that

$$\begin{aligned}\frac{\varrho\sqrt{\varrho^{p+1}-1}}{k-\varrho^{p+1}} &= \frac{m\varrho'}{\sqrt{\varrho^{p+1}-1}} > \frac{p-1}{2}\left(\int_1^{\varrho}\frac{dV}{\sqrt{V^{p+1}-1}} + \int_1^{\frac{R_2(m)}{m}}\frac{dV}{\sqrt{V^{p+1}-1}}\right)\\ &\ge (p-1)\int_1^{\varrho}\frac{dV}{\sqrt{V^{p+1}-1}} \ge 2\frac{p-1}{p+1}\frac{\sqrt{\varrho^{p+1}-1}}{\varrho^p}.\end{aligned}$$

This implies that

$$\varrho^{p+1} \ge \frac{2(p-1)}{3p-1}k = \frac{4(p-1)q}{(3p-1)(2q-p-1)}.$$

By (3.6) we have

$$\varrho^{p+1}\left(1 - \frac{p+1}{2a}R_1(m)^{2q-p-1}\right) = 1,$$

hence

$$y_1^{\frac{2q-p-1}{q+1}} = R_1(m)^{2q-p-1} \geq \frac{2a}{p+1}\Big(1 - \frac{(3p-1)(2q-p-1)}{4q(p-1)}\Big). \tag{3.45}$$

If we show that $H'''(y) < 0$ for $y \in [y_1, y_2]$ then we arrive at a contradiction. A straightforward calculation yields that

$$\frac{(q+1)^3}{p+1} y^{\frac{q+3}{q+1}} H'''(y) = -\frac{2q}{a}(q-1) + (p-q)(2q-p+1)y^{-\frac{2q-p-1}{q+1}}. \tag{3.46}$$

Taking (3.45) into account we see, that we need only to consider

$$y^{-\frac{2q-p-1}{q+1}} \leq \frac{2q(p-1)}{a(3p-2q-1)}.$$

The right hand side of (3.46) is then nonpositive if

$$(p-q)(2q-p+1)\frac{p-1}{3p-2q-1} \leq q-1.$$

By straightforward calculations it can be shown that the last inequality holds if and only if

$$(2q-p-1)(p^2-pq+2q-3p+1) \leq 0.$$

The first term is positive and the second one is nonpositive if and only if

$$q(p-2) \geq p^2 - 3p + 1. \tag{3.47}$$

If $p > 4$ and $q \geq p - 1 - \frac{1}{p-2}$, then (3.47) is easily seen to hold. Consider now $p \leq 4$. If $p \leq 2$, then $q(p-2) \geq p(p-2) > p^2 - 3p + 1$. If $2 < p \leq 4$, then $q(p-2) > \frac{1}{2}(p+1)(p-2) \geq p^2 - 3p + 1$. □

Remark 3.1. The method of proof of Lemma 3.6 does not work for any $p < 2q - 1$, since for $q > 3$ there exists $p \in (q, 2q-1)$ such that $H'''(y_o) > 0$.

Theorem 3.4. *Assume that $p < 2q - 1$. Then the following holds:*

(i) *There is a number $L^{**} \in (0, L_{(a)}]$ (which depends on a, p, q) such that for any $l > L^{**}$ the problem (3.4) has at least one pair of positive nonsymmetric solutions u_1, u_2, $u_1(x) = u_2(-x)$ for $x \in [-l, l]$, while for $l < L^{**}$ there are no positive nonsymmetric solutions.*

(ii) *If $p \leq 4$ or $p > 4$, $q \geq p - 1 - \frac{1}{p-2}$, then $L^{**} = L_{(a)}$ and the pair of nonsymmetric positive solutions is unique.*

Proof. In order to prove (i) we need only to show that the range of $\frac{1}{2}(L_1 + L_2)$ contains the interval $(L_{(a)}, \infty)$. This follows from (3.20), (3.25).

Lemma 3.6 implies (ii). □

In sections 4 and 5 it will be important to know how are the stationary solutions ordered. For $p \geq 2q-1$ we have shown that there is at most one positive solution, for $p \leq q$ it follows from Theorem 2.1(i) that any two positive solutions cross each other. Concerning the remaining case $q < p < 2q-1$ we have the following result.

Proposition 3.1. *Assume that $q < p < 2q-1$, $l > L^*$. Let u_1, u_2 be the two symmetric solutions from Theorem* 3.3(iii)*, $m_1 = u_1(0) < u_2(0) = m_2$. Then*

(i) *$u_1 < v$ (i.e. $u_1(x) < v(x)$, $x \in [-l, l]$) for any positive solution v, $v \not\equiv u_1$,*
(ii) *any nonsymmetric positive solution crosses u_2,*
(iii) *any two nonsymmetric positive solutions cross each other.*

Proof. (i) We show first that $u_1 < u_2$. Suppose there is a point $x_o \in (0, l]$ such that $u_1(x_o) = u_2(x_o)$, $u_1(x) < u_2(x)$ for $x \in [0, x_o)$. Set $w := u_2 - u_1$. Then $w_x(0) = 0$ and $w_{xx}(x) > 0$ for $x \in [0, x_o)$, hence $w_x(x) > 0$ for $x \in (0, x_o)$. But then $w(x_o) > w(0) > 0$, a contradiction.

Let now v be an arbitrary nonsymmetric solution. If $v \geq u_1$ then $v > u$ by the maximum principle. Suppose there is a point $x_o \in [-l, l]$ such that $u_1(x_o) > v(x_o)$. Set $\underline{w}(x) := \min\bigl(v(x), u_1(x)\bigr)$. Then

$$\underline{w} \leq u_1, \quad \underline{w} \not\equiv u_1, \quad \underline{w} \text{ is a supersolution.} \tag{3.47a}$$

The problem (1.1) generates a strongly monotone compact local semiflow in $C^+ := \{v \in W^{1,2}(\Omega)\,;\, v \geq 0\}$ (see Proposition 5.1) and it is easily seen that the subset $C_s^+ := \{v \in C^+\,;\, v(x) = v(-x)\}$ is invariant. The zero solution is unstable from above (Theorem 2.1(iii)), the $W^{1,2}$–norm of any orbit can be estimated in terms of its sup–norm (see (4.10)), therefore there is an orbit lying in C_s^+ which connects 0 to u_1 ([**M,** Theorem 8]), a contradiction to (3.47a). (It is not difficult to see that Theorem 8 from [**M**] is applicable in our case, although it was formulated in [**M**] only for semiflows on whole Banach spaces.)

(ii) Let v be an arbitrary nonsymmetric solution. Suppose v does not cross u_2, i.e. either $v \geq u_2$ or $v \leq u_2$. In both cases we arrive at a contradiction, because according to [**M,** Theorem 8] there are orbits (in C_s^+) which connect u_2 to u_1 and to ∞.

(iii) Let v_1, v_2 be nonsymmetric solutions, $v_1 \not\equiv v_2$. If $v_1(x) = v_2(-x)$, then they cross at $x = 0$. Assume now that there is a point x_o such that $v_1(x_o) \neq v_2(-x_o)$. Then v_1, v_2 lie on two different trajectories of the planar system

$$\begin{aligned} u' &= w, \\ w' &= au^p. \end{aligned}$$

The phase portrait for this problem is depicted in Figure 10.

Trajectories going through the points $(m, 0)$, $m < c_{(a)}$ (cf. (3.17)), cross both of the curves $u' = u^q$, $u' = -u^q$ exactly twice. The trajectory going through $(c_{(a)}, 0)$ hits any of the curves $u' = u^q$, $u' = -u^q$ exactly once. Trajectories going through the points $(m, 0)$, $m > c_{(a)}$, cannot yield solutions to (3.4).

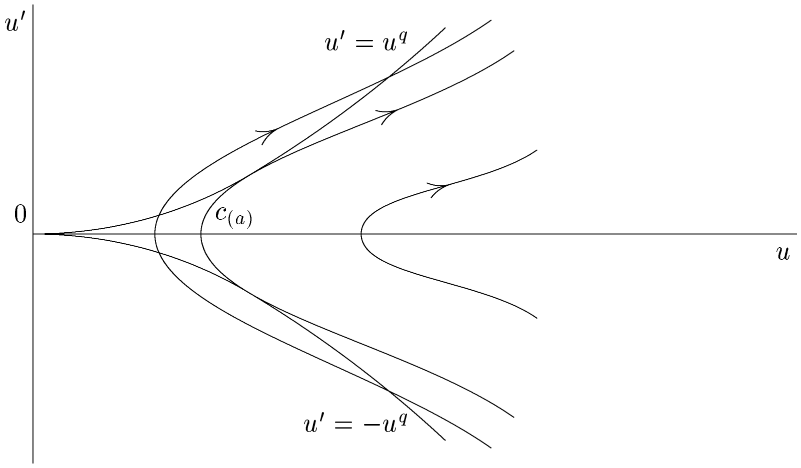


Figure 10. The phase portrait for $u' = v$, $v' = au^p$.

If a pair of nonsymmetric solutions is not unique, then there are two trajectories that need the same "time" to go from the first intersection with $u' = -u^q$ to the first intersection with $u' = u^q$. It is easy to see that it is sufficient to consider v_1, v_2 as depicted in the following Figure 11.

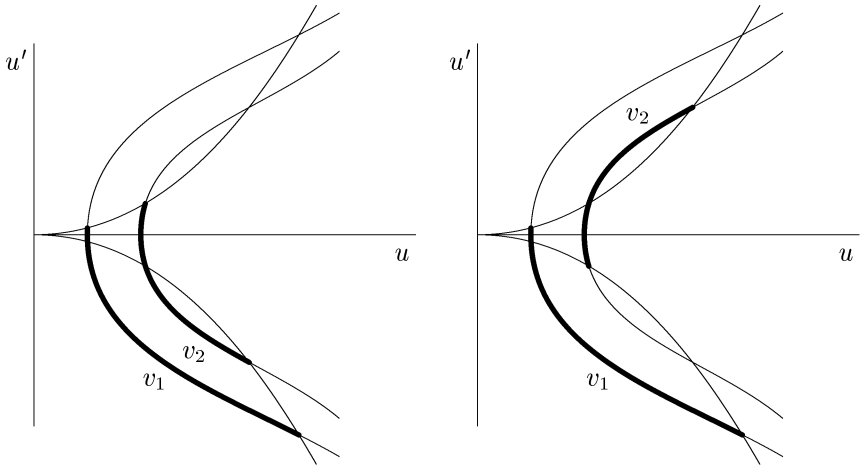


Figure 11. Two nonsymmetric solutions v_1, v_2.

In both cases $v_1(-l) > v_2(-l)$ and $v_1(l) < v_2(l)$, i.e. v_1, v_2 cross each other. $\square$

Now we turn to the investigation of the Morse indices of the stationary solutions. These results will be used in Section 5.

Theorem 3.5

(i) *The symmetric stationary solutions are hyperbolic except of the cases*

$$p < 2q - 1, \quad m = c_{(a)} \qquad \textit{or} \qquad q < p < 2q - 1, \quad l = L^*.$$

(ii) *The nonsymmetric stationary solutions are hyperbolic if they correspond to m such that $L_1'(m) + L_2'(m) \neq 0$ (cf. Lemma 3.6).*

Proof. Let $v(x;\mu)$ be a solution of (3.1) with

$$v(-l,\mu) = \mu > 0, \qquad v_x(-l,\mu) = -\mu^q.$$

We have to show that the linearized problem

$$w'' = apv^{p-1}w, \qquad x \in (-l,l) \tag{3.48}$$

$$w'(-l) = -qv(-l)^{q-1}w(-l) \tag{3.49}$$

$$w'(l) = qv(l)^{q-1}w(l) \tag{3.50}$$

cannot have a nontrivial solution, if μ is such that

$$v_x(l;\mu) = v^q(l,\mu).$$

Obviously, $w(x) = v_\mu(x;\mu)$ satisfies (3.48), (3.49). Since (3.48) is a linear second order equation, any solution of (3.48), (3.49) must be a scalar multiple of $v_\mu(x;\mu)$.

Assume now that v corresponds to a symmetric solution u, hence $l = L_i(m)$ for $i = 1$ or 2 (or $l = L(m)$). The numbers $m = v(0;\mu)$ and μ are related (cf. (3.6)) by the equation

$$m^{p+1}(\mu) = \mu^{p+1} - \frac{p+1}{2a}\mu^{2q}.$$

Differentiating the equality

$$v_x(L_i(m);\mu) = v^q(L_i(m);\mu)$$

with respect to μ, we obtain that

$$v_{x\mu} - qv^{q-1}v_\mu = \frac{-1}{m^p}L_i'(\mu^p - \frac{q}{a}\mu^{2q-1})(v_{xx} - qv^{q-1}v_x). \tag{3.51}$$

Since

$$v_{xx}(L_i(m);\mu) - qv^{q-1}v_x(L_i(m);\mu) = av^p(L_i(m);\mu) - qv^{2q-1}(L_i(m);\mu),$$

we see that the right hand side of (3.51) is nonzero under the assumptions of the first part of the theorem (L_1' vanishes if and only if $q < p < 2q - 1$, $l = L^*$; $a\mu^p - q\mu^{2q-1} = av^p(L_i(m);\mu) - qv^{2q-1}(L_i(m);\mu)$ vanishes if and only if $m = c_{(a)}$). Hence, (3.48)–(3.50) has no nontrivial solution.

