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HILBERT–SPACE–VALUED MEASURES ON BOOLEAN ALGEBRAS (EXTENSIONS)

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Abstract. We prove that if B_1 is a Boolean subalgebra of B_2 and if $m\colon B_1 \to H$ is a bounded finitely additive measure, where H is a Hilbert space, then m admits an extension over B_2. This result generalizes the well-known result for real-valued measures (see e.g. [**1**]). Then we consider orthogonal measures as a generalization of two-valued measures. We show that the latter result remains valid for $\dim H < \infty$. If $\dim H = \infty$, we are only able to prove a weaker result: If B_1 is a Boolean subalgebra of B_2 and $m\colon B_1 \to H$ is an orthogonal measure, then we can find a Hilbert space K such that $H \subset K$ and such that there is an orthogonal measure $k\colon B_2 \to K$ with $k/B_1 = m$.

Notions and Results

Definition 1. Let B be a Boolean algebra and let H be a real Hilbert space. A mapping $m\colon B \to H$ is called a measure if the following two conditions are satisfied (the symbol $\| \ \|$ stands for the norm of H induced by the scalar product $\langle \cdot, \cdot \rangle$):

(i) $\sup_{a \in B} \|m(a)\| < \infty$,
(ii) if $a, b \in B$ and $a \wedge b = 0$, then $m(a \vee b) = m(a) + m(b)$.

Obviously, if $H = \mathbb{R}$ then we obtain an ordinary real-valued bounded measure. For a given Boolean algebra B and a given Hilbert space H, let us denote by $\mathcal{M}(B,H)$ the set of all measures on B ranging in H, and let us denote by $\mathcal{M}_c(B,H)$ the set of all $m \in \mathcal{M}(B,H)$ such that $\sup_{a \in B} \|m(a)\| \le c$. The following simple proposition says that the set $\mathcal{M}_c(B,H)$ $(c \ge 0)$ is quite large. (Recall that two elements $a_1, a_2 \in B$ are called disjoint if $a_1 \wedge a_2 = 0$.)

Proposition 2. *Suppose that we are given a nonnegative $c \in \mathbb{R}$ and suppose that $\{a_1, a_2, \dots, a_n\}$ is a subset of B consisting of mutually disjoint nonzero elements in B. Suppose further that $\{v_1, v_2, \dots, v_n\}$ is a subset of H such that $\| \sum_{i \le n} \varepsilon_i v_i \| \le c$ for any choice of $\varepsilon_i \in \{0,1\}$ $(i \le n)$. Then there exists a measure $m \in \mathcal{M}_c(B,H)$ such that $m(a_i) = v_i$. Moreover, if the vectors v_i $(i \le n)$*

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are mutually orthogonal then m can be chosen so that $\langle m(a), m(b)\rangle = 0$ for any disjoint pair $a, b \in B$.

Proof. We may (and shall) assume that B is a collection of subset of a set A. Viewing B this way, the set $\{a_1, a_2, \ldots, a_n\}$ becomes a set of mutually disjoint nonvoid subsets of A, say $\{A_1, A_2, \ldots, A_n\}$. Choose points $p_i \in A_i$ $(i \leq n)$ and write, for any subset D of A which belongs to B, $P(D) = \{v_i \,|\, p_i \in D\}$. If we put $m(D) = \sum_{i \in P(D)} v_i$, then m is the desired measure. □

The following proposition sheds light on the topological structure of $\mathcal{M}_c(B, H)$.

