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A THEOREM CONCERNING SYSTEMS OF RESIDUE CLASSES

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We first introduce some notation. As usual $(n_1, \dots, n_k)$ (resp. $[n_1, \dots, n_k]$) stands for the greatest common divisor (resp. least common multiple) of $n_1, \dots, n_k$. By system we mean a multi-set whose elements are unordered but may occur repeatedly. Following Š. Znám [**8**] we use $a(n)$ to denote the residue class

$$\{\, x \in \mathbb{Z} \colon x \equiv a \pmod{n} \,\} \ .$$

For a system

$$A = \{a_i(n_i)\}_{i=1}^k \tag{1}$$

of residue classes, the n_i are called its moduli.

Definition. An integer T is said to be a covering period of (1) if it is a period of the characteristic function of the set $\bigcup_{i=1}^k a_i(n_i)$.

It is clear that $[n_1, \dots, n_k]$ is a covering period of (1), and that any covering period is a multiple of the smallest positive one.

For any set S of integers we use $d(S)$ to denote the asymptotic density

$$\lim_{N \to \infty} \frac{1}{N} \left|\{\, 0 \le x < N \colon x \in S \,\}\right| \ .$$

($|A|$ is the cardinality of A.) The limit obviously exists if S is a union of finitely many residue classes. In fact

$$d\left(\bigcup_{i=1}^k a_i(n_i)\right) = \frac{1}{N} \left|\{\, 0 \le x < N \colon x \in a_i(n_i) \text{ for some } i \,\}\right|$$

where N is any positive common multiple of $n_1, \dots, n_k$.

Our main result is

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Theorem. *Let T be the smallest positive covering period of* (1). *Then we have*

$$\frac{(n_1,\dots,n_k)}{(T,n_1,\dots,n_k)} \le \max_{n\in\mathbb{Z}^+} |\{\,1\le i\le k\colon n_i=n\,\}| \sum_{d\mid\frac{[n_1,\dots,n_k]}{(n_1,\dots,n_k)}} \frac{1}{d}\,. \tag{2}$$

To prove it we need two lemmas.

Lemma 1. $d\left(\bigcup_{i=1}^k a_i(n_i)\right) \ge d\left(\bigcup_{i=1}^k 0(n_i)\right)$.

This is Lemma 2.3 of R.J.Simpson [**6**]. We can also prove it by using Theorem 1 of [**2**].

Lemma 2. *Let $n_1,\dots,n_k\in\mathbb{Z}^+$, and let P be a finite set of primes such that all the n_i are contained in*

$$\overline{P}=\{\,n\in\mathbb{Z}^+\colon \textit{all prime divisors of } n \textit{ belong to } P\,\}\ .$$

Then

$$d\left(\bigcup_{i=1}^k 0(n_i)\right)=\left(\prod_{p\in P}\frac{p-1}{p}\right)\sum_{n\in\overline{P}\cap\bigcup_{i=1}^k 0(n_i)}\frac{1}{n}\ .$$

Proof. We note first that

$$\sum_{n\in\overline{P}\cap\bigcup_{i=1}^k 0(n_i)}\frac{1}{n}\le\sum_{n\in\overline{P}}\frac{1}{n}=\prod_{p\in P}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right)=\prod_{p\in P}\frac{p}{p-1}\ .$$

Let $N=[n_1,\dots,n_k]$ and $N_m=\left(\prod_{p\in P}p\right)^m$. For sufficiently large m we have $N|N_m$. From the inclusion-exclusion principle it follows that

$$\begin{aligned}
d\left(\bigcup_{i=1}^k 0(n_i)\right)&=\frac{1}{N}\left|\left\{\,0\le x<N\colon x\in\bigcup_{i=1}^k 0(n_i)\,\right\}\right|\\
&=\frac{1}{N}\left(\sum_{i=1}^k\Big|\{\,0\le x<N\colon n_i|x\,\}\Big|-\sum_{1\le i<j\le k}\Big|\{\,0\le x<N\colon [n_i,n_j]|x\,\}\Big|+\cdots\right.\\
&\qquad\left.+(-1)^{k-1}\Big|\{\,0\le x<N\colon [n_1,\dots,n_k]|x\,\}\Big|\right)\\
&=\frac{1}{N}\left(\sum_{i=1}^k\frac{N}{n_i}-\sum_{1\le i<j\le k}\frac{N}{[n_i,n_j]}+\cdots+(-1)^{k-1}\frac{N}{[n_1,\dots,n_k]}\right)
\end{aligned}$$