To prove (ii) we can argue exactly as before with the only difference that now $2l = L_1(m) + L_2(m)$. $\square$

In the remainder of this section we shall work with a fixed length l, we shall vary the parameter a and we shall use the notations introduced in Section 2. We want to use the bifurcation diagrams shown in Section 1; their correctness is shown (except of the case $2q-1 > p > \max(4, q+1+\frac{1}{p-2})$) by Theorems 3.1–3.4 and by the following lemma:

Lemma 3.7. *Let u_a be (any) positive solution of* (3.4) *and let $L_{(a)}$ be as in* (3.20) *(if $p < 2q-1$). Then we have*

(i) *If $p < 2q-1$, then $\frac{d}{da}L_{(a)} < 0$, $\lim_{a\to\infty} L_{(a)} = 0$, $\lim_{a\to 0+} L_{(a)} = +\infty$.*

(ii) *If $p < q$ and $a \to 0+$ or if $p = q$ and $a \to \frac{1}{l}+$, then $\|u_a\| \to 0$.*

(iii) *If $q < p < 2q-1$ and $m \in (0, c_{(a)})$ is fixed, then $\frac{d}{da}L_1(m) < 0$,*

$$\lim_{a\to 0+}\Big(\min_{0<m<c_{(a)}} L_1(m)\Big) = +\infty.$$

(iv) *If $p > 2q-1$ (or if $p = 2q-1$ and $a > q$), then $\lim_{a\to 0+}\|u_a\| = +\infty$ (or $\lim_{a\to q+}\|u_a\| = +\infty$) and $\lim_{a\to\infty}\|u_a\| = 0$.*

(v) *If $p < 2q-1$, then $u_a \to \overline{u_{a_1}}$ in $W^{1,2}(\Omega)$ as $a \to a_1+$, where $\overline{u_{a_1}}$ is the maximal positive solution of (3.4) and a_1 is as in Figs. 1–3.*

Proof. (i) Follows immediately from (3.10) and (3.17).

(ii) By the same way as in (2.29) we obtain

$$u^{2q-p-1}(l) \le \frac{2a}{p+1}, \tag{3.52}$$

hence $\|u_a\| \to 0$ if $a \to 0+$ and $\|u_a\|$ is bounded if $a \to \frac{1}{l}+$. If $p = q$, $a \to \frac{1}{l}+$ and $u_a \not\to 0$, choose a sequence $a_n \downarrow \frac{1}{l}$ such that $\|u_n\| \to c > 0$ (where $u_n := u_{a_n}$). We may suppose $u_n \rightharpoonup u$ (weak convergence) and passing to the limit in the equality

$$u_n = Ku_n - a_nP(u_n) + Q(u_n)$$

we get $u_n \to u = Ku + \frac{1}{l}P(u) + Q(u)$, which contradicts Theorem 3.3(ii).

(iii) Using (3.7) we obtain

$$\frac{d}{da}L_1(m) = -\frac{1}{2a}L_1(m) + \sqrt{\frac{p+1}{2a}}\,m^{-(p+1)/2}\frac{1}{\sqrt{\left(\frac{R_1(m)}{m}\right)^{p+1}-1}}\frac{d}{da}R_1(m)$$

and differentiating (3.6) we get

$$\frac{d}{da}R_1(m) = \frac{p+1}{2a^2}\frac{R_1(m)^{2q}}{\mathcal{F}'\big(R_1(m)\big)} < 0,$$

hence $\frac{d}{da}L_1(m) < 0$.

Now suppose $a_k \to 0$ and $\min_{0<m<c_k} L_1(m) < l < +\infty$, where $c_k = c_{a_k}$. Then $c_k \to 0$ by (3.17), hence there exists a sequence u_k of positive solutions of (3.4) with $a = a_k$ and $u_k(0) = m_k \to 0$. Since $a_k \to 0$, this implies $\|u_k\| \to 0$, which contradicts Lemma 2.2.

(iv) The estimate (3.52) implies $\|u_a\| \to +\infty$ for $a \to 0+$ and $p > 2q-1$. If $p = 2q-1$, $a_k \to q+$ and $\|u_k\| < c$ (where $u_k = u_{a_k}$ and c is a constant), then choosing a weakly convergent subsequence we get (as in the proof of (ii)) $u_k \to u$, where u is a positive solution corresponding to $a = q$. Now Theorem 3.2(i) and Lemma 2.2 yield a contradiction.

Finally, choose $\varepsilon > 0$ and choose any positive function $u : [-l, l] \to (0, \varepsilon)$ fulfilling the boundary conditions in (3.4). Then $u_{xx} \le au^p$ for sufficiently large a, hence $u_a \le u < \varepsilon$, which implies $\|u_a\| \to 0$ for $a \to \infty$.

(v) This follows from the continuous dependence of $L_i(m), L_{(a)}$ and $c_{(a)}$ on a and from the continuous dependence of the solution of (3.1)–(3.2) on m. □

If u is a solution of (3.4) (or, equivalently, (2.2)), then the number of the negative or zero eigenvalues of the operator $F'(u) = I - K + aP'(u) - Q'(u)$ (where I denotes the identity), will be denoted by $M^-(u)$ or $M^o(u)$, respectively. Recall that any eigenvalue $\lambda \ne 1$ of $F'(u)$ is simple since the corresponding eigenvector is a solution of a second order linear differential equation with a fixed boundary condition. Moreover, the variational characterization of eigenvalues of $F'(u)$ gives us immediately the continuous dependence of these eigenvalues on the solution u, which implies

$$M^-(u_n) \to M^-(u) \qquad \text{if } u_n \to u \text{ and } M^o(u) = 0 \tag{3.53}$$

$$0 \le \lim_{n\to\infty} \big(M^-(u_n) - M^-(u)\big) \le 1 \qquad \text{if } u_n \to u \text{ and } M^o(u) \ne 0 \tag{3.54}$$

Finally, if $M^o(u) = 0$, then the degree $d(u)$ is well defined and $d(u) = (-1)^{M^-(u)}$.

Theorem 3.6. *Let a_o and a_1 be as in Figs.* 1–5.

(i) *Let $p \le q$. If u is a positive symmetric solution of* (3.4)*, then $M^-(u) = 1$ for $a \le a_1$ and $M^-(u) = 2$ for $a > a_1$. Moreover, $M^o(u) = 0$ if $a \ne a_1$. If u is a positive nonsymmetric solution of* (3.4)*, then $M^-(u) = 1$, $M^o(u) = 0$.*

(ii) *Let $q < p < 2q-1$ and let the assumptions of Theorem* 3.4(ii) *be fulfilled. Let $a > a_o$ and let $u_1 < u_2$ be the two corresponding symmetric positive solutions of* (3.4)*. Then $M^-(u_1) = M^o(u_1) = 0$, $M^-(u_2) = 1$ for $a < a_1$, $M^-(u_2) = 2$ for $a > a_1$ and $M^o(u) = 0$ if $a \ne a_1$. If u is a positive nonsymmetric solution of* (3.4)*, then $M^-(u) = 1$, $M^o(u) = 0$.*

(iii) *Let $p \ge 2q-1$ and let u be a positive solution of* (3.4)*. Then $M^-(u) = M^o(u) = 0$.*

(iv) *$M^-(0) = 0$ and $M^o(0) = 1$ for any p, q.*

Proof. (i) Choose $a > a_o$ sufficiently close to a_o. Using Lemma 2.6, Lemma 2.3 and the homotopy invariance property of the degree one easily gets

$$0 = d^+_{a_o}(0) = d^+_a(u) + d^+_a(0) = d^+_a(u) + 1,$$

where u is the unique positive solution of (3.4), hence $d^+_a(u) = d_a(u) = -1$. Since $u \to 0$ for $a \to a_o$ and $M^-(0) = 0 \neq M^o(0)$, we get using (3.54) (which is easily seen to hold also for varying a) and Theorem 3.5 that $M^-(u) = 1$ and $M^o(u) = 0$. By Theorem 3.5 this will hold for any $a < a_1$.

Now choose $a^- < a_1 < a^+$ close to a_1 and let u^-, u^+ be the corresponding positive symmetric solutions and u^+_1, u^+_2 the corresponding positive nonsymmetric solutions (for $a = a^+$). Using the homotopy invariance of the degree we get

$$-1 = d_{a^-}(u^-) = d_{a^+}(u^+) + d_{a^+}(u^+_1) + d_{a^+}(u^+_2) = d_{a^+}(u^+) + 2d_{a^+}(u^+_1) \tag{3.55}$$

due to the symmetry $u^+_1(x) = u^+_2(-x)$. Theorem 3.5 implies

$$|d_{a^+}(u^+)| = |d_{a^+}(u^+_1)| = 1,$$

so that (3.55) yields $d_{a^+}(u^+) = 1$, $d_{a^+}(u^+_1) = -1$.

Repeating our considerations with a close to a_o for F/X_s, where X_s is the space of symmetric functions from X, we get $d^s_a(u) = -1$ and $(M^s)^-(u) = 1$ for any $a > a_o$ and any positive symmetric solution u (where d^s and M^s is the degree and the Morse index corresponding to F/X_s, respectively), hence $M^-(u^+) \geq (M^s)^-(u^+) = 1$. Since $d_{a^+}(u^+) = 1$ and $M^-(u^+) \leq 2$ by (3.54), we have $M^-(u^+) = 2$. Finally, (3.54) and $d_{a^+}(u^+_1) = d_{a^+}(u^+_2) = -1$ imply $M^-(u^+_1) = M^-(u^+_2) = 1$.

(ii) We have $M^o(u_1) = 0$ by Theorem 3.5. If $M^-(u_1) > 0$, then this would imply the existence of a positive (symmetric) solution lying between 0 and u_1 (see Lemma 2.4 and the proof of Theorem 2.2(ii)). Hence $M^-(u_1) = 0$, $d(u_1) = d_a(u_1) = 1$. Let u_o be the unique positive solution for $a = a_o$ and choose $a^- < a_o < a^+$ sufficiently close to a_o, $\varepsilon > 0$ small. If $u^+_1 < u^+_2$ are the positive solutions corresponding to a^+, we have

$$\begin{aligned} 0 &= \deg(F_{a^-}, 0, B_\varepsilon(u_o)) = \deg(F_{a^+}, 0, B_\varepsilon(u_o)) = d_{a^+}(u^+_1) + d_{a^+}(u^+_2) \\ &= 1 + d_{a^+}(u^+_2), \end{aligned}$$

hence $d_{a^+}(u^+_2) = -1$ and using (3.54) and Theorem 3.5 we get $M^-(u^+_2) = 1$. The rest of the assertion can be proved analogously as in (i).

(iii) The assertion can be proved by the same way as the equality $M^-(u_1) = M^o(u_1) = 0$ in the proof of (ii).

(iv) This is trivial. □

Corollary. *If $p < 2q - 1$ and $\overline{u}$ is a maximal non-negative solution, then $\overline{u}$ is unstable from above.*

Proof. If $M^-(\overline{u}) > 0$, then the instability from above of $\overline{u}$ follows by the same way as in the proof of Theorem 2.1(i), since the eigenvector corresponding to the first eigenvalue of $F'(\overline{u})$ is positive.

If $M^-(\overline{u}) = 0$, then $p \geq q$, $a \leq a_o$ and either $\overline{u} = 0$ or $p > q$, $a = a_o$. In both cases $d(\overline{u}) \neq 1$, hence $\overline{u}$ is unstable (see [**Q3**]). Assume now that $\overline{u}$ is stable both from above and from below and choose $\varepsilon > 0$. Then there exists $\delta > 0$ such that the solution $u(t, u_o)$ of (1.1) stays in $B_\varepsilon(\overline{u}) = \{v \in W^{1,2}(-l,l)\,;\, \|v - \overline{u}\| < \varepsilon\}$ whenever $u_o \in B_\delta(\overline{u})$ and either $u_o \geq \overline{u}$ or $u_o \leq \overline{u}$. Choosing $u^-, u^+ \in B_\delta(\overline{u})$ such that $u^- < \overline{u} < u^+$ in $[-l,l]$ we can find $\nu > 0$ such that $u^- < v < u^+$ for any $v \in B_\nu(\overline{u})$, since $W^{1,2}(-l,l) \hookrightarrow C([-l,l])$. The monotonicity of the flow (see the proof of Proposition 5.1) implies

$$\|u(t,v)-\overline{u}\|_{L^\infty(-l,l)} \leq \|u(t,u^+)-u(t,u^-)\|_{L^\infty(-l,l)} \leq c\|u(t,u^+)-u(t,u^-)\| \leq 2\varepsilon c$$

for any $v \in B_\nu(\overline{u})$, and the variation-of-constants formula from [**A1**] implies

$$\|u(t,v) - \overline{u}\| \leq C(\varrho) \sup_{t-\varrho\leq\tau\leq t} \|u(\tau,v) - \overline{u}\|_{L^\infty(-l,l)} \leq 2\varepsilon c C(\varrho) \tag{3.56}$$

for any $v \in B_\nu(\overline{u})$, where $C(\varrho) \to +\infty$ as $\varrho \to 0+$. Fix $\varrho > 0$. Taking $\nu > 0$ smaller, if necessary, we may suppose $u(t,v) \in B_\varepsilon(\overline{u})$ for $v \in B_\nu(\overline{u})$ and $t \in [0,\varrho]$ (see Proposition 5.1). This estimate together with (3.56) imply $u(t,v) \in B_{\tilde\varepsilon}(\overline{u})$ for $v \in B_\nu(\overline{u})$ and any $t > 0$ (where $\tilde\varepsilon = \varepsilon \max(1, 2cC(\varrho))$), which gives us a contradiction with the instability of u.

Consequently, $\overline{u}$ is unstable from above or from below. If $\overline{u} = 0$ then the instability from above follows from the fact that the functional Φ corresponding to (2.1) is even. If $p > q$, $a = a_o$, then $\overline{u}$ is stable from below by [**M,** Theorem 8], hence it is unstable from above. ☐

4. Blow up and global existence

In this section we consider the problem

$$\begin{cases} u_t = \triangle u - au^p & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial n} = u^q & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_o(x) \geq 0 & x \in \overline{\Omega}, \end{cases} \tag{4.1}$$

with $p, q > 1$, $a > 0$, $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain. We assume that $u_o \geq 0$ is smooth enough and that the compatibility condition $\dfrac{\partial u_o}{\partial n} = u_o^q$, $x \in \partial\Omega$ is satisfied. By a solution we mean a nonnegative classical solution.