Proposition 3. *Suppose that w denotes the weak topology of H (thus, w is the weakest topology making all the functions $\langle \cdot, x\rangle$ $(x \in H)$ continuous). If we understand $\mathcal{M}_c(B, H)$ $(c \geq 0)$ as a subset of the (topological) product $(H, w)^B$, then $\mathcal{M}_c(B, H)$ becomes a compact space.*

Proof. Let us denote by $S_c(0)$ the closed ball in H with the radius c and with the centre in 0. As known (see e.g. [**2**]), the set $S_c(0)$ is compact in the topology w. Since $\mathcal{M}_c(B, H)$ is a subset of the topological product $(S_c(0), w)^B$ and since $(S_c(0), w)^B$ is compact (Tychonoff's theorem), we only have to verify that $\mathcal{M}_c(B, H)$ is closed in $(S_c(0), w)^B$. But the pointwise limits of (finitely additive) measures obviously remain measures. This completes the proof of Proposition 3.□

Theorem 4. *Let B_1 be a Boolean subalgebra of B_2 and let H be a real Hilbert space. Let $m\colon B_1 \to H$ be a measure. Then m admits an extension over B_2.*

Proof. Let us denote by $\mathcal{P}$ the set of all finite partitions of B_1. (By a partition we mean a set $P = \{p_1, p_2, \ldots, p_n\}$ such that p_i $(i \leq n)$ are mutually disjoint and $\vee_{i \leq n} p_i = 1$.) Let us consider the set $\mathcal{P}$ together with the refinement relation $\leq$. Thus, we put $P \leq R$ $(P, R \in \mathcal{P})$ if for any $p \in \mathcal{P}$ there exists $r \in R$ such that $p \leq r$. Obviously, the relation $\leq$ is a partial ordering of $\mathcal{P}$ and we may view $\mathcal{P}$ as a partial ordered set.

Put $c = \sup_{a \in B_1} \|m(a)\|$ and set, for any $P = \{p_1, p_2, \ldots, p_n\} \in \mathcal{P}$, $\mathcal{F}_P = \{\overline{m} \in \mathcal{M}_c(B_2, H) \,|\, \overline{m}(p_i) = m(p_i)$ for any i, $i \leq n\}$. By Prop. 3, $\mathcal{F}_P \neq \emptyset$ for any $P \in \mathcal{P}$. Put further $\mathcal{F} = \{\mathcal{F}_P | P \in \mathcal{P}\}$. We shall show that $\mathcal{F}$ is a filter base consisting of closed sets in $\mathcal{M}_c(B_2, H)$.
Indeed, since $\mathcal{M}_c(B_2, H)$ is considered with the weak topology, every $\mathcal{F}_P$ is closed in $\mathcal{M}_c(B_2, H)$. Further, suppose that $P, R \in \mathcal{P}$, where $P = \{p_1, p_2, \ldots, p_n\}$ and $R = \{r_1, r_2, \ldots, r_m\}$. put $Q = P \wedge R = \{p_i \wedge r_j | i \leq n, j \leq m\}$. Obviously, $\mathcal{F}_Q \subset \mathcal{F}_P \bigcap \mathcal{F}_R$ and thus, as a simple consequence, every intersection of a finite subset of $\mathcal{F}$ is nonvoid. Hence $\mathcal{F}$ is a filter base as we were to show.

We shall now use the compactness argument. Since $\mathcal{M}_c(B_2, H)$ is compact, we infer that $\bigcap \mathcal{F} \neq \emptyset$. Choose a measure in the intersection $\bigcap \mathcal{F}$, some $k \in \mathcal{M}_c(B_2, H)$. By the definition of the sets $\mathcal{F}_P$, we immediately obtain that $k(a) =$

$m(a)$ for any $a \in B_1$. Therefore k extends m and the proof of Theorem 4 is complete. □

Let us now take up a special class of measures. It should be noted that in a slightly more general setup they have been considered in [**3**].

Definition 5. Let B be a Boolean algebra and let H be a real Hilbert space. A measure $m\colon B \to H$ is called orthogonal if for any two disjoint elements $a, b \in B$ we have $\langle m(a), m(b)\rangle = 0$.