$$= \frac{1}{\prod_{p\in P}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right)}\left(\sum_{i=1}^{k}\frac{1}{n_i}\sum_{d\in\overline{P}}\frac{1}{d}-\sum_{1\le i<j\le k}\frac{1}{[n_i,n_j]}\sum_{d\in\overline{P}}\frac{1}{d}+\cdots\right.$$

$$\left.+(-1)^{k-1}\frac{1}{[n_1,\ldots,n_k]}\sum_{d\in\overline{P}}\frac{1}{d}\right)$$

$$=\prod_{p\in P}\left(1-\frac{1}{p}\right)\lim_{m\to\infty}\left(\sum_{i=1}^{k}\frac{1}{n_i}\sum_{d|\frac{N_m}{n_i}}\frac{1}{d}-\sum_{1\le i<j\le k}\frac{1}{[n_i,n_j]}\sum_{d|\frac{N_m}{[n_i,n_j]}}\frac{1}{d}+\cdots\right.$$

$$\left.+(-1)^{k-1}\frac{1}{[n_1,\ldots,n_k]}\sum_{d|\frac{N_m}{[n_1,\ldots,n_k]}}\frac{1}{d}\right)$$

$$=\left(\prod_{p\in P}\frac{p-1}{p}\right)\lim_{m\to\infty}\left(\sum_{i=1}^{k}\sum_{n_i|n|N_m}\frac{1}{n}-\sum_{1\le i<j\le k}\sum_{[n_i,n_j]|n|N_m}\frac{1}{n}+\cdots\right.$$

$$\left.+(-1)^{k-1}\sum_{[n_1,\ldots,n_k]|n|N_m}\frac{1}{n}\right)$$

($d|n|m$ stands for "$d|n$ and $n|m$ ")

$$=\left(\prod_{p\in P}\frac{p-1}{p}\right)\lim_{m\to\infty}\sum_{\substack{n_i|n|N_m\\ \text{for some } i}}\frac{1}{n}=\left(\prod_{p\in P}\frac{p-1}{p}\right)\sum_{n\in\bigcup_{i=1}^{k}0(n_i)\cap\overline{P}}\frac{1}{n}\,.$$

This concludes the proof. □

Proof of Theorem. Since T is a covering period (1), we have

$$\begin{aligned}\bigcup_{i=1}^{k}a_i(n_i) &= \left\{z+Ty\colon z\in\bigcup_{i=1}^{k}a_i(n_i)\text{ and }y\in\mathbb{Z}\right\}\\ &=\bigcup_{i=1}^{k}\{\,a_i+n_ix+Ty\colon x,y\in\mathbb{Z}\,\}=\bigcup_{i=1}^{k}a_i((n_i,T))\,.\end{aligned}$$

Let S denote the set $\{n_1,\ldots,n_k\}$ and P be the set of all prime divisors of $[n_1,\ldots,n_k]$. Since

$$\frac{(n_1,\ldots,n_k)}{(T,n_1,\ldots,n_k)}=\frac{[(n_1,\ldots,n_k),T]}{T}\quad\text{and}\quad\frac{n_i}{(T,n_i)}=\frac{[n_i,T]}{T}\ ,$$

we have

$$\frac{(n_1,\ldots,n_k)}{(T,n_1,\ldots,n_k)}\ \Bigg|\ \frac{n_i}{(T,n_i)}$$

and hence
$$\frac{n_i}{(n_1,\dots,n_k)/(T,n_1,\dots,n_k)} \in 0((T,n_i))\cap \overline{P}\ .$$

Obviously, $\frac{[n_1,\dots,n_k]}{(n_1,\dots,n_k)}$ can be written in the form $\prod_{p\in P} p^{\delta_p}$ where $\delta_p \geq 0$. And it is clear that
$$|ord_p n_i - ord_p n_j| \leq ord_p[n_1,\dots,n_k] - ord_p(n_1,\dots,n_k) = \delta_p\ .$$
(We use $ord_p n$ to denote the greatest integer α such that p^α divides n.) So, if $n,n'\in S$ and
$$n\prod_{p\in P} p^{k_p(1+\delta_p)} = n'\prod_{p\in P} p^{l_p(1+\delta_p)}\ ,$$
then $k_p = l_p$ for all $p\in P$ and hence $n=n'$.