Theorem 4.1. *If Ω is the unit ball $B_1(0)$ and*

$$p > 2q - 1 \qquad \textit{or} \qquad p = 2q - 1 \quad \textit{and} \quad a > q$$

then the solution u exists globally and stays uniformly bounded for any u_o.

Proof. For any u_o there is a smooth function v_o satisfying the compatibility condition and such that

$$u_o(x) \le v_o(x) = V_o(|x|) \qquad \text{for } |x| \le 1.$$

We shall construct a sequence $\{w_n(r)\}$ such that for n large enough

$$w_n(r) \ge V_o(r) \tag{4.2}$$

and

$$w_n''(r) + \frac{N-1}{r} w_n'(r) - a w_n^p(r) \le 0, \qquad r \in (0,1), \tag{4.3}$$

$$w_n'(0) = 0, \quad w_n'(1) = w_n^q(1). \tag{4.4}$$

The maximum principle implies then that the solution emanating from u_o stays below w_n for $t > 0$.

Put $w_n(r) := \Big(C_n - \dfrac{q-1}{n} r^n\Big)^{1/(1-q)}$, $C_n := \dfrac{q-1}{n} + \varepsilon_n$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$. Then $w_n(r) \ge C_n^{1/(1-q)}$, hence (4.2) holds for n large enough.

Set $\varphi_n(r) := C_n - \dfrac{q-1}{n} r^n$. Then

$$w_n'(r) = \varphi_n(r)^{q/(1-q)} r^{n-1}$$

and (4.4) follows. Since

$$w_n''(r) = q\varphi_n(r)^{(2q-1)/(1-q)} r^{2n-2} + (n-1)\varphi_n(r)^{q/(1-q)} r^{n-2}$$

it suffices to show that

$$q\varphi_n(r)^{(2q-1)/(1-q)} r^{2n-2} + (n+N-2)\varphi_n(r)^{q/(1-q)} r^{n-2} \le a\varphi_n(r)^{p/(1-q)}. \tag{4.5}$$

Multiplying (4.5) by $\varphi_n(r)^{p/(q-1)}$, we obtain

$$q\varphi_n(r)^{(p-2q+1)/(q-1)} r^{2n-2} + (n+N-2)\varphi_n(r)^{(p-q)/(q-1)} r^{n-2} \le a. \tag{4.6}$$

If $p > 2q-1$, then the left hand side of (4.6) is easily seen converge to zero as $n \to \infty$.

If $p = 2q-1$, $a > q$ then (4.6) has the form

$$\Big(1 - \frac{N-2}{n}(q-1)\Big) r^{2n-2} + nC_n \Big(1 + \frac{N-2}{n}\Big) r^{n-2} \le a. \tag{4.7}$$

It is obvious that it is sufficient to prove (4.7) for $r = 1$. But for $r = 1$, (4.7) reduces to

$$q + \varepsilon_n(n + N - 2) < a,$$

therefore we only need to choose $\varepsilon_n < \dfrac{a-q}{n+N-2}$ and we are done. □

In what follows we show that Theorem 3.1 is sharp. More precisely, we prove that for $\Omega = B_1(0)$, $p < 2q - 1$ or $p = 2q - 1$, $a < q$ blow up occurs, while for $N = 1$, $p = 2q - 1$, $a = q$ all solutions are global but unbounded.

We are also interested in all possible types of behaviour of solutions to (4.1). Three possibilities are conceivable:

(i) global existence and boundedness,
(ii) blow up in finite time,
(iii) global existence without uniform boundedness.

In several cases we will be able to prove that the third possibility cannot occur.

Theorem 4.2. *Assume that* $N = 1$ $(\Omega = (-l, l))$ *and*

$$p < 2q - 1 \qquad or \qquad p = 2q - 1,\ a < q.$$

Then

(i) *any global solution is uniformly bounded in* $X = W^{1,2}$, *i.e.* $\sup_{t>0} \|u(\cdot, t)\|_X < \infty$.
(ii) *If* $u_o \geq v$, $u_o \not\equiv v$, *where* v *is any maximal stationary solution, then* u *blows up in a finite time.*

Remark 4.1. Under the assumptions of Theorem 4.2 we have the following list of maximal stationary solutions.

The trivial solution is maximal if

$$\begin{aligned} & p = q,\ a > 0,\ l \leq \frac{1}{a} && \text{(Theorem 3.3(ii))}, \\ \text{or} \quad & q < p < 2q - 1,\ a > 0,\ l < L^* && \text{(Theorem 3.3(iii))}, \\ \text{or} \quad & p = 2q - 1,\ a < q,\ l > 0 && \text{(Theorem 3.2(i))}. \end{aligned}$$

Any positive solution is maximal if

$$\begin{aligned} & p < q,\ a > 0,\ l > 0 && \text{(Theorem 3.3(i))}, \\ \text{or} \quad & p = q,\ a > 0,\ l > \frac{1}{a} && \text{(Theorem 3.3(ii))}, \\ \text{or} \quad & q < p < 2q - 1,\ a > 0,\ l = L^* && \text{(Theorem 3.3(iii))}. \end{aligned}$$

Except of the minimal symmetric solution, any nontrivial solution is maximal if

$$q < p < 2q - 1,\ a > 0,\ l > L^* \qquad \text{(Proposition 3.1)}.$$

Proof of Theorem 4.2. (i) We proceed by contradiction. Suppose that u is a global solution which is unbounded in the $W^{1,2}$–norm (which we denote similarly as in Section 2 by $\|\cdot\|$). Then one of the following possibilities must occur:

$$\lim_{t\to\infty} \|u(\cdot,t)\| = \infty \tag{4.8}$$

or

$$\limsup_{t\to\infty} \|u(\cdot,t)\| = \infty, \quad \liminf_{t\to\infty} \|u(\cdot,t)\| < \infty. \tag{4.9}$$

Exactly in the same manner as in [**F**], the variation of constants formula from [**A1**] can be used to prove that for any constant C large enough, (4.9) implies the existence of a positive stationary solution v with $\|v\| = C$. This is impossible, since under our assumptions we have an apriori bound for stationary solutions due to (2.29), (3.52) and Theorem 3.2(i).

Suppose now that (4.8) holds. The solution u satisfies the well known energy identity

$$\int_0^t \int_{-l}^{l} u_t^2 \, dx \, dt + \Phi\big(u(\cdot,t)\big) = \Phi(u_o), \quad t > 0, \tag{4.10}$$

where Φ is the energy functional introduced in Section 2, i.e.

$$\Phi(v) = \int_{-l}^{l} \Big(\frac{1}{2}v_x^2 + \frac{a}{p+1}v^{p+1}\Big)dx - \frac{1}{q+1}\Big(v(l)^{q+1} + v(-l)^{q+1}\Big).$$

If we set

$$\mathcal{F}(v) := v(l)^{2q} + v(-l)^{2q}$$

for $v \in C([-l,l])$, then we get from (4.10) and (4.8) that

$$\mathcal{F}\big(u(\cdot,t)\big) \to \infty \quad \text{as} \quad t \to \infty.$$

Our next aim is to show that there is a constant $c_o > 0$ such that

$$\mathcal{F}\big(u(\cdot,t)\big) \le c_o \int_{-l}^{l} u_t^2 dx \tag{4.11}$$

for t large enough. To do this we first choose for any $t > 0$ a point $x_o = x_o(t)$ such that $u_x(x_o,t) = 0$ and observe that

$$\int_{x_o}^{l} u_t u_x dx \ge \frac{1}{2}u(l,t)^{2q} - \frac{a}{p+1}u(l,t)^{p+1}, \tag{4.12}$$

since $u_t u_x = \big(\frac{1}{2}u_x^2 - \frac{a}{p+1}u^{p+1}\big)_x$.

Analogously,

$$-\int_{-l}^{x_o} u_t u_x dx \geq \frac{1}{2}u(-l,t)^{2q} - \frac{a}{p+1}u(-l,t)^{p+1}. \tag{4.13}$$

Adding (4.12) to (4.13) and using the assumption on p, q, a we get, that for t large enough ($t > 0$ if $p = 2q-1$, $a < q$) there exists an $\varepsilon > 0$ such that

$$\begin{aligned} \varepsilon\mathcal{F}\big(u(\cdot,t)\big) &\leq \int_{-l}^{l} |u_t u_x|\, dx \leq \Big(\int_{-l}^{l} u_t^2\, dx\Big)^{1/2}\Big(\int_{-l}^{l} u_x^2\, dx\Big)^{1/2} \\ &\leq \frac{\varepsilon}{8l}\int_{-l}^{l} u_x^2\, dx + C_\varepsilon \int_{-l}^{l} u_t^2\, dx. \end{aligned} \tag{4.14}$$

But

$$u_x(l,t) - u_x(x,t) = \int_x^l u_{xx} dx \geq \int_x^l u_t\, dx \geq -\Big(2l\int_{-l}^{l} u_t^2\, dx\Big)^{1/2},$$

hence

$$u_x(x,t) \leq u(l,t)^q + \Big(2l\int_{-l}^{l} u_t^2\, dx\Big)^{1/2}$$

and

$$\int_{-l}^{l} u_x^2\, dx \leq 4l\Big(u(l,t)^{2q} + 2l\int_{-l}^{l} u_t^2\Big). \tag{4.15}$$

Using (4.14) and (4.15) we obtain (4.11). If we now set

$$f(t) := \int_0^t \mathcal{F}\big(u(\cdot,s)\big)\, ds,$$

then (4.11) and (4.10) yield that

$$\begin{aligned} f(t) &\leq c_o \int_0^t \int_{-l}^{l} u_t^2\, dx\, dt = c_o\Big(\Phi(u_o) - \Phi\big(u(\cdot,t)\big)\Big) \\ &\leq c_1 + c_2\Big(\mathcal{F}\big(u(\cdot,t)\big)\Big)^{\frac{q+1}{2q}} \end{aligned}$$

for some positive constants c_1, c_2. But this means that there is a constant $c_3 > 0$ such that

$$f'(t) \geq c_3 f(t)^{\frac{2q}{q+1}}$$

for t large enough. Hence f blows up in a finite time what is a contradiction.

(ii) We show that (ii) is a consequence of (i). We recall from Section 3 (Corollary of Theorem 3.6) that under our assumptions any maximal solution v is unstable from above. By [**A1**], (4.1) defines a local semiflow in $X = W^{1,2}(-l,l)$. The maximum principle implies the strong monotonicity of this semiflow. Hence, according to [**M,** Theorem 5] there is a function w defined on $[-l,l]\times(-\infty,T)$, $T>0$, with the following properties: w satisfies the equation

$$w_t = w_{xx} - aw^p \qquad \text{in} \;\; (-l,l)\times(-\infty,T),$$

together with the boundary condition

$$w_x(\pm l,t) = \pm w^q(\pm l,t) \qquad \text{for} \;\; t\in(-\infty,T),$$

further

$$w(x,t_1) < w(x,t_2) \qquad \text{for} \;\; x\in[-l,l], \;\; -\infty < t_1 < t_2 < T, \tag{4.16}$$

and

$$w(\cdot,t)\to v \qquad \text{in } X \;\; \text{as} \;\; t\to-\infty. \tag{4.17}$$

Suppose that $T=\infty$. Then

$$\sup_{t>0} \|w(\cdot,t)\| < \infty,$$

hence w tends to a stationary solution which is by (4.16), (4.17) greater than v — a contradiction. By the maximum principle $u(\cdot,t)>v$ for $t>0$, therefore (4.17) implies the assertion. □

Concerning the localization of blow up points we have the following result.

Theorem 4.3. *Assume that* $N=1$, $\Omega=(-1,1)$ *and*

$$p < 2q-1 \qquad or \qquad p = 2q-1, \;\; a<q.$$

Let $u_o \not\equiv 0$ *satisfy the conditions :*

$$u_o(x) = u_o(-x) \qquad for \;\; x\in[-1,1],$$
$$u_o'(x) \ge 0 \qquad for \;\; x\in[0,1].$$

If $p>q$ *assume further that* u_o *is a subsolution, i.e.*

$$u_o'' - au_o^p \ge 0 \qquad for \;\; x\in(-1,1),$$
$$u_o'(\pm1) = \pm u_o^q(\pm1).$$

Let u *blow up. Then* u *blows up only at the points* $-1, 1$.