In other words, an orthogonal measure maps disjoint elements into orthogonal ones. Let us denote by $\mathcal{O}(B,H)$ the set of all orthogonal measures on B which range in H. Analogously, we define $\mathcal{O}_c(B,H)$ $(c \geq 0)$ as the set of all $m \in \mathcal{O}(B,H)$ such that $\sup_{a\in B} \|m(a)\| \leq c$. Thus, $\mathcal{O}_c(B,H) \subset \mathcal{M}_c(B,H)$. The following proposition gives us a useful characterization of orthogonal measures.

Proposition 6. *Let $m \in \mathcal{M}(B,H)$ be a measure. Put $\mathcal{R}(m) = \{p \in H \mid p = m(a)$ for some $a \in B\}$ and denote by $\mathcal{S}(m)$ the sphere in H which is centred in $\frac{m(1)}{2}$ and which has the radius $\frac{\|m(1)\|}{2}$. Then m is an orthogonal measure if and only if $\mathcal{R}(m) \subset \mathcal{S}(m)$.*

Proof. Suppose that m is an orthogonal measure. Suppose further that $p \in \mathcal{R}(m)$. It means that $p = m(a)$ for some $a \in B$. We have $\langle m(a), m(1) - m(a)\rangle = \langle m(a), m(a')\rangle = 0$ and therefore we obtain

$$\begin{aligned}\|m(a) - \frac{m(1)}{2}\|^2 &= \frac{1}{4}\langle m(a) - (m(1) - m(a)), m(a) - (m(1) - m(a))\rangle \\ &= \frac{1}{4}(\|m(a)\|^2 + \|m(a')\|^2) = \frac{\|m(1)\|^2}{4}.\end{aligned}$$

It follows that $\|m(a) - \frac{m(1)}{2}\| = \frac{\|m(1)\|}{2}$ and therefore $m(a) \in \mathcal{S}(m)$. Thus, $\mathcal{R}(m) \subset \mathcal{S}(m)$.

Suppose on the contrary that $\mathcal{R}(m) \subset \mathcal{S}(m)$. Suppose that $a \wedge b = 0$ for some elements $a, b \in B$. Put $x_1 = m(a)$, $x_2 = m(b)$ and $x_3 = m(a \vee b)$. Then $x_i \in \mathcal{R}(m)$ $(i = 1,2,3)$ and $x_3 = x_1 + x_2$. By an easy computation we obtain

$$\begin{aligned}\|x_i\|^2 &= \|x_i - \frac{m(1)}{2} + \frac{m(1)}{2}\|^2 \\ &= \|x_i - \frac{m(1)}{2}\|^2 + \|\frac{m(1)}{2}\|^2 + 2\left\langle x_i - \frac{m(1)}{2}, \frac{m(1)}{2}\right\rangle \\ &= \langle x_i, m(1)\rangle \qquad (i = 1,2,3).\end{aligned}$$

Thus, $\|x_1\|^2 + \|x_2\|^2 = \|x_1 + x_2\|^2$ and therefore

$$\langle m(a), m(b)\rangle = \langle x_1, x_2\rangle = 0.$$

□

The foregoing proposition has the following two corollaries.

Corollary 7 (see also [**3**]). *If $H = \mathbb{R}$, then $\mathcal{O}(B, H)$ consists of all two-valued measures.*

Proof. A measure $m: B \to \mathbb{R}$ is orthogonal if and only if $\mathcal{R}(m) = \mathcal{S}(m) = \{0, m(1)\}$ (Prop. 6). It follows that m is orthogonal if and only if it is two-valued.□

Corollary 8. *Suppose that* $\dim H < \infty$ *and suppose that we are given nonnegative $c \in \mathbb{R}$. Then $\mathcal{O}_c(B, H)$ $(c \geq 0)$ is a compact space when viewed as a subspace of $\mathcal{M}_c(B, H)$ (as always, $\mathcal{M}_c(B, H)$ is endowed with the weak topology).*

Proof. We have to show that if m is a pointwise limit of elements of $\mathcal{O}_c(B, H)$, then $\|m(a) - \frac{m(1)}{2}\| = \frac{\|m(1)\|}{2}$ for any $a \in B$. But this is obvious since the norm topology and the weak topology coincide on every finite dimensional normed space. □

Let us consider the extension problem for the orthogonal measures.

