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A NOTE ON CHARACTERIZATIONS OF COMPACTNESS

D. S. JANKOVIĆ and C. KONSTADILAKI-SAVVOPOULOU

It is well known that a function f from a topological space X into a compact space Y is continuous if the graph of f, i.e. the set $\{(x, f(x)) : x \in X\}$, is a closed subset of the product space $X \times Y$. J. Joseph [**3**] showed that the converse is true for T_1 spaces. A T_1 space Y is compact if for every space X every function $f : X \to Y$ with a closed graph is continuous. This result was originally obtained in [**1**], [**6**], and [**4**]. In this note we improve the mentioned result and also offer some new characterizations of compact T_1 spaces. In doing so, we utilize the concept of somewhat nearly continuous functions recently introduced by Z.Piotrowski [**5**]. A function $f : X \to Y$ is called **somewhat nearly continuous** if $\mathrm{Int}_X \mathrm{Cl}_X f^{-1}(V) \neq \emptyset$ for every nonvoid open set V in Y ($\mathrm{Cl}_X A$ and $\mathrm{Int}_X A$ denote the closure and interior of a set A in a space X, respectively). Recall now that a function $f : X \to Y$ is **locally closed** [**2**] if for every $x \in X$ and for every neighbourhood U of x there exists a neighbourhood V of x such that $V \subset U$ and $f(V)$ is closed. A space X is called **hyperconnected** [**7**] if every two nonempty open subsets of X have a nonempty intersection.

Theorem. *The following statements are equivalent for a T_1 space Y.*

(a) *Y is compact.*
(b) *Every function from a T_1 space into Y with all inverse images of compact sets closed is somewhat nearly continuous.*
(c) *Every closed graph function from a T_1 space into Y is somewhat nearly continuous.*
(d) *Every locally closed function from a T_1 space into Y with all point inverses closed is somewhat nearly continuous.*
(e) *Every locally closed bijection from a T_1 space onto Y is somewhat nearly continuous.*

Proof. It is clearly that if $f : X \to Y$ is a function with all inverse images of compact sets closed and Y is compact, then f is continuous. Thus, (a) implies (b). In [**2**] it is shown that the inverse images of compact sets under closed graph functions are closed. Hence (b) implies (c). Since locally closed functions with all point inverses closed have closed graphs [**2**], (c) implies (d). Obviously, (d) implies (e). To show that (e) implies (a), suppose that a T_1 space $Y = (Y, \tau)$ is

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not compact. Then there exists a filter base $\mathcal{F}$ of closed sets in Y with $\cap\mathcal{F} = \emptyset$. Let $\mathcal{B} = \{F \cup \{x\}: x \in Y \text{ and } F \in \mathcal{F}\}$. It is not difficult to check that $\mathcal{B}$ is an open base for a topology τ^* on Y. Let $x, y \in Y$ with $x \neq y$. Sine $\cap\mathcal{F} = \emptyset$, there exists an $F \in \mathcal{F}$ with $y \notin F \cup \{x\}$. Therefore, the space $Y^* = (Y, \tau^*)$ is T_1. Since each pair of elements of $\mathcal{B}$ has a nonempty intersection, Y^* is hyperconnected. Let $f: Y^* \to Y$ be the identity function, let $x \in Y$,and let U be a neighbourhood of x in Y^*. Clearly, there exists an $F \in \mathcal{F}$ such that $F \cup \{x\} \subset U$. Since F is closed in Y and Y is T_1, $F \cup \{x\}$ is closed in Y. Thus, $f(F \cup \{x\})$ is closed in Y. This shows that f is locally closed. By hypothesis, f is somewhat nearly continuous and so $\mathrm{Int}_{Y^*}\mathrm{Cl}_{Y^*} f^{-1}(V) \neq \emptyset$ for every nonvoid $V \in \tau$. Let $F \in \mathcal{F}$ with $F \neq Y$. Then F is closed in Y, and consequently, $\mathrm{Cl}_{Y^*}\mathrm{Int}_{Y*} f^{-1}(F) \neq Y$. This implies $\mathrm{Cl}_{Y^*} F \neq Y$ since F is open in Y^*, but $\mathrm{Cl}_{Y^*} F = Y$ since Y^* is hyperconnected. This contradition completes the proof. □

Corollary. *A T_1 space Y is compact if and only if each locally closed bijection from a T_1 space onto Y is continuous.*

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D. S. Janković, Department of Mathematics, East Central University, Ada, Oklahoma 74820, U.S.A.

Ch. Konstadilaki-Savvopoulou, Department of Mathematics, Faculty of Sciences, Aristotle University of Thessaloniki, 54006 Thessaloniki, Greece