Proof. We shall use an idea from [**FML**]. We set

$$J(x,t) := u_x(x,t) - \varphi(x)u^q(x,t)$$

and show that $J \geq 0$ in $[0,1] \times [0,T)$ for a suitably chosen function φ; T is the blow up time of u. For given u_o we choose φ smooth as follows:

$$\begin{aligned} &\varphi = 0 \qquad \text{on } [0, 1-\varepsilon],\ 0 < \varepsilon < 1, \\ &\varphi, \varphi', \varphi'' > 0 \qquad \text{on } (1-\varepsilon, 1], \qquad \varphi(1) = \varepsilon, \\ &u_o' \geq \varphi u_o^q \qquad \text{on } (1-\varepsilon, 1]. \end{aligned}$$

With this choice of φ we have

$$\begin{aligned} &J(x,0) \geq 0 \qquad \text{for } x \in [0,1], \\ &J(0,t) = 0, \quad J(1,t) > 0 \qquad \text{for } t \in [0,T). \end{aligned}$$

If we derive for J a linear parabolic inequality such that the maximum principle enables us to conclude that $J \geq 0$ in $[0,1] \times [0,T)$ then we are done, since then

$$u_x(x,t) \geq \varphi(1 - \frac{\varepsilon}{2}) u^q(x,t) \qquad \text{for } x \in [1 - \frac{\varepsilon}{2}, 1)$$

and integrating this, we obtain

$$u(x,t) \leq k(1-x)^{1/(1-q)} \qquad \text{for } x \in [1 - \frac{\varepsilon}{2}, 1), \quad k := \big((q-1)\varphi(1 - \frac{\varepsilon}{2})\big)^{1/(1-q)}.$$

Obvious calculations yield

$$\begin{aligned} J_t - J_{xx} &= u_{xt} - u_{xxx} - q\varphi u^{q-1}(u_t - u_{xx}) + \varphi'' u^q + 2q\varphi' u^{q-1} u_x + q(q-1)\varphi u^{q-2} u_x^2 \\ &= -apu^{p-1} u_x + aq\varphi u^{p+q-1} + \varphi'' u^q + 2q\varphi' u^{q-1} u_x + q(q-1)\varphi u^{q-2} u_x^2. \end{aligned}$$

From the definition of J we have

$$-apu^{p-1} u_x + aq\varphi u^{p+q-1} = -aqu^{p-1} J - a(p-q) u^{p-1} u_x.$$

If $p \leq q$, then

$$J_t - J_{xx} + aqu^{p-1} J \geq 0$$

and we are done.

If $p > q$, then we obtain

$$-a(p-q)u^{p-1}u_x = -(p-q)\frac{u_x}{u} J_x + (p-q)\frac{u_x}{u} u_t - (p-q)\varphi' u^{q-1} u_x - (p-q) q\varphi u^{q-2} u_x^2.$$

Since $u_t \geq 0$ by the maximum principle, we arrive at

$$\begin{aligned} &J_t - J_{xx} + (p-q)\frac{u_x}{u} J_x + aqu^{p-1} J \geq \\ &\qquad \geq (3q - p)\varphi' u^{q-1} u_x + q(2q - 1 - p)\varphi u^{q-2} u_x^2 \geq 0 \end{aligned}$$

and the proof is finished. $\square$

Remark 4.2. If $a = 0$, $\Omega = (-1,1)$ and $u_o \geq 0$ fulfils some additional assumptions, then one can use the similarity variables

$$w(y,s) = (T-t)^{\lambda} u(x,t), \quad y = \frac{1-x}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad \lambda = \frac{1}{2(q-1)}$$

(where T is the blow up time) in order to show that for any $y \geq 0$,

$$(T-t)^{\lambda} u(1 - y\sqrt{T-t}, t) \to w_o(y) \qquad \text{as } \ t \to T,$$

where w_o is the unique positive bounded solution of

$$\begin{aligned} w'' &= \frac{y}{2} w' + \lambda w \qquad \text{in } \ (0,\infty), \\ w'(0) &= -w^q(0), \end{aligned}$$

see [**FQ**]. Repeating **formally** these considerations also for $a > 0$, we get the same result if $p < 2q-1$, while for $p = 2q-1$ we obtain

$$(T-t)^{\lambda} u(1 - y\sqrt{T-t}, t) \to w_a(y) \qquad \text{as } \ t \to T,$$

where w_a is a positive solution of

$$\begin{aligned} w'' &= \frac{y}{2} w' + \lambda w + a w^p \qquad \text{in } \ (0,\infty), \\ w'(0) &= -w^q(0). \end{aligned}$$

The existence of such a solution for small $a > 0$ can be shown e.g. by investigation of critical points of the functional

$$E(v) := \int_0^{\infty} \Big(\frac{1}{2} \varrho v_y^2 + \frac{\lambda}{2} \varrho v^2 + \frac{a}{p+1} \varrho v^{p+1} \Big)\, dy - \frac{1}{q+1} v^{q+1}(0),$$

where $\varrho(y) = e^{-y^2/4}$ (cf. [**FQ**]). Notice also that this problem does not have positive solution for $a > 0$ large, since in this case we have $E'(v)v > 0$ for any $v \neq 0$.

Next we turn to the higher dimensional radially symmetric case.

Theorem 4.4. *Assume that $N > 1$, $\Omega = B_1(0)$ and*

$$p < 2q-1 \qquad \textit{or} \qquad p = 2q-1, \ a < q.$$

(i) *If u_o $(=u_o(r))$ is such that*

$$u_o'' + \frac{N-1}{r} u_o' - a u_o^p \geq 0 \qquad \text{for } r \in (0,1), \tag{4.18}$$

$$u_o'(0) = 0, \quad u_o'(1) = u_o^q(1), \tag{4.19}$$

$$u_o(1) > 1 \quad \text{and} \quad u_o(1) > \Big(\frac{2a}{p+1}\Big)^{\frac{1}{2q-p-1}} \ \text{if } p < 2q-1, \tag{4.20}$$

then u blows up in a finite time.

(ii) *Initial data u_o with the properties required in* (i) *exist.*

Proof. (i) According to the maximum principle it follows from (4.18),(4.19) that $u_t \geq 0$. On the other hand, (4.18) yields that

$$(r^{N-1}u_o')' \geq ar^{N-1}u^p \geq 0,$$

hence $u_o' \geq 0$ on $[0,1]$ because $u_o'(0) = 0$. The maximum principle implies then that $u_r \geq 0$.

We want to show that $U(t) := \int_0^1 u(r,t)\,dr$ satisfies an O.D.E. which has the property that all its positive solutions blow up in finite time. To do this we first derive some estimates.

$$\begin{aligned}\int_0^1 u_t u_r\,dr &\geq \int_0^1 \big(u_{rr}u_r - au^p u_r\big)\,dr \\ &= \int_0^1 \Big(\frac{1}{2}u_r^2 - \frac{a}{p+1}u^{p+1}\Big)_r\,dr \\ &\geq \frac{1}{2}u^{2q}(1,t) - \frac{a}{p+1}u^{p+1}(1,t).\end{aligned}$$

If $p = 2q-1$, $a < q$, then

$$\frac{1}{2}u^{2q}(1,t) - \frac{a}{p+1}u^{p+1}(1,t) = \delta u^{2q}(1,t), \qquad \delta := \frac{1}{2}\Big(1 - \frac{a}{q}\Big).$$

If $p < 2q-1$, then we use (4.20) to obtain

$$\frac{1}{2}u^{2q}(1,t) - \frac{a}{p+1}u^{p+1}(1,t) \geq \delta u^{2q}(1,t)$$

for some $\delta > 0$. In both cases

$$\int_0^1 u_t u_r\,dr \geq \delta u^{2q}(1,t). \tag{4.21}$$

On the other hand

(4.22)

$$\begin{aligned}\int_0^1 u_t u_r\,dr &\leq \frac{u^q(1,t)}{\varepsilon^{N-1}}\int_\varepsilon^1 u_t\,dr + \int_0^\varepsilon u_t\Big(\int_0^r u_t\,dr\Big)\,dr + \int_0^\varepsilon u_t\Big(\int_0^r au^p\,dr\Big)\,dr \\ &=: I_1 + I_2 + I_3,\end{aligned}$$

where we used the facts that

$$u_r \leq \frac{u_r(1,t)}{r^{N-1}}, \qquad \text{since} \quad \big(r^{N-1}u_r\big)_r = (u_t + au^p)r^{N-1} \geq 0,$$

and that

$$u_r = \int_0^r u_{rr}\,dr \leq \int_0^r (u_t + au^p)\,dr.$$

Put now

$$\varepsilon := u^{-\alpha}(1,t), \qquad \alpha = \frac{q-1-\eta}{N-1}, \quad 0 < \eta < q-1.$$

Then

$$\begin{aligned}
I_1 &\le u^{2q-1-\eta}(1,t) \int_0^1 u_t \, dr, \\
I_2 &\le \Big(\int_0^1 u_t \, dr \Big)^2, \\
I_3 &\le a\varepsilon u^p(1,t) \int_0^\varepsilon u_t \, dr \le a u^{p-\alpha}(1,t) \int_0^1 u_t \, dr \\
&\le a u^{2q-1-\alpha}(1,t) \int_0^1 u_t \, dr,
\end{aligned}$$

hence

$$\begin{aligned}
&I_1 + I_2 + I_3 \le \big(2Au^{2q-1-\xi}(1,t) + \int_0^1 u_t \, dr \big) \int_0^1 u_t \, dr, \\
&A := \max\{1, a\}, \qquad \xi := \min\{\alpha, \eta\}.
\end{aligned}$$

For t such that

$$2Au^{2q-1-\xi}(1,t) \le \int_0^1 u_t \, dr$$

we obtain

$$I_1 + I_2 + I_3 \le 2 \Big(\int_0^1 u_t \, dr \Big)^2.$$

If

$$2Au^{2q-1-\xi}(1,t) \ge \int_0^1 u_t \, dr,$$

then

$$I_1 + I_2 + I_3 \le 4Au^{2q-1-\xi}(1,t) \int_0^1 u_t \, dr.$$

In both cases we get from (4.21), (4.22) that

$$\frac{d}{dt} U(t) \ge \lambda U(t)^\mu$$

for some $\lambda > 0$, $\mu > 1$ which means that u must blow up in a finite time.

(ii) We show that there is a number $\tilde{a} > a$ such that a solution of the equation

$$u_o'' = \tilde{a} u_o^p, \qquad x \in (0,1) \tag{4.23}$$

satisfies (4.19), (4.20).

Consider first the case $p < 2q - 1$. By Lemma 3.7 we have $L_{(a)} \to 0$ as $a \to \infty$. Hence, for any $\tilde{a}$ large enough the equation

$$L_2(m; \tilde{a}) = 1$$

has a solution $\tilde{m}$. The value at 1 of the solution to (4.23), (4.19) which corresponds to $\tilde{m}$ is equal to $R_2(\tilde{m})$. But

$$R_2(\tilde{m}) > \left(\frac{\tilde{a}}{q}\right)^{\frac{1}{2q-p-1}}$$

and the assertion follows.

If $p = 2q - 1$, $a < q$, then the assertion follows from Lemma 3.7(iv). □

For general domains we have the following result.

Theorem 4.5. *Assume that $p \le q$. Then*

(i) *u blows up if $\Phi(u_o) < 0$.*

(ii) *If u is global, then*

$$\sup_{t\ge 0} \|u(\cdot,t)\|_{C(\overline{\Omega})} < \infty, \tag{4.24}$$

$$\sup_{t\ge 0} \|u(\cdot,t)\|_X < \infty, \tag{4.25}$$

provided

$$N \le 2 \quad \text{or} \quad N > 2,\ q < \frac{N}{N-2}. \tag{4.26}$$

In the case $p = q$ we assume in addition that $a \ne a_\Omega$.

(iii) *Assume that (4.26) holds. If $p = q$ and $a < a_\Omega$ then any solution blows up.*

(iv) *Assume that (4.26) holds. If $p = q$ assume in addition that $a > a_\Omega$. Then u blows up provided $u_o \ge v$, $u_o \not\equiv v$, v is any positive stationary solution.*

Remark 4.3. Under the assumption (4.26), the assertion (i) was proved already in [**E,** Theorem 1.1(a)]. The proof there is similar to ours, it is based on the classical concavity method (see [**L**]).

Proof of Theorem 4.5. We prove first the assertion (ii). Observe that (4.25) implies (4.24), since the trace operator $\mathrm{Tr} : W^{1,2}(\Omega) \to L^{q+1}(\partial\Omega)$ is continuous and according to [**Fo**]

$$\|u(\cdot,t)\|_{C(\overline{\Omega})} \le c(\|u_o\|_{C(\overline{\Omega})}, \sup_{0\le s\le t} \|u(\cdot,s)\|_{L^r(\partial\Omega)})$$

if $r > (q-1)(N-1)$, $N > 1$.

Exactly by the same reasoning as at the beginning of the proof of Theorem 4.2, it can be seen that we have only to prove that (4.8) leads to a contradiction.

To do this we proceed similarly as in [**F,** Lemma 1]. Put

$$M(t) := \int_0^t \int_\Omega u^2 \, dx\, dt.$$

Then

$$M'(t) = \int_\Omega u^2 dx = \int_0^t \int_\Omega (u^2)_t \, dx \, dt + \int_\Omega u_o^2 \, dx.$$

Assuming that $p < q$ and setting $\varepsilon := p - 1$ we obtain that

$$\frac{1}{2} M''(t) = -(2+\varepsilon)\Phi(u) + \frac{\varepsilon}{2} \int_\Omega |\nabla u|^2 dx + \frac{q-1-\varepsilon}{q+1} \int_{\partial\Omega} u^{q+1} dS. \tag{4.27}$$

In what follows, positive constants depending only on a, p, q, Ω, u_o will be denoted by c_i $(i = 1, 2, \dots)$. From (4.27) we get

$$M''(t) \geq c_1 \|u(\cdot, t)\|_X^2 - c_2, \tag{4.28}$$

therefore

$$M'(t) \to \infty \qquad \text{as} \quad t \to \infty. \tag{4.29}$$

On the other hand, using (4.10) we obtain from (4.27) that

$$M''(t) \geq 2\Big((2+\varepsilon) \int_0^t \int_\Omega u_t^2 \, dx \, dt + c_3 M'(t) - c_4\Big),$$

hence

$$\begin{aligned} MM'' - &\Big(1 + \frac{\varepsilon}{2}\Big) \big(M'\big)^2 \geq \\ &\geq 2(2+\varepsilon)\bigg(\Big(\int_0^t \int_\Omega u^2 \, dx \, dt\Big)\Big(\int_0^t \int_\Omega (u_t)^2 \, dx \, dt\Big) - \Big(\int_0^t \int_\Omega u u_t \, dx \, dt\Big)^2\bigg) + \\ &\qquad + 2M(c_3 M' - c_4) - c_5 M'. \end{aligned}$$

The first term on the right hand side is nonnegative according to the Cauchy inequality and the second one tends to infinity as $t \to \infty$. Thus, there is a $t_o \geq 0$ such that the right hand side is positive for $t \geq t_o$. This implies that

$$\big(M^{-\varepsilon/2}\big)'' < 0 \qquad \text{for} \quad t \geq t_o. \tag{4.30}$$

Since $M^{-\varepsilon/2}$ is decreasing, it must have a root – a contradiction. This proves (ii) for $p < q$.