Let $M = \max_{n\in\mathbb{Z}^+} |\{\,1\leq i\leq k\colon n_i = n\,\}|$. From Lemmas 1,2 and the above, we have

$$\begin{aligned}
\sum_{i=1}^k \frac{1}{n_i} &= \sum_{i=1}^k d(a_i(n_i)) \geq d\left(\bigcup_{i=1}^k a_i(n_i)\right) = d\left(\bigcup_{i=1}^k a_i((n_i,T))\right)\\
&\geq d\left(\bigcup_{i=1}^k 0((n_i,T))\right) = \left(\prod_{p\in P}\frac{p-1}{p}\right)\sum_{m\in \bigcup_{i=1}^k 0((n_i,T))\cap\overline{P}} \frac{1}{m}\\
&\geq \left(\prod_{p\in P}\frac{p-1}{p}\right)\sum_{n\in S}\left(\frac{n}{(n_1,\dots,n_k)/(T,n_1,\dots,n_k)}\right)^{-1}\\
&\quad\cdot\prod_{p\in P}\left(1+\frac{1}{p^{1+\delta_p}}+\frac{1}{p^{2(1+\delta_p)}}+\cdots\right)\\
&= \frac{(n_1,\dots,n_k)}{(T,n_1,\dots,n_k)}\left(\prod_{p\in P}\frac{p-1}{p}\cdot\frac{1}{1-\frac{1}{p^{1+\delta_p}}}\right)\sum_{n\in S}\frac{1}{n}\\
&= \frac{(n_1,\dots,n_k)}{(T,n_1,\dots,n_k)}\left(\frac{1}{M}\prod_{p\in P}\frac{p^{\delta_p}}{1+p+\cdots+p^{\delta_p}}\right)\sum_{n\in S}\frac{M}{n}\\
&\geq \frac{1}{M}\cdot\frac{(n_1,\dots,n_k)}{(T,n_1,\dots,n_k)}\prod_{p\in P}\frac{1}{1+\frac{1}{p}+\cdots+\frac{1}{p^{\delta_p}}}\sum_{i=1}^k\frac{1}{n_i}\ .
\end{aligned}$$

Therefore
$$\frac{(n_1,\dots,n_k)}{(T,n_1,\dots,n_k)} \leq M\prod_{p\in P}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{\delta_p}}\right) = M\sum_{d\mid\frac{[n_1,\dots,n_k]}{(n_1,\dots,n_k)}}\frac{1}{d}\ ,$$

which is the desired result. □

Remark 1. By checking the proof we see that (2) is implied by

$$\sum_{i=1}^{k}\frac{1}{n_i}\geq d\left(\bigcup_{i=1}^{k}0((n_i,T))\right) \tag{3}$$

which holds if T is a covering period of (1).

We now say a few words about the theorem. If $(n_1,\dots,n_k)|T$ then (2) holds trivially. Note that (2) can be written in the form

$$\frac{1}{(T,n_1,\dots,n_k)}\leq \max_{n\in\mathbb{Z}^+}|\{\,1\leq i\leq k\colon n_i=n\,\}|\sum_{(n_1,\dots,n_k)|d|[n_1,\dots,n_k]}\frac{1}{d} \tag{2'}$$

which is implied by

$$\sum_{i=1}^{k}\frac{1}{n_i}\geq\frac{1}{(T,n_1,\dots,n_k)}\ . \tag{4}$$

If $T|(n_1,\dots,n_k)$ then (4) holds, for

$$\sum_{i=1}^{k}\frac{1}{n_i}\geq d\left(\bigcup_{i=1}^{k}a_i(n_i)\right)\geq d(a_1(T))=\frac{1}{(T,n_1,\dots,n_k)}\ .$$

However (4) fails to hold in general, for example, the smallest positive covering period of $\{0(2),0(3)\}$ is $T=6$, but $\frac{1}{2}+\frac{1}{3}\not\geq\frac{1}{(6,2,3)}$.

Corollary. *Let n_0 be the smallest positive covering period of* (1)*, and* $[n_1,\dots,n_k]$ *have the prime factorization*

$$[n_1,\dots,n_k]=\prod_{i=1}^{r}p_i^{\alpha_i}\ ,\qquad p_1<p_2<\cdots<p_r\ .$$

Suppose that $p_t^{\alpha}\nmid n_0$ and $p_t^{\alpha}|n_s$ for some $s=1,\dots,k$, and that $a_i(n_i)\cap a_j(n_j)=\emptyset$ whenever $p_t^{\alpha}|n_i$ and $p_t^{\alpha}\nmid n_j$ $(1\leq i,j\leq k)$. Then we have

$$p_t^{\delta_t(\alpha)}\leq\varepsilon_t(\alpha)\max_{\substack{1\leq s\leq k\\ p_t^{\alpha}|n_s}}|\{\,1\leq i\leq k\colon n_i=n_s\,\}|\prod_{i=1}^{r}\frac{p_i}{p_i-1}\ , \tag{5}$$

where

$$\delta_t(\alpha)=\min\{\,\delta\geq 1\colon p_t^{\alpha-\delta}\|n_i\quad\text{for some}\quad 0\leq i\leq k\,\}$$