Theorem 9. *Let B_1 be a Boolean subalgebra of B_2 and let $m: B_1 \to H$ be an orthogonal measure. If* $\dim H < \infty$ *then m admits an orthogonal extension over B_2.*

Proof. Suppose that $c = \sup_{a \in B} \|m(a)\|$. In order to copy the proof of Theorem 4, we have to establish that the corresponding collection $\mathcal{F}_P$ consists of nonvoid sets and that the filter base $\mathcal{F}$ converges in $\mathcal{O}_c(B, H)$. The former property is guaranteed by the last statement of Prop. 2 (the measure m of Prop. 2 can be chosen orthogonal) and the latter property follows from the compactness of $\mathcal{O}_c(B, H)$ (Prop. 8).

To our regret, we have not be able to prove Theorem 9 for an arbitrary Hilbert space. We rather suspect that it cannot be proved at all. The thing is that in this case the space $\mathcal{O}_c(B, H)$ does not have to be compact. (For instance, take $B = \{0, a, a', 1\}$ and $H = l_2(\mathbb{N})$, where $\mathbb{N}$ means the set of natural numbers. Let δ_j $(j \in \mathbb{N})$ denote the element of H such that $\delta_j(i) = \delta_{ji}$ $(i \in \mathbb{N})$. Let M be the sequence of orthogonal measures $(m_n)_{n \in \mathbb{N}}$ determined by the following equalities:

$$m_n(a) = \frac{\delta_1 + \delta_n}{2}, \quad m_n(a') = \frac{\delta_1 - \delta_n}{2}.$$

Then the set M has no cluster point in $\mathcal{O}_1(B, H)$.) □

In the conclusion, let us relax the initial extension problem by allowing "enlargements" of the range Hilbert space of the orthogonal measure in question. In this case we have a positive result. Unfortunately, we cannot avoid the use of deeper results of the operator theory.

Theorem 10. *Let B_1 be a Boolean subalgebra of B_2 and let H be a Hilbert space. Let $m: B_1 \to H$ be an orthogonal measure. Then there is a Hilbert space K*

which contains H as a Hilbert subspace and which fulfils the following property: There is an orthogonal measure $k\colon B_2 \to K$ such that $k|B_1 = m$.

Proof. We may (and shall) identity B_2 with the lattice of all projectors $L(C)$ of a commutative C^*-algebra C. We assume that C acts on a (complex) Hilbert space F.

Take now the mapping $f\colon B_1 \to \mathbb{R}$ defined by putting $f(P) = \|m(P)\|^2$ ($P \in B_1$). It is easy to check that f is a bounded finitely additive nonnegative measure on B_1. According to [**4**], f can be extended to a finitely additive nonnegative bounded measure $\tilde{f}\colon L(\tilde{C}) \to R$, where $\tilde{C}$ is a commutative von Neumann algebra containing C. Going on further, $\tilde{f}$ can be extended to a nonnegative bounded functional $\hat{f}$ on the entire $\tilde{C}$ (we take the integral of the measure $\tilde{f}|L(\tilde{C})$ by making use of the spectral theorem in $\tilde{C}$). Put $\overline{f} = \hat{f}|C$. Using the GNS-construction (see [**2**]), we infer that there is a (complex) Hilbert space G and a $*$-homomorphism ϕ of C into a C^*-algebra $\mathcal{B}(G)$ of all bounded operators acting on G such that $\overline{f}(A) = \langle \phi(A)x, x\rangle$ ($A \in C$) for a suitable $x \in G$. Let us define a mapping $p\colon B_2 \to G$ by the equality