Let us now prove (i) for $p < q$. From (4.27) and (4.10) we obtain that

$$M''(t) \geq -2(2+\varepsilon)\Phi(u_o).$$

This again yields (4.29), hence also (4.30) and (i) follows.

Now we turn to the case $p = q$. We again prove first (ii) showing that (4.8) leads to a contradiction. Choose $0 < \varepsilon < p - 1$ and set $\tilde{\varepsilon} = 1 - \dfrac{2+\varepsilon}{p+1}$. Then

$$\begin{aligned}\frac{1}{2}M''(t) &= -(2+\varepsilon)\Phi(u) + \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 dx - a\tilde{\varepsilon}\int_\Omega u^{p+1} dx + \tilde{\varepsilon}\int_{\partial\Omega} u^{p+1} dS \\ &= -(2+\varepsilon)\Phi(u) + \left(\frac{\varepsilon}{2} + \tilde{\varepsilon}\right)\int_\Omega |\nabla u|^2 dx + \frac{\tilde{\varepsilon}}{2}M''(t),\end{aligned}$$

hence

$$\frac{1-\tilde{\varepsilon}}{2}M''(t) \geq -(2+\varepsilon)\Phi(u) + \left(\frac{\varepsilon}{2} + \tilde{\varepsilon}\right)\int_\Omega |\nabla u|^2\, dx. \tag{4.31}$$

Using (4.10) we get

$$\frac{1-\tilde{\varepsilon}}{2}M''(t) \geq -(2+\varepsilon)\Phi(u_o) + (2+\varepsilon)\int_0^t \int_\Omega (u_t)^2 dx\, dt + \left(\frac{\varepsilon}{2} + \tilde{\varepsilon}\right)\int_\Omega |\nabla u|^2\, dx. \tag{4.32}$$

Our next aim is to show by contradiction that $M''(t) \to \infty$ as $t \to \infty$. Suppose that there exist a $c > 0$ and a sequence $\{t_n\}$, $t_n \to \infty$, such that

$$M''(t_n) \leq c \qquad \text{for} \quad n \in \mathbb{N}. \tag{4.33}$$

Set $d(t) := \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u(x,t)\, dx$ and write u in the form $u = d + u^\perp$, where $u^\perp$ belongs to the subspace of functions in X which are orthogonal to constants. From [**N,** Theorem 7.1] it follows that $(\int_\Omega |\nabla v|^2 dx)^{1/2}$ is an equivalent norm for v from this subspace. Thus, (4.32) yields that $\|u^\perp(\cdot, t_n)\|_X$ is bounded because

$$\int_\Omega |\nabla u^\perp|^2 dx = \int_\Omega |\nabla u|^2 dx.$$

Therefore (4.8) implies that $d(t_n) \to \infty$. Now (4.10) yields that $\Phi(u(\cdot, t_n))$ is bounded from above and according to (4.31) it is also bounded from below. Thus

$$\frac{\Phi\big(u(\cdot, t_n)\big)}{d(t_n)^{p+1}} \to 0. \tag{4.34}$$

Setting $v_n := \dfrac{u(\cdot, t_n)}{d(t_n)}$ we have that $v_n \to 1$ in X since

$$\|v_n - 1\|_X = \frac{\|u^\perp(\cdot, t_n)\|_X}{d(t_n)}.$$

Under our assumption on p, $X = W^{1,2}(\Omega)$ is continuously embedded into $L^{p+1}(\Omega)$ and the trace operator $\mathrm{Tr} : X \to L^{p+1}(\partial\Omega)$ is also continuous. Hence, (4.34) implies that

$$0 = \lim_{n\to\infty} \big(a \int_\Omega v_n^{p+1} dx - \int_{\partial\Omega} v_n^{p+1} dS\big) = a|\Omega| - |\partial\Omega|$$

what is a contradiction with our assumption on a.

Since $M''(t) \to \infty$ as $t \to \infty$, it is possible to find for any $C > 0$ a $\tau \geq 0$ such that

$$M'(t) \geq Ct, \qquad M(t) \geq Ct \qquad \text{for} \quad t \geq \tau. \tag{4.35}$$

From (4.32) it follows that

$$M''(t) \geq \big(2(2+\varepsilon) + c_6\big) \int_0^t \int_\Omega (u_t)^2 dx\, dt - c_7,$$

thus

$$MM'' - \Big(1 + \frac{\varepsilon}{2}\Big)\big(M'\big)^2 \geq M\big(c_6 \int_0^t \int_\Omega (u_t)^2 dx\, dt - c_7\big) - c_8 M'. \tag{4.36}$$

We want to show that the right hand side of (4.36) is positive for t large enough. To do this, we use the estimate

$$M'(t) = \int_\Omega u^2\, dx \leq 2\Big(\int_\Omega u_o^2\, dx + t \int_0^t \int_\Omega (u_t)^2 dx\, dt\Big) \tag{4.37}$$

which follows from the next simple observation:

$$u(x,t) = u_o(x) + \int_0^t u_t(x,s)\, ds \leq u_o(x) + \sqrt{t}\Big(\int_0^t \big(u_t(x,s)\big)^2 ds\Big)^{\frac{1}{2}}.$$

According to (4.37) we obtain

$$\begin{aligned} M\big(c_6 \int_0^t \int_\Omega (u_t)^2 - c_7\big) - c_8 M' &\geq \frac{c_9}{t} MM' - c_{10} M - c_8 M' \\ &= M'\Big(\frac{c_9}{2t} M - c_8\Big) + M\Big(\frac{c_9}{2t} M' - c_{10}\Big). \end{aligned}$$

Now (4.35) yields that the right hand side of (4.36) is positive for t large enough and the proof of (ii) for $p = q$ can be finished in the same way as in the case $p < q$.

To prove (i) for $p = q$ we recall (4.32). It implies now that

$$M''(t) \geq \big(2(2+\varepsilon) + c_6\big) \int_0^t \int_\Omega (u_t)^2 dx\, dt + c_7.$$

Hence, for any $C > 0$ there is a $\tau \geq 0$ such that

$$M(t) \geq Ct \qquad \text{for} \quad t \geq \tau. \tag{4.38}$$

The right hand side of (4.36) reads now

$$M\Big(c_6 \int_0^t \int_\Omega (u_t)^2 dx\, dt + c_7\Big) - c_8 M'$$

and using (4.37) we get

$$M\Big(c_6 \int_0^t \int_\Omega (u_t)^2 dx\, dt + c_7\Big) - c_8 M' \geq M\Big(c_7 - \frac{1}{t} c_6 \int_\Omega u_o^2\, dx\Big) + M'\Big(\frac{c_6}{2t} M - c_8\Big).$$

The estimate (4.38) ensures positivity of the last expression for large t. This completes the proof of (i) and (ii).

Proof of (iii). Suppose there is a global solution u. According to (ii) it is bounded, hence its ω–limit set consist of stationary solutions. But the only stationary solution is the unstable trivial solution (see Theorem 2.1(ii)), a contradiction.

To prove (iv) we argue similarly. If u were global then its ω–limit set would consist of stationary solutions. But according to Theorem 2.1(i) there is no stationary solution larger than v and v is unstable from above. $\square$

Remark 4.4. If $p \leq q$ then initial functions u_o with $\Phi(u_o) < 0$ always exist. If $p < q$ then for any v ($\operatorname{Tr} v \not\equiv 0$), $\Phi(\lambda v) < 0$ provided λ is large enough. If $p = q$ then we choose v such that

$$a \int_\Omega v^{p+1} dx < \int_{\partial\Omega} v^{p+1} dS.$$

Concerning global existence for general domains we have the following result.

Theorem 4.6. *Let p, q be such as in Theorem* 2.1(v)*. Then all solutions are global and bounded.*

Proof. Using the energy identity (4.10), we derive an apriori estimate for $\|u(\cdot,t)\|_X$. This implies (by the same argument as at the beginning of the proof of Theorem 4.5) that $\|u(\cdot,t)\|_{C(\overline{\Omega})}$ is bounded, too.

By Lemma 2.7 and Hölder inequality we have that for any $\varepsilon > 0$ small there exists a constant C_ε such that

$$\int_{\partial\Omega} u^{q+1} dS \leq \varepsilon \|u\|^2 + C_\varepsilon \|u\|_{q+1}^{p+1-\varepsilon} \leq \varepsilon\|u\|^2 + \varepsilon \|u\|_{p+1}^{p+1} + C_\varepsilon, \tag{4.39}$$

where $\|\cdot\|_r$ denotes the norm in $L^r(\Omega)$. Since Hölder inequality implies also

$$\|u\|_2^2 \leq \varepsilon \|u\|_{p+1}^{p+1} + C_\varepsilon,$$

we get using (4.39)

$$\Phi(u) \geq \Big(\frac{1}{2} - \varepsilon\Big)\|u\|^2 - C_\varepsilon. \tag{4.40}$$

Since $\Phi(u) \leq \Phi(u_o)$, the estimate (4.40) proves our assertion. □

Let us now turn to the interesting case $p = 2q - 1$, $a = q$.

Theorem 4.7. *Assume that $p = 2q - 1$, $a = q$, $N = 1$, $\Omega = (-1, 1)$. Then there exists a unique function w which satisfies the equation*

$$w'' - qw^{2q-1} = 0 \qquad in \;\; (-1,1) \tag{4.41}$$

together with the boundary condition

$$w(\pm 1) = \infty. \tag{4.42}$$

All nontrivial solutions of (4.1) *are global and tend pointwise to w as $t \to \infty$.*

Remark 4.5. It is known (see [**KN**] and the references there), that positive solutions of the problem

$$\triangle u = au^p \qquad x \in \Omega, \tag{4.43}$$

$$u = \infty \qquad x \in \partial\Omega \tag{4.44}$$

exist for $a > 0$, $p > 1$.

In [**KN**] it is shown that $u(x)$ behaves near $\partial\Omega$ like

$$\Big(\frac{a(p-1)^2}{2(p+1)}\Big)^{\frac{1}{1-p}} \big(\mathrm{dist}(x, \partial\Omega)\big)^{\frac{2}{1-p}}. \tag{4.45}$$

From this it follows that solutions to (4.43), (4.44) are not singular stationary solutions to the problem (1.1), except of the case $a = q$, $p = 2q - 1$, when $\dfrac{\partial u}{\partial n}$ behaves like u^q near $\partial\Omega$.

In [**KN**] also uniqueness of solutions to (4.43), (4.44) is shown for $p \geq 3$.

We prove Theorem 4.7 in the following series of lemmas.

Lemma 4.1. *There is a unique function w which satisfies* (4.41)*,* (4.42)*. Moreover,*

$$w(x) = w(-x) \qquad for \quad x \in [-1, 1]. \tag{4.46}$$

Proof. Denote by φ_α the solution of the initial value problem

$$\varphi'' = q\varphi^{2q-1}$$
$$\varphi(0) = \alpha > 0$$
$$\varphi'(0) = 0.$$

Then φ_α is given by the formula (cf. (3.5))

$$\int_\alpha^{\varphi_\alpha(x)} \frac{dv}{\sqrt{v^{2q}-\alpha^{2q}}} = |x|. \tag{4.47}$$

The function φ_α exists for $|x| \leq L_{\max}(\alpha) < \infty$, $L_{\max}(\alpha)$ is given by the formula

$$L_{\max}(\alpha) = \int_\alpha^\infty \frac{dv}{\sqrt{v^{2q}-\alpha^{2q}}} = \frac{1}{\alpha^{q-1}} \int_1^\infty \frac{dz}{\sqrt{z^{2q}-1}}.$$

$L_{\max}(\alpha)$ is decreasing, it tends to zero as $\alpha \to \infty$ and there is a unique α_o such that $L_{\max}(\alpha_o) = 1$, namely

$$\alpha_o = \left(\int_0^\infty \frac{dz}{\sqrt{z^{2q}-1}} \right)^{\frac{1}{q-1}}.$$

Hence, $w = \varphi_{\alpha_o}$ is the unique solution to (4.41), (4.42) with $w'(0) = 0$. The function φ_{α_o} obviously satisfies (4.46).