$(p^{\alpha}\|n$ *stands for* "$p^{\alpha}|n$ *and* $p^{\alpha+1}\nmid n$" .)

and

$$\varepsilon_t(\alpha) = \left(1 - \frac{1}{p_t^{\alpha_t-\alpha+1}}\right) \prod_{\substack{i=1\\ i\neq t}}^{r} \left(1 - \frac{1}{p_i^{\alpha_i+1}}\right) .$$

Proof. Let $I = \{\, 1 \le i \le k \colon p_t^{\alpha} | n_i \,\}$ and $J = \{0, 1, \dots, k\} - I$. Obviously $I \neq \emptyset$, $0 \in J$ and $p_t^{\alpha} \not| n_j$ for every $j \in J$. If $i \in I$ and $j \in J - \{0\}$ then $a_i(n_i) \cap a_j(n_j) = \emptyset$. From this it follows that

$$x \in \bigcup_{i\in I} a_i(n_i) \text{ implies } x \pm [n_j]_{j\in J} \in \bigcup_{i=1}^{k} a_i(n_i) - \bigcup_{j\in J-\{0\}} a_j(n_j) = \bigcup_{i\in I} a_i(n_i) .$$

Hence the smallest positive covering period of $\{a_i(n_i)\}_{i\in I}$ must be a divisor of $[n_j]_{j\in J}$.

Applying the theorem we get

$$\frac{(n_i)_{i\in I}}{((n_i)_{i\in I}, [n_j]_{j\in J})} \le \max_{s\in I} |\{\, 1 \le i \le k \colon n_i = n_s \,\}| \sum_{d | \frac{[n_i]_{i\in I}}{(n_i)_{i\in I}}} \frac{1}{d} . \tag{6}$$

(Notice that $i \in I$ if $1 \le i \le k$ and $n_i = n_s$ for some $s \in I$.) Since $p_t^{\alpha} | (n_i)_{i\in I}$ we have

$$\frac{\left[[n_j]_{j\in J}, p_t^{\alpha}\right]}{[n_j]_{j\in J}} \;\Bigg|\; \frac{\left[[n_j]_{j\in J}, (n_i)_{i\in I}\right]}{[n_j]_{j\in J}}$$

and thus the left side of (6) is a multiple of $p_t^{\alpha}/(p_t^{\alpha}, [n_j]_{j\in J}) = p_t^{\delta_t(\alpha)}$. As for the right side of (6), we note that

$$\begin{aligned}
\sum_{d | \frac{[n_i]_{i\in I}}{(n_i)_{i\in I}}} \frac{1}{d} \le \sum_{d | \frac{[n_i]_{i\in I}}{p_t^{\alpha}}} \frac{1}{d} &\le \sum_{d | p_t^{\alpha_t-\alpha} \prod_{i=1, i\neq t}^{r} p_i^{\alpha_i}} \frac{1}{d} \\
&= \left(\left(1 + \frac{1}{p_t} + \frac{1}{p_t^2} + \cdots\right) - \frac{1}{p_t^{\alpha_t-\alpha+1}}\left(1 + \frac{1}{p_t} + \frac{1}{p_t^2} + \cdots\right)\right) \\
&\quad \cdot \prod_{\substack{i=1\\ i\neq t}}^{r} \left(\left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots\right) - \frac{1}{p_i^{\alpha_i+1}}\left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots\right)\right) \\
&= \varepsilon_t(\alpha) \prod_{i=1}^{r} \frac{p_i}{p_i - 1} .
\end{aligned}$$

Combining the above we obtain (5) from (6). □

Remark 2. $1 \le \delta_t(\alpha) \le \alpha$, $0 < \varepsilon_t(\alpha) < 1$.

Suppose that (1) is a disjoint system (i.e. $a_1(n_1), \dots, a_k(n_k)$ are pairwise disjoint). If $p_r^{\alpha_r}$ does not divide (the smallest positive covering period) n_0, then by the corollary we have

$$p_r^{\delta_r(\alpha_r)} \le \max_{\substack{1\le s\le k\\ p_r^{\alpha_r}\|n_s}} |\{\,1 \le i \le k\colon n_i = n_s\,\}| \prod_{i=1}^{r-1} \frac{p_i}{p_i-1}\,. \tag{7}$$

(Note that $\varepsilon_r(\alpha_r) \le \frac{p_r-1}{p_r}$.) This is the first result announced in Sun [**7**].