$$p(P) = \phi(P)x \qquad (P \in B_2).$$

Then p can be viewed as an orthogonal measure with values in a real Hilbert subspace L of G generated by the set $\{p(P)|P \in B_2\}$. Indeed, if $P, Q \in B_2$, then $PQ \in B_2$ and $\langle p(P), p(Q)\rangle = \langle \phi(PQ)x, x\rangle \geq 0$. Moreover, for any $P \in B_2$ we have $\|p(P)\|^2 = \langle \phi(P)x, \phi(P)x\rangle = \langle \phi(P)x, x\rangle = \overline{f}(P)$. Further, for any $P, Q \in B_1$ we have $\langle p(P), p(Q)\rangle = \|p(P \wedge Q)\|^2 = f(P \wedge Q) = \|m(P \wedge Q)\|^2 = \langle m(P), m(Q)\rangle$. Put $K_p = \{p(P)|P \in B_1\}$ and $H_m = \{m(P)|P \in B_1\}$. Define a mapping $\mathcal{U}\colon K_p \to H_m$ by putting $\mathcal{U}(p(P)) = m(P)$. The definition of $\mathcal{U}$ is correct and moreover, $\mathcal{U}$ can be linearly extended over $\overline{\operatorname{span}}\, K_p$ (here $\overline{\operatorname{span}}\, K_p$ means the closed linear hull in L). Indeed, if $\sum_{i \leq n} \alpha_i p(P_i) = p(Q)$ ($\alpha_i \in \mathbb{R}$), then using the equality $\langle p(P), p(Q)\rangle = \langle m(P), m(Q)\rangle$ ($P, Q \in B_1$) we obtain

$$\begin{aligned}\|\sum_{i \leq n} \alpha_i m(P_i) - m(Q)\|^2 &= \|\sum_{i \leq n} \alpha_i m(P_i)\|^2 + \|m(Q)\|^2 - 2\left\langle \sum_{i \leq n} \alpha_i m(P_i), m(Q)\right\rangle \\ &= \|\sum_{i \leq n} \alpha_i p(P_i) - p(Q)\|^2 = 0.\end{aligned}$$

Let us denote again by $\mathcal{U}$ the extended mapping. Thus, we have a unitary mapping $\mathcal{U}\colon \overline{\operatorname{span}}\, K_p \to \overline{\operatorname{span}}\, H_m$. Let K be a (real) Hilbert space such that $K \supset H$ and $\dim L \leq \dim K$. Extend $\mathcal{U}$ to a unitary mapping from L into K (we set $\mathcal{U}|K_p^{\perp} = \mathcal{V}$, where $\mathcal{V}$ is a unitary mapping of $K_p^{\perp}$ into $H_m^{\perp}$). Denoting finally the extended mapping again by $\mathcal{U}$ and putting $k = \mathcal{U} \circ p$, we easily check that k is an orthogonal measure on B_2 which extends m. The proof is complete. □

Let us note in the conclusion of this paper that the proof of the latter theorem yields an "absolute" extension theorem provided the dimension of H is sufficiently large. Indeed, if C is the C^*-algebra associated with B_2 in the proof and if we put $d(B_2) = dens\ C$, where $dens\ C$ denotes the density character of the (topological) space C, then we easily check that $\dim L \leq d(B_2)$ (the representation ϕ in the proof of Theorem 10 can be chosen cyclic). Applying now Theorem 10, we obtain the following corollary.

Corollary 11. *Let B_1 be a Boolean subalgebra of B_2 and let H be a real Hilbert space. Suppose that $m\colon B_1 \to H$ be an orthogonal measure. If $d(B_2) \leq \dim H$ then m admits an orthogonal extension over B_2.*

Remark 12. After finding the proof of Theorem 10, we became aware of the paper "Orthogonally scattered dilation of Hilbert space valued set functions", Lecture Notes in Math. 945, 269–281, Springer-Villey 1982 by S. D. Chatterji. Quite deep main result of the latter paper together with our Theorem 4 provides an alternative proof of Theorem 10. As technical as our proof of Theorem 10 may be, we believe that it is still simpler.

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