Suppose there is a nonsymmetric solution. Let m be its minimum attained at $0 \neq x_o \in (-1,1)$. Instead of (4.47) we obtain now the formula

$$\int_m^{\varphi_m(x)} \frac{dv}{\sqrt{v^{2q}-m^{2q}}} = |x - x_o|.$$

(4.42) implies that

$$\int_m^\infty \frac{dv}{\sqrt{v^{2q}-m^{2q}}} = |1 - x_o| = |-1 - x_o|$$

what is a contradiction. $\square$

Lemma 4.2. *Let p, q, a, N be as in Theorem 4.7 and let α_o, φ_α be as in Lemma 4.1. If $0 \leq u_o \leq w$, $u_o \not\equiv 0$, then there exist $\alpha_1, \alpha_2 \in (0, \alpha_o)$ and functions g_1, g_2 such that $\psi_i = \varphi_{\alpha_i} + g_i$ satisfy the conditions*

$$\psi_1(x) \leq u(x,t_o) \leq \psi_2(x) \qquad \textit{for some} \quad t_o > 0, \tag{4.48}$$

$$\psi_i'' - q\psi_i^{2q-1} \geq 0 \qquad \textit{in} \quad (-1,1), \tag{4.49}$$

$$\psi_i'(\pm 1) = \pm \psi_i^q(\pm 1), \tag{4.50}$$

$$\psi_i(x) = \psi_i(-x) \qquad \textit{for} \quad |x| \leq 1, \tag{4.51}$$

$$\psi_i'(x) \geq 0 \qquad \textit{for} \quad x \in [0,1]. \tag{4.52}$$

Proof. By the maximum principle

$$0 < u(x,t_o) < w(x) \qquad \text{for any} \quad t_o \in \big(0, t_{\max}(u_o)\big),$$

where $t_{\max}(u_o)$ is the maximal existence time. From (4.47) it is easily seen that $\varphi_\alpha(1) \to 0$ as $\alpha \to 0$. Therefore, there is a $\alpha_1 > 0$ such that $\varphi_{\alpha_1} < u(\cdot, t_o)$. On the other hand $\varphi_\alpha(1) \to \infty$ as $\alpha \to \alpha_o$ and $\varphi_\alpha(x) \to \varphi_{\alpha_o}(x)$ pointwise in $(-1,1)$, hence, there is a $\alpha_2 > 0$ such that $\varphi_{\alpha_2} > u(\cdot, t_o)$. The functions φ_{α_i} satisfy all conditions except of (4.50). We shall show that it is possible to find functions g_i such that $\psi_i = \varphi_{\alpha_i} + g_i$ satisfies (4.48)–(4.52). Observe first that $\varphi'_\alpha(1) < \varphi^q_\alpha(1)$ for any $\alpha \in (0, \alpha_o)$. This is easy to see if α is small, because then $\varphi_\alpha(1)$ is small, let us say $\varphi_\alpha(1) = \varepsilon$ and

$$\varphi'_\alpha(1) = \int_0^1 \varphi''_\alpha(x)\,dx < q\varepsilon^{2q-1} < \varepsilon^q = \varphi^q_\alpha(1).$$

But the mapping $\alpha \mapsto \varphi'_\alpha(1) - \varphi^q_\alpha(1)$ is continuous, therefore its values must be negative for all $\alpha \in (0, \alpha_o)$ since there is no β with $\varphi'_\beta(1) - \varphi^q_\beta(1) = 0$.

Set

$$g_{\eta,n}(x) = \begin{cases} 0 & \text{for} \quad |x| \le 1-\eta, \quad 0<\eta<1 \\ \big(|x| - 1 + \eta\big)^n & \text{for} \quad 1-\eta < |x| \le 1 \end{cases}$$

Taking $g_i = g_{\eta_i, n_i}$ with n_i sufficiently large and suitable η_i (η_i small), it is not difficult to check that the conditions (4.48)–(4.52) are satisfied. □

Lemma 4.3. *Assume that p, q, a, Ω are as in Theorem 4.7. Let*

$$u_o(x) = u_o(-x) \qquad \textit{for} \quad |x| \le 1,$$
$$u'_o(x) \ge 0 \qquad \textit{for} \quad x \in [0,1].$$

Assume further that

$$u(0,t) \le K \qquad \textit{on} \quad [0, t_{\max}(u_o)) \quad \textit{for some} \quad K > 0$$

and either

(i) $u_t \ge 0$ *in* $[-1,1] \times [0, t_{\max})$

or

(ii) *for any $t \in [0, t_{\max})$ there is a unique point $y(t) \in (0,1)$ such that*

$$u_t(x,t) < 0 \quad \textit{for} \quad 0 \le x < y(t), \qquad u_t(x,t) > 0 \quad \textit{for} \quad y(t) < x \le 1.$$

Then $t_{\max} = \infty$.

Proof. Consider the case (ii). Since $u_x(0,t) = 0$ and $u_x(x,t) \ge 0$ for $x \in (0,1]$, we have $u_{xx}(0,t) \ge 0$, hence

$$u_t(0,t) \ge -qu^{2q-1}(0,t) \ge -qK^{2q-1}.$$

By the maximum principle, there is a constant $c_1 > 0$ such that

$$u_t(x,t) \ge -c_1 \qquad \text{for} \quad |x| \le 1, \ t \in [0, t_{\max}).$$

Therefore, for $z \in (0,1)$ we get

$$\int_0^z u_t u_x \, dx \geq -c_1 \int_0^z u_x \, dx = -c_1 u(z,t). \tag{4.53}$$

Further,

$$\int_0^1 u_t u_x \, dx = \frac{1}{2}\int_0^1 \big(u_x^2 - u^{2q}\big)_x \, dx = \frac{1}{2}u^{2q}(0,t) \leq \frac{1}{2}K^{2q} \tag{4.54}$$

and

$$\int_z^1 u_t u_x \, dx = \int_0^1 u_t u_x \, dx - \int_0^z u_t u_x \, dx \leq c_2\big(1 + u(z,t)\big). \tag{4.55}$$

The inequalities (4.53)–(4.55) will be used to derive an apriori estimate of $u(1,t)$. Using (4.55) we get

$$\begin{aligned} u_x^2(z,t) &= u^{2q}(z,t) - 2\int_z^1 u_t u_x \, dx \geq u^{2q}(z,t) - 2c_2\big(u(z,t)+1\big) \\ &\geq \frac{1}{2}u^{2q}(z,t) - c_3. \end{aligned}$$

Hence

$$u_x(z,t) \geq \frac{1}{2}u^q(z,t) - c_4. \tag{4.56}$$

Using (4.54), (4.53) we get

$$\frac{1}{2}K^{2q} \geq \int_0^1 u_t u_x \, dx \geq -c_1 u\big(y(t),t\big) + \int_{y(t)}^1 u_t u_x \, dx. \tag{4.57}$$

By (4.56) we have

$$\int_{y(t)}^1 u_t u_x \, dx \geq \int_{y(t)}^1 \Big(\frac{1}{2}u^q - c_4\Big) u_t \, dx = \int_0^1 \Big(\frac{1}{2}u^q - c_4\Big) u_t \, dx - \int_0^{y(t)} \Big(\frac{1}{2}u^q - c_4\Big) u_t \, dx.$$

Since $-c_1 \leq u_t(x,t) \leq 0$ for $x \in [0, y(t))$, we obtain

$$\int_0^{y(t)} \Big(\frac{1}{2}u^q - c_4\Big) u_t \, dx \leq c_1 c_4$$

thus

$$\int_{y(t)}^1 u_t u_x \, dx \geq \frac{d}{dt}\int_0^1 \Big(\frac{u^{q+1}}{2(q+1)} - c_4 u\Big) \, dx - c_5. \tag{4.58}$$

Combining (4.57), (4.58) we get

$$\frac{d}{dt}\int_0^1 \Big(\frac{u^{q+1}}{2(q+1)} - c_4 u\Big)\,dx \le c_6\big(1+u(1,t)\big). \tag{4.59}$$

Integrating from 0 to $T < t_{\max}$ and taking into account that $u_t(1,t) > 0$, it follows that

$$\frac{1}{2(q+1)}\int_0^1 u^{q+1}(x,T)\,dx - c_4\int_0^1 u(x,T)\,dx \le c_6\big(1+u(1,T)\big)T + c_7. \tag{4.60}$$

By Hölder and Young inequalities

$$c_4\int_0^1 u(x,T)\,dx \le \eta\int_0^1 u^{q+1}(x,T)\,dx + c_\eta, \quad \eta > 0.$$

If we take $\eta < \dfrac{1}{2(q+1)}$, then (4.60) yields

$$\int_0^1 u^{q+1}(x,T)\,dx \le c_8\big(1+u(1,T)\big)T + c_9 \qquad \text{for } \ T \in (0, t_{\max}).$$

Suppose that $t_{\max} < \infty$. Then $u(1,t) \to \infty$ as $t \to t_{\max}$. Therefore, there is a $\tau \in (0, t_{\max})$ such that

$$\int_0^1 u^{q+1}(x,T)\,dx \le c_{10} T u(1,T) \qquad \text{for } \ T \in (\tau, t_{\max}). \tag{4.61}$$

Using (4.53), let us now estimate u_x from above in the following way:

$$\begin{aligned} u_x^2(z,t) &= u^{2q}(z,t) - 2\int_z^1 u_t u_x\,dx \le u^{2q}(1,t) + 2c_2\big(1+u(z,t)\big) \\ &\le 2u^{2q}(1,t) + c_{11}, \end{aligned}$$

hence

$$u_x(z,T) \le 2u^q(1,T)$$

if $u(1,T)$ is large enough. By the mean value theorem

$$u(1,T) - u(1-\varepsilon,T) = \varepsilon u_x(\xi,T) \le 2\varepsilon u^q(1,T),$$

thus

$$u(1-\varepsilon,T) \ge u(1,T) - 2\varepsilon u^q(1,T) \ge \frac{1}{2}u(1,T) \tag{4.62}$$

if $\varepsilon := \frac{1}{4}u^{1-q}(1,T)$. Using (4.61), (4.62) we obtain

$$\begin{aligned} c_{10}Tu(1,T) &\geq \int_0^1 u^{q+1}(x,T)\,dx \geq \int_{1-\varepsilon}^1 u^{q+1}(x,T)\,dx \\ &\geq \int_{1-\varepsilon}^1 \Big(\frac{u(1,T)}{2}\Big)^{q+1} dx = \frac{u^2(1,T)}{2^{q+3}}. \end{aligned}$$

This means that

$$u(1,T) \leq c_{11}T,$$

c_{11} does not depend on T. This is a contradiction.

In the case (i), the proof is slightly simpler. We only mention that $c_1 = 0$, hence (4.57), (4.58) are not needed to derive (4.59). The estimate

$$\frac{d}{dt}\int_0^1 \Big(\frac{u^{q+1}}{2(q+1)} - c_4 u\Big)\,dx \leq c_6$$

follows from (4.54), (4.56) in the following way:

$$\frac{1}{2}K^{2q} \geq \int_0^1 u_t u_x\,dx \geq \int_0^1 u_t\Big(\frac{1}{2}u^q - c_4\Big) = \frac{d}{dt}\int_0^1 \Big(\frac{u^{q+1}}{2(q+1)} - c_4 u\Big)\,dx.$$

□

Lemma 4.4 *Let* p, q, a, Ω *be as in Theorem* 4.7. *Then* u *is global and* $u(\cdot,t) \to w$ *pointwise as* $t \to \infty$, *provided* $0 \leq u_o \leq w$, $u_o \not\equiv 0$.

Proof. According to Lemma 4.2 we need only to prove that u is global and tends to w if $u_o = \varphi_\alpha + g_{\eta,n}$, $\alpha \in (0,\alpha_o)$ (with suitable η, n). We first show that the assumptions of Lemma 4.3 are fulfilled. We have

$$u(0,t) \leq \alpha_o \qquad \text{for } \ t \in (0, t_{\max})$$

since

$$u(x,t) \leq w(x) \qquad \text{for } \ |x| \leq 1,\ t \in [0, t_{\max}) \tag{4.63}$$

according to the maximum principle. We have also

$$u_t \geq 0 \qquad \text{in } \ [-1,1] \times [0, t_{\max}), \tag{4.64}$$

because

$$\begin{aligned} u_o'' - q u_o^{2q-1} &\geq 0 \qquad \text{in } \ (-1,1), \\ u_o'(\pm 1) &= \pm\, u_o^q(\pm 1). \end{aligned}$$

Hence, u is global.

From the fact that there are no stationary solutions it follows that u cannot be bounded. By (4.63), (4.64) the pointwise limit V exists in $(-1,1)$. But V must satisfy (4.41), (4.42), hence $V = w$. $\square$

Lemma 4.5. *Let p, q, a, Ω be as in Theorem 4.7. Then u is global and $u(\cdot, t) \to w$ pointwise as $t \to \infty$, provided $u_o = k + g_{\eta,n}$, where k is any positive constant and $g_{\eta,n}$ is from Lemma 4.2, η, n are suitably chosen.*

Proof. According to Lemma 4.4, we need only to consider $k > \alpha_o$. Obviously, $u_o'(x) \ge 0$ for $x \in [0,1]$ and it is not difficult to verify that there are η, n such that $u_o'' - qu_o^{2q-1}$ has exactly one sign change in $[0,1]$ and u_o satisfies the compatibility condition. From the maximum principle it follows that there is at most one sign change of $u_t(\cdot, t)$ in $(0,1)$ for $t \in (0, t_{\max})$. Take any $t_1 \in (0, t_{\max})$.

If $u_t(x, t_1) \ge 0$ for $x \in [0,1]$, then $u(\cdot, t_1) \le w$ by the maximum principle and Lemma 4.4 yields the assertion.

If $u_t(x, t_1) \le 0$ for $x \in [0,1]$, then $u_t \le 0$ in $[0,1] \times [t_1, t_{\max})$, hence $t_{\max} = \infty$ and u tends to a stationary solution from above. But the only stationary solution is 0 and 0 is unstable from above, a contradiction.