Assume that each modulus of the disjoint system (1) occurs at most M times (i.e.
$|\{\,1 \le i \le k\colon n_i = n_s\,\}| \le M$ for every $s = 1, \dots, k$). By Merten's theorem (cf.[**5**]), we have

$$\frac{1}{x}\prod_{\substack{p<x\\ p \text{ prime}}} \frac{p}{p-1} \sim e^{\gamma}\frac{\ln x}{x} \qquad \text{where } \gamma \text{ is the Euler constant}\,,$$

and thus

$$\frac{1}{x}\prod_{\substack{p<x\\ p \text{ prime}}} \frac{p}{p-1} < \frac{1}{M} \qquad \text{for sufficiently large } x\,.$$

Let p^* be the smallest prime such that

$$p^* > M \prod_{\substack{p<p^*\\ p \text{ prime}}} \frac{p}{p-1}\,.$$

If $p_r^{\alpha_r} \nmid n_0$, in view of (7), we have

$$p_r \le M \prod_{i=1}^{r-1} \frac{p_i}{p_i-1} \le M \prod_{\substack{p<p_r\\ p \text{ prime}}} \frac{p}{p-1}\,,$$

and hence p^* is an upper bound of prime divisors of $n_1, \dots, n_k$. If $p_r \ge p^*$ we must have $p_r^{\alpha_r}\|n_0$.

Now let's suppose the disjoint system (1) is also a covering, that is to say, (1) is a disjoint covering system (i.e. $a_i(n_i)$, $1 \le i \le k$, form a partition of $\mathbb{Z}$). By the corollary,

$$p_t \le p_t^{\delta_t(\alpha)} < M \prod_{i=1}^{r} \frac{p_i}{p_i-1} \qquad \text{for all} \quad t = 1, \dots, r \quad \text{and} \quad \alpha = 1, \dots, \alpha_t\,.$$

(Notice that $n_0 = 1$ and $\varepsilon_t(\alpha) < 1$.) This establishes Burshtein's conjecture ([**4**]). (The original conjecture is that $p_r \le M \prod_{i=1}^{r} \frac{p_i}{p_i-1}$.)

Let $1 \le t \le r$,

$$\delta_t = \delta_t(\alpha_t) = \min\{\,\delta \ge 1\colon p_t^{\alpha_t-\delta}\|n_i \quad \text{for some} \quad 0 \le i \le k\,\}$$

and

$$M_t = \begin{cases} 1 + \left[p_t^{\delta_t} \prod\limits_{\substack{i=1 \\ i \neq t}}^{r} \frac{p_i - 1}{p_i} \right] & \text{if } r > 1, \\ p_t^{\delta_t} & \text{if } r = 1. \end{cases}$$

($[\cdot]$ is the greatest integer function.) In [**3**] Berger, Felzenbaum and Fraenkel showed that

$$M \geq 1 + \left[(p_t - 1) \prod_{\substack{i=1 \\ i \neq t}}^{r} \frac{p_i - 1}{p_i} \right], \quad \text{i.e.} \quad p_t \prod_{i=1}^{r} \frac{p_i - 1}{p_i} < M \ .$$

In [**6**] R.J.Simpson proved that

$$M \geq p_r \prod_{i=1}^{r-1} \frac{p_i - 1}{p_i} \ ,$$

and then he derived that there exists a number $B(M)$ such that, in any disjoint covering system whose moduli are repeated at most M times, the least modulus is less that $B(M)$. It is obvious that

$$M_r \geq p_r \prod_{i=1}^{r-1} \frac{p_i - 1}{p_i} \ .$$

Given $1 \leq t \leq r$, (since $\varepsilon_t(\alpha_t) < \frac{p_t - 1}{p_t}$ if $r > 1$, and $\varepsilon_t(\alpha_t) = \frac{p_t - 1}{p_t}$ if $r = 1$) we have from the corollary $M \geq M_t$, moreover there exists a modulus divided by $p_t^{\alpha_t}$ and not by $p_t^{\alpha_t + 1}$ which is repeated at least M_t times. If $r \geq 2$ then

$$M_r > p_r \prod_{i=1}^{r-1} \frac{p_i - 1}{p_i} \geq p_{r-1} \prod_{i=1}^{r-2} \frac{p_i - 1}{p_i} \geq \cdots \geq p_2 \frac{p_1 - 1}{p_1}$$

and thus

$$M \geq \left[p_2 (1 - p_1^{-1}) \right] + 1 \ .$$

The last inequality was first proved by Berger, Felzenbaum and Fraenkel [**1**]. There something was said about which modulus must occur at least $[p_2(1 - p_1^{-1})] + 1$ times.

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