We only need to consider the case when there is a function $y(t)$ such that

$$u_t(x,t) < 0 \quad \text{for} \quad x < y(t), \qquad u_t(x,t) > 0 \quad \text{for} \quad x > y(t), \qquad t \in [0, t_{\max}).$$

Lemma 4.3(ii) can be applied if we show that

$$u(0,t) \le K \qquad \text{on} \quad [0, t_{\max}) \quad \text{for some} \quad K > 0.$$

Take any $t_o < t_{\max}$ and choose x_o such that $w(x_o) > u(1,t)$ for $t \le t_o$. By the maximum principle

$$\max_{\substack{0 \le x \le x_o \\ 0 \le t \le t_o}} (u - w) = \max_{\substack{x=0 \\ 0 \le t \le t_o}} (u - w).$$

If $u(0,t) - w(0) = \max_{\tau \le t} \big(u(0,\tau) - w(0) \big)$, then we use the fact that $u_x(0,t) - w_x(0) = 0$ which implies $u_{xx}(0,t) - w_{xx}(0) \le 0$ and

$$u_{xx}(0,t) \le w_{xx}(0) = qw^{2q-1}(0).$$

This yields

$$u_t(0,t) = u_{xx}(0,t) - qu^{2q-1}(0,t) \le q\big(w^{2q-1}(0) - u^{2q-1}(0,t) \big) < 0.$$

This means that $\max_{t \le t_o} \big(u(0,t) - w(0) \big) = u_o(0) - w(0) = k - \alpha_o$, therefore $u(0,t) \le k$ for $t \in [0, t_{\max})$. $\square$

Proof of Theorem 4.7. Theorem 4.7 is an immediate consequence of Lemma 4.5, since for any $u_o \ge 0$, $u_o \not\equiv 0$ and any t_o small, there are constants $k_1, k_2 > 0$ and functions g_1, g_2 as in Lemma 4.2 such that $k_1 + g_1 \le u(\cdot, t_o) \le k_2 + g_2$. $\square$

5. Convergence to equilibria

The aim of this section is to study the problem (1.1) from the point of view of dynamical systems.

The solution starting from u_o will be often denoted by $u(t, u_o)$, by $\|\cdot\|$ we mean the norm in $X = W^{1,2}(\Omega)$.

Proposition 5.1. *The problem* (1.1) *defines a compact local semiflow in* $C^+ = \{v \in W^{1,2}(\Omega)\,;\, v \geq 0$ *a.e.*$\}$ *provided* $N = 1, 2$ *or* $N > 2$, $q < N/(N-2)$, $p < (N+2)/(N-2)$. *This local semiflow is monotone in the following sense: if* $u_o \leq \tilde{u}_o$ *a.e.,* $u_o \not\equiv \tilde{u}_o$, *then* $u(t, u_o) < u(t, \tilde{u}_o)$ *in* $\overline{\Omega}$ *for any* $t \in (0, t_{\max}(\tilde{u}_o))$. *If* $N = 1$, *this is the strong monotonicity* ($u(t, \tilde{u}_o) - u(t, u_o)$ *lies in the interior of* C^+*).*

Proof. A straightforward modification of [**A1,** Lemma 14.3] in virtue of [**A1,** Remark 14.7(b)] shows that [**A1,** Theorem 12.3] is applicable for $W^{\beta}_{\mathcal{B}} = W^{1,2}(\Omega)$, i.e. (1.1) (where $u^p := |u|^{p-1}u$ and $u^q := |u|^{q-1}u$) defines a local semiflow in $W^{1,2}(\Omega)$. Moreover, a repeated use of the variation-of-constants formula [**A1,** Corollary 12.2] with suitable $W^{\beta}_{\mathcal{B}} = W^{1,r_i}(\Omega)$, $2 =: r_o < r_1 < \cdots < r_m =: r > N$ shows the continuity and boundedness on bounded sets of $u(t,\cdot) : W^{1,r_i}(\Omega) \to W^{1+\varepsilon,r_i}(\Omega) \hookrightarrow W^{1,r_{i+1}}(\Omega)$, hence $u(t,\cdot) : W^{1,2}(\Omega) \to W^{1,r}(\Omega)$ is continuous and the flow in $W^{1,2}(\Omega)$ is compact.

Suppose now $u_o \leq \tilde{u}_o$. It can be easily shown that we may find $u_n, \tilde{u}_n \in C^2(\overline{\Omega})$ such that $u_n \leq \tilde{u}_n$, $u_n \to u_o$, $\tilde{u}_n \to \tilde{u}_o$ in $W^{1,2}(\Omega)$ and $u_n, \tilde{u}_n$ fulfil the boundary condition $\dfrac{\partial u}{\partial n} = |u|^{q-1}u$. Using the maximum principle we get $u(t, u_n) \leq u(t, \tilde{u}_n)$ for t small enough. The continuous dependence on initial values implies now $u(t, u_o) \leq u(t, \tilde{u}_o)$.

If $u_o \geq 0$, $u_o \not\equiv 0$, then $u_1 := u(\tau, u_o) \in W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ is nonnegative and $u_1 \not\equiv 0$ for $\tau > 0$ small enough, hence we may find $\varphi_1 \in \mathcal{D}(\Omega)$ (a smooth function with compact support in Ω) such that $\varphi_1 \not\equiv 0$, $0 \leq \varphi_1 \leq u_1$. The maximum principle implies $u(t, \varphi_1) > 0$ for $t \in (0, t_{\max}(\varphi_1))$, hence

$$u(t + \tau, u_o) = u(t, u_1) \geq u(t, \varphi_1) > 0.$$

Consequently, $u(t, u_o)$ is positive for $t \in (0, t_{\max}(u_o))$ and [**A3,** Corollary 9.3 or 9.4] implies that $u(\cdot, u_o)$ is a classical solution of (1.1) for $t \in (0, t_{\max}(u_o))$.

Finally, let $0 \leq u_o \leq \tilde{u}_o$, $u_o \not\equiv \tilde{u}_o$. Then we have $u(\tau, u_o) \leq u(\tau, \tilde{u}_o)$ and $u(\tau, u_o) \not\equiv u(\tau, \tilde{u}_o)$ for τ small enough. Moreover, $u(\cdot, u_o), u(\cdot, \tilde{u}_o)$ are classical solutions, hence the maximum principle implies $u(t + \tau, u_o) < u(t + \tau, \tilde{u}_o)$ on the time interval where both solutions exist. (4.10) implies then that $t_{\max}(u_o) \geq t_{\max}(\tilde{u}_o)$ and the assertion follows. $\square$

Remark 5.1. The problem (1.1) defines a strongly monotone compact local semiflow in $C^+ \cap W^{1,r}(\Omega)$ for any $r > N$.

Theorem 5.1. *Denote the set of initial nonnegative data for which the solutions exist globally by G. Then G is star-shaped with respect to zero. Moreover, G is closed in C^+ provided one of the following conditions holds:*

(i) $p < q$ *and* $q < \dfrac{N+1}{N-1}$ *if* $N > 1$,

(ii) $p = q < \min\Big(2, \dfrac{N+2}{N}\Big)$.

Proof. The fact that G is star-shaped follows from Proposition 5.1. To prove that G is closed we proceed by contradiction. Suppose that $u(t, u_o)$ blows up in a finite time T and that there is a sequence $\{u_n\} \subset G$, $u_n \to u_o$ in X. By continuous dependence on initial values, it is possible to choose for any $K > 0$ a $t_1 < T$ and an n_o such that

$$\|u(t_1, u_{n_o})\| = \max_{t \le t_1} \|u(t, u_{n_o})\| > K. \tag{5.1}$$

Differentiating the equation with respect to t, multiplying it by u_t and integrating, we obtain

(5.2)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2\,dx = \int_\Omega u_{tt}u_t\,dx = -\int_\Omega |\nabla u_t|^2 dx - ap\int_\Omega u^{p-1}u_t^2\,dx + q\int_{\partial\Omega} u^{q-1}u_t^2\,dS.$$

By Hölder inequality we have

(5.3)

$$\int_{\partial\Omega} u^{q-1}u_t^2\,dS \le \Big(\int_{\partial\Omega} u_t^{2s}\,dS\Big)^{\frac{1}{s}}\Big(\int_{\partial\Omega} u^{s'(q-1)}\,dS\Big)^{\frac{1}{s'}} \qquad \text{for } s, s' > 1,\ \frac{1}{s} + \frac{1}{s'} = 1.$$

Consider first the case $N > 1$ or $N = 1$ and $p = q$.

If $N > 1$, set $s := \dfrac{N-1}{N-2\theta}$, $\theta := \frac{1}{2} + \varepsilon$. Then the trace operator $\mathrm{Tr} : W^{\theta,2}(\Omega) \to L^{2s}(\partial\Omega)$ is continuous, $s' = \dfrac{N-1}{2\theta - 1}$ and the trace operator $\mathrm{Tr} : W^{1,2}(\Omega) \to L^{s'(q-1)}(\partial\Omega)$ is continuous provided $N = 2$ or $N > 2$, $s'(q-1) \le 2\dfrac{N-1}{N-2}$, i.e. $\theta \ge \dfrac{1}{2} + \dfrac{(N-2)(q-1)}{4}$. Hence, if $N > 2$, then we take $\varepsilon = \dfrac{(N-2)(q-1)}{4}$.

If $N = 1$, then we choose arbitrary $s > 1$.

With this choice of s, ε, we obtain

$$\int_{\partial\Omega} u^{q-1}u_t^2\,dS \le c_1\|u\|^{q-1}\|u_t\|_{\theta,2}^2 \le c_2\|u\|^{q-1}\|u_t\|^{2\theta}\|u_t\|_2^{2(1-\theta)},$$

where $\|\cdot\|_{\theta,2}$ or $\|\cdot\|_2$ denotes the norm in $W^{\theta,2}(\Omega)$ or $L^2(\Omega)$, respectively. Using Young inequality we obtain

$$\int_{\partial\Omega} u^{q-1}u_t^2\,dS \le c_2\|u\|^{q-1}\Big(\eta\|u_t\|^2 + \eta^{-\frac{\theta}{1-\theta}}\int_\Omega u_t^2\,dx\Big).$$

Now (5.2) yields that

(5.4)
$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2\,dx &\le -\|u_t\|^2+\int_\Omega u_t^2\,dx+c_2\|u\|^{q-1}\Big(\eta\|u_t\|^2+\eta^{-\frac{\theta}{1-\theta}}\int_\Omega u_t^2\,dx\Big)\\ &\le -\frac{1}{2}\|u_t\|^2+c_3\|u\|^{\frac{q-1}{1-\theta}}\int_\Omega u_t^2\,dx\end{aligned}$$

if we put $\eta := \big(2c_2\|u\|^{q-1}\big)^{-1}$. Since u is global, we know from Theorem 4.5(i) that $\Phi\big(u(\cdot,t)\big)\ge 0$ for $t\ge 0$, hence

$$\int_0^t\int_\Omega u_t^2\,dx\,dt=\Phi\big(u(\cdot,0)\big)-\Phi\big(u(\cdot,t)\big)\le\Phi\big(u(\cdot,0)\big). \tag{5.5}$$

Therefore, integrating (5.4) we get

$$\int_0^\tau\|u_t(\cdot,t)\|^2\,dt+\int_\Omega u_t^2(x,\tau)\,dx\le c_4\Big(\max_{t\le\tau}\|u(\cdot,t)\|^{\frac{q-1}{1-\theta}}+1\Big). \tag{5.6}$$

Estimating both $\int_\Omega u^2dx$ and $\int_\Omega|\nabla u|^2dx$ as in (4.37) we see that

$$\|u(\cdot,\tau)\|^2\le 2\Big(\|u(\cdot,0)\|^2+\tau\int_0^\tau\|u_t(\cdot,t)\|^2dt\Big).$$

Hence, (5.6) yields

$$\|u(\cdot,\tau)\|^2\le 2\tau c_4\Big(\max_{t\le\tau}\|u(\cdot,t)\|^{\frac{q-1}{1-\theta}}+1\Big)+c_5. \tag{5.7}$$

In the case $p=q$ it is easy to check that $\dfrac{q-1}{1-\theta}<2$ if $N=1,2$ and $\varepsilon<\dfrac{2-q}{2}$ or if $N>2$, $\varepsilon=\dfrac{(N-2)(q-1)}{4}$. Therefore (5.1), (5.7) yield a contradiction if we choose K large enough and $\tau=t_1$, $u(\cdot,0)=u_{n_o}$.

In the case $N>1$, $p<q$, we proceed slightly differently. According to (4.28)

$$\|u\|^2\le c_6M''+c_7=2c_6\int_\Omega uu_t\,dx+c_7. \tag{5.8}$$

By Hölder inequality, (4.37) and (5.5), we obtain

$$\begin{aligned}\int_\Omega uu_t\,dx &\le\sqrt{2}\Big(\int_\Omega u_o^2\,dx+t\int_0^t\int_\Omega u_t^2\,dx\,dt\Big)^{1/2}\Big(\int_\Omega u_t^2\,dx\Big)^{1/2}\\ &\le(c_8+c_9t)^{1/2}\Big(\int_\Omega u_t^2\,dx\Big)^{1/2}.\end{aligned}$$

Now (5.8) implies that

$$\|u(\cdot,\tau)\|^2 \le c_{10}\Big(\int_\Omega u_t^2(x,\tau)\,dx\Big)^{1/2} + c_7, \tag{5.9}$$

where c_{10} depends on τ. But taking $u(\cdot,0)=u_{n_o}$, $\tau=t_1$ we obtain from (5.6), (5.9) that

$$\|u(t_1,u_{n_o})\|^2 \le c_{11}\Big(\|u(t_1,u_{n_o})\|^{\frac{q-1}{2(1-\theta)}}+1\Big) \tag{5.10}$$

what is a contradiction to (5.1), since $\dfrac{q-1}{2(1-\theta)}<2$ under the assumption $N>1$, $p<q<\dfrac{N+1}{N-1}$.

Consider now the case $N=1$, $p<q$. From (4.10) it follows that

$$c_{12}\|u\| \le \frac12\int_\Omega |\nabla u|^2\,dx + \frac{a}{p+1}\int_\Omega u^{p+1}\,dx \le \Phi(u_o) + \frac{1}{q+1}\int_{\partial\Omega} u^{q+1}\,dS$$

hence

$$\|u(\cdot,t)\| \le c_{13}\Big(\sup_{x\in\overline{\Omega}} u(x,t)+1\Big).$$

This means that for K large enough $\max\limits_{0\le t\le t_1}\|u(t,u_{n_o})\|_{C(\overline{\Omega})}$ cannot be attained for $t=0$. Therefore, there is a $t_2\in(0,t_1]$ for which

$$\max_{\substack{x\in\overline{\Omega}\\ 0\le t\le t_1}} u(x,t) = \max_{x\in\overline{\Omega}} u(x,t_2) = \max_{x\in\partial\Omega} u(x,t_2) =: U.$$

If $U > \Big(a\dfrac{q+1}{p+1}|\Omega|\Big)^{\frac{1}{q-p}}$ then there is a $\delta\in(0,1)$ such that

$$\frac{a}{p+1}\int_\Omega u^{p+1}(x,t_2)\,dx \le \frac{\delta}{q+1}\int_{\partial\Omega} u^{q+1}(x,t_2)\,dS. \tag{5.10a}$$

Since $\Phi(u)\ge 0$ by Theorem 4.5(i), we obtain from (5.10a) that

$$\frac{1-\delta}{q+1}\int_{\partial\Omega} u^{q+1}(x,t_2)\,dS \le \frac12\int_\Omega |\nabla u(x,t_2)|^2\,dx. \tag{5.11}$$

Taking $s'=\dfrac{q+1}{q-1}$, (5.11) yields

$$\begin{aligned}\Big(\int_{\partial\Omega} u^{s'(q-1)}(x,t)\,dS\Big)^{\frac{1}{s'}} &\le c_{14}\Big(\int_\Omega |\nabla u(x,t_2)|^2\,dx\Big)^{\frac{q-1}{q+1}}\\ &\le c_{14}\max_{t\le t_1}\|u(\cdot,t)\|^{2\frac{q-1}{q+1}} \qquad \text{for } t\le t_1.\end{aligned}$$

By (5.3) we have

$$\int_{\partial\Omega} u^{q-1}u_t^2\,dS \le c_{14}\Big(\int_{\partial\Omega} u_t^{q+1}\,dS\Big)^{\frac{2}{q+1}} \max_{t\le t_1}\|u(\cdot,t)\|^{2\frac{q-1}{q+1}}$$
$$\le c_{15}\|u_t\|_{\theta,2}^2 \max_{t\le t_1}\|u(\cdot,t)\|^{2\frac{q-1}{q+1}}, \quad \theta = \frac{1}{2}+\varepsilon.$$

Instead of (5.10) we obtain now by the same arguments as before that

$$\|u(t_1,u_{n_o})\|^2 \le c_{16}\Big(\|u(t_1,u_{n_o})\|^{\frac{1}{1-\theta}\frac{q-1}{q+1}} + 1\Big),$$

where $\dfrac{1}{1-\theta}\dfrac{q-1}{q+1} < 2$ if $\varepsilon < \dfrac{1}{q+1}$ and we arrive at a contradiction.

The proof is trivial if $U \le \Big(a\dfrac{q+1}{p+1}|\Omega|\Big)^{\frac{1}{q-p}}$, since then

$$\int_{\partial\Omega} u^{s'(q-1)}\,dS \le c_{17}.$$

□

Theorem 5.2. *Assume that $N=1$ and $p<q$. Then for any $u_o \in C^+$, $u_o \not\equiv 0$ on $\partial\Omega$, there is a $\lambda_o > 0$ such that $u(t,\lambda u_o) \to 0$ in X as $t\to\infty$ for $\lambda<\lambda_o$; $u(t,\lambda_o u_o)$ tends to a positive stationary solution as $t\to\infty$, while $u(t,\lambda u_o)$ blows up in finite time for $\lambda>\lambda_o$.*

Proof. Set
$$\lambda_o = \sup\{\lambda>0\,;\, u(t,\lambda u_o) \text{ exists globally}\}.$$

From Theorem 4.5(i) and Remark 4.2 it follows that $\lambda_o<\infty$. Set

$$\lambda_1 = \sup\{\lambda>0\,;\, u(t,\lambda u_o)\to 0 \ \text{ in } X \ \text{ as } t\to\infty\}.$$

According to Theorem 2.1(i), zero is a stable stationary solution, therefore $\lambda_1 > 0$. Obviously, $\lambda_1 \le \lambda_o$. The domain of attraction of 0 is open in X, hence $u(t,\lambda_1 u_o)$ cannot tend to zero. By Theorem 5.1, $u(t,\lambda_1 u_o)$ is global and according to Theorem 4.5(ii), it is bounded in X. Therefore the ω–limit set $\omega(\lambda_1 u_o)$ is nonempty and consists of positive stationary solutions. Since $\omega(\lambda_1 u_o)$ is connected and the positive stationary solutions are isolated (Theorems 3.3(i) and 3.4(ii)), $\omega(\lambda_1 u_o) = \{v\}$, where v is a positive stationary solution.

The proof will be finished if we show that $\lambda_1 = \lambda_o$.

Suppose $\lambda_1 < \lambda_o$. By Theorem 5.1, $u(t,\lambda_o u_o)$ is global, hence $u(t,\lambda_o u_o)\to w$ as $t\to\infty$, where w is a positive stationary solution. Since

$$u(t,\lambda_o u_o) > u(t,\lambda_1 u_o) \quad \text{for} \ \ t>0, \tag{5.13}$$

we have that $w = v$, because any two positive stationary solutions must intersect (Theorem 2.1(i)). The stationary solution v is hyperbolic (Theorem 3.5) and unstable (Theorem 3.6(i)). Its stable manifold $W^s(v)$ is an immersed submanifold of X with codimension ≥ 1, therefore it cannot contain an open set – a contradiction to (5.13). (For the existence of the local (un)stable manifold see [**S,** Theorem 5.2], for the globalization see [**H,** Theorem 6.1.9].) $\square$

Let v_1 and v_2 be stationary solutions. We say that v_1 connects to v_2, iff there is an orbit $\{u(t)\,;\, t \in \mathbb{R}\}$ such that $u(t) \to v_1$ in X as $t \to -\infty$, $u(t) \to v_2$ in X as $t \to +\infty$.

For semilinear parabolic equations with homogeneous Dirichlet or Neumann boundary conditions, the connecting orbits problem was solved completely in [**BF1**],[**BF2**]. But nonlinear boundary conditions were not considered there.

Theorem 5.3. *Assume that $N = 1$, $p \leq q$ and $a > a_1$. Let v_1 denote the symmetric positive stationary solution and v_2, v_3 the nonsymmetric positive stationary solutions. Then*

(i) *v_i connects to 0 for $i = 1, 2, 3$;*
(ii) *v_1 connects to v_2 and v_3.*

Proof. (i) follows from [**M,** Theorem 8] and Theorem 2.1(i).

To prove (ii) we first recall that $M^-(v_1) = 2$ (Theorem 3.6(i)) and that there exists an orbit w lying in the unstable manifold $W^u(v_1)$ such that w blows up in a finite time T (see the proof of Theorem 4.2(ii)).

Let $\Gamma = \{\gamma(s)\,;\, s \in [0,1]\}$ be a Jordan curve in $W^u(v_1)$ around v_1 such that $\gamma(0) = \gamma(1) =: \gamma_o$ lies on the orbit which connects v_1 to 0. Set

$$s_o := \sup\{s\,;\, u(t, \gamma(\sigma)) \to 0 \text{ for } \sigma \in [0, s]\}.$$

Then $s_o > 0$, because the domain of attraction of 0 is open. There is a $s_1 \in (0,1)$ such that $\gamma(s_1) \in w$, hence $s_o < s_1$. By Theorem 5.1, $u(t, \gamma(s_o))$ exists for $t \geq 0$ and $u(t, \gamma(s_o))$ converges to a stationary solution v_o. The semiflow is gradient like with respect to the functional Φ, therefore $v_o \neq v_1$. This means that v_1 connects to v_2 or v_3. But if $\{u(x,t)\,;\, t \in \mathbb{R}\}$ is an orbit connecting v_1 to v_2, then $\{u(-x,t)\,;\, t \in \mathbb{R}\}$ connects v_1 to v_3. $\square$

Theorem 5.4. *Assume that $N = 1$, $q < p < 2q - 1$, $a > a_o$. Let u_1 denote the smaller symmetric positive stationary solution. Then the following holds.*

(i) *Any positive stationary solution v ($v \not\equiv u_1$) connects to u_1.*
(ii) *Let v be a stationary solution, $v > u_1$. If $0 \leq u_o \leq v$, $u_o \not\equiv 0$, $u_o \not\equiv v$, then $u(t, u_o) \to u_1$.*
(iii) *Let all nonsymmetric positive stationary solutions be hyperbolic. Then for any $u_o \in C^+$, $u_o > 0$, there is a $\lambda_o > 0$ such that $u(t, \lambda u_o) \to u_1$ as $t \to \infty$ for $\lambda < \lambda_o$; while $u(t, \lambda u_o)$ blows up in finite time for $\lambda > \lambda_o$.*

Remark 5.2. The nonsymmetric positive stationary solutions are hyperbolic if e.g. $p \leq 4$ or $p > 4$, $q \geq p - 1 - \frac{1}{p-2}$ (see Theorem 3.5 and Lemma 3.6).

Proof of Theorem 5.4. (i) Since u_1 is stable (see the proof of Theorem 3.6(i)), the assertion follows from Proposition 3.1 and [**M,** Theorem 8].

(ii) is an immediate consequence of (i).

To prove (iii) we define λ_o, λ_1 as in the proof of Theorem 5.2. Since the set of stationary solutions is bounded in $L^\infty(\Omega)$ (cf. (2.29), (3.52)), Theorem 4.2(ii) implies that $\lambda_o < \infty$. Obviously, $\lambda_1 \leq \lambda_o$ and according to (ii), $\lambda_1 > 0$.

Suppose that $\lambda_1 < \lambda_o$. Take $\lambda_2 \in (\lambda_1, \lambda_o)$. Then $u(t, \lambda_1 u_o)$ and $u(t, \lambda_2 u_o)$ converge to the same stationary solution by Proposition 3.1. But this stationary solution is unstable and hyperbolic. Hence, we arrive at a contradiction as in the proof of Theorem 5.2. ☐

We formulate our next (and last) result as a remark, since we only indicate some possible proofs. To give a complete proof is out of the scope of this paper.

Remark 5.3. *If $N = 1$, p, q are as in Lemma* 3.6 *and $a > a_1$ then the larger symmetric stationary solution u_2 connects to both of the nonsymmetric stationary solutions.*

There are several possibilities to prove this fact. We shall sketch two of them.

(i) $\partial W^s(u_1)$ (the boundary of the domain of attraction of the smaller symmetric stationary solution u_1) is an invariant Lipschitz manifold with codimension one. (This might be shown in the same way as Theorem 5.5 in [**P**], see also [**T,** Propositions 1.2 and 1.3].) From Theorem 5.4(i) it follows that $u_2 \in \partial W^s(u_1)$. Since $W^u(u_2)$ contains initial data for which blow up occurs and also initial data for which the corresponding solutions tend to u_1 and $\dim W^u(u_2) = 2$, there is a $u_o \in W^u(u_2) \cap \partial W^s(u_1)$, $u_o \not\equiv u_2$. Analogously as in [**P,** Theorem 5.4(v)] it could be shown that any two functions in $\partial W^s(u_1)$ cross each other. Hence, if we denote by $z(f)$ the number of sign changes (zero number) of $f \in C([-l, l])$, then $z(u_o - u_2) \geq 1$. But we also know that

$$\lim_{t \to -\infty} z\big(u(t, u_o) - u_2\big) = 1,$$

since

$$\lim_{t \to -\infty} \frac{u(t, u_o) - u_2}{\|u(t, u_o) - u_2\|} = \pm\varphi_2,$$

φ_2 being the second eigenfunction of the linearized (at u_2) stationary problem (which is a Sturm–Liouville problem). It is well known that $z\big(u(t, u_o) - u_2\big)$ is nonincreasing in t for $t \in (-\infty, T)$, $T \leq \infty$ being the maximal existence time. Therefore

$$z\big(u(t, u_o) - u_2\big) = 1 \qquad \text{for } \ t \in (-\infty, T). \tag{5.14}$$

Assume that

$$u(-l, t; u_o) > u_2(-l), \qquad u(l, t; u_o) < u_2(l) \tag{5.15}$$

for t close to $-\infty$. From (5.14) it follows that

$$u(l,t;u_o) \leq u_2(l) \qquad \text{for } t \in (-\infty, T). \tag{5.16}$$

Let v_1 be the nonsymmetric stationary solution with

$$v_1(-l) > u_2(-l), \qquad v_1(l) < u_2(l).$$

Since $z(v_1 - u_2) = 1$, we have $z\big(u(t,u_o) - v_1\big) = 1$ for t close to $-\infty$. But then

$$u(-l,t;u_o) \leq v_1(-l) \qquad \text{for } t \in (-\infty, T) \tag{5.17}$$

according to the nonincrease of z.

(5.16), (5.17) and (4.10) imply that $\|u(t,u_o)\|$ is bounded for $t \in (-\infty, T)$ which means that $T = \infty$ and $u(t,u_o) \to v_1$ as $t \to \infty$.

If the inequalities in (5.15) are reversed, we can argue exactly as before to prove that u_2 connects to v_2, $v_2(x) = v_1(-x)$.

(ii) Another possibility to prove the connections is to use the y–map (see [**BF1,** Section 2]). This tool enables us to show the existence of an orbit $\{w(t)\,;\, t \in (-\infty, T)\}$ lying in $W^u(u_2)$ with $z\big(w(t) - u_2\big) = 1$ for $t \in (-\infty, T)$. As before, we can conclude that $T = \infty$ and $w(t)$ converges to a nonsymmetric stationary solution as $t \to \infty$.

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