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EXISTENCE OF OSCILLATORY SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Dedicated to Professor Valter Šeda on the occasion of his sixtieth birthday

1. Introduction

One of the important problems in the qualitative theory of functional differential equations of neutral type, which has currently received wide attention, is to discuss the existence of solutions, either oscillatory or nonoscillatory, of the equation under consideration. As far as the nonoscillatory solutions are concerned, there has been a lot of systematic investigations, and numerous results can be found in the literature regarding the construction of such solutions with specified asymptotic behavior at infinity; see e.g. the papers [**1–6**]. However, virtually nothing is known about the existence of oscillatory solutions of neutral functional differential equations even for the first-order case.

The objective of this paper is establish sufficient conditions for the existence of solutions for neutral equations of the form

$$\frac{d^n}{dt^n}\left[x(t) \pm \lambda x(t-\tau)\right] + f(t, x(g_1(t)), \dots, x(g_N(t))) = 0, \tag{$\mathrm{A}_\pm$}$$

where $n \geq 1$, λ and τ are positive constants, and $g_i(t)$, $1 \leq i \leq N$, and $f(t, u_1, \dots, u_N)$ are continuous functions on $[t_0, \infty)$ and $[t_0, \infty) \times \mathbb{R}^N$, respectively. In particular, our results provide sufficient conditions under which equation (A_+) has infinitely many oscillatory solutions which behave like

$$\lambda^{t/\tau}\left[a \cos((2m-1)\pi t/\tau) + b \sin((2m-1)\pi t/\tau)\right], \quad m = 1, 2, \dots$$

at $t \to \infty$, and equation (A_-) has infinitely many oscillatory solutions which behave like

$$\lambda^{t/\tau}\left[a \cos(2m\pi t/\tau) + b \sin(2m\pi t/\tau)\right], \quad m = 1, 2, \dots$$

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at $t \to \infty$, where a and b are constants with $a^2 + b^2 > 0$. The desired solutions of $(A_\pm)$ will be obtained as solutions of the following "integral-difference" equations

$$(B_\pm) \qquad x(t) \pm \lambda x(t-\tau) + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(g(s)))\, ds = 0, \quad t \geq T,$$

$$(C_\pm) \qquad x(t) \pm \lambda x(t-\tau) + \int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x(g(s)))\, ds = 0, \quad t \geq T,$$

where $f(s, x(g(s))) \equiv f(s, x(g_1(s)), \ldots, x(g_N(s)))$ and $T > T_0$ is a constant, and a fixed point analysis combined with a device of Ruan [**6**] will be applied for solving $(B_\pm)$ or $(C_\pm)$.

We note that by a solution of equation (A_+) or (A_-) is meant a continuous function $x\colon [T_x, \infty) \to \mathbb{R}$ such that $x(t) + \lambda x(t-\tau)$ or $x(t) - \lambda x(t-\tau)$ is n-times continuously differentiable and satisfies (A_+) or (A_-) for all sufficiently large $t > T_x$. Such a solution is said to be nonoscillatory if it is either eventually positive or eventually negative. A solution is said to be oscillatory if it has a sequence of zeros tending to infinity.

2. Main Results

The following conditions on $g_i(t)$, $1 \leq i \leq N$, and $f(t, u_1, \ldots, u_N)$ in $(A_\pm)$ are assumed to hold without further mentioning: $\lim_{t\to\infty} g_i(t) = \infty$, $1 \leq i \leq N$; and there is a continuous function $F(t, v_1, \ldots, v_N)$ on $[T_0, \infty) \times \mathbb{R}_+^N$ which is nondecreasing in each v_i, $1 \leq i \leq N$, and satisfies

$$|f(t, u_1, \ldots, u_N)| \leq F(t, |u_1|, \ldots, |u_N|), \quad (t, u_1, \ldots, u_N) \in [t_0, \infty) \times \mathbb{R}^N.$$

The main results of this paper are as follows.

Theorem 1. *Suppose that $0 < \lambda \leq 1$ and that there exist constants $\mu \in (0, \lambda)$ and $a > 0$ such that*

$$(1) \qquad \int^\infty t^{n-1} \mu^{-t/\tau} F(t, a\lambda^{g_1(t)/\tau}, \ldots, a\lambda^{g_N(t)/\tau})\, dt < \infty.$$

Then

(i) *for any continuous periodic oscillatory function $\omega_-(t)$ with period τ, equation (A_-) has a bounded oscillatory solution $x_-(t)$ such that*

$$(2_-) \qquad x_-(t) = \lambda^{t/\tau} \omega_-(t) + o(\lambda^{t/\tau}) \quad \text{as} \quad t \to \infty,$$

(ii) *for any continuous oscillatory function $\omega_+(t)$ such that $\omega_+(t+\tau) = -\omega_+(t)$ for all t, equation (A_+) has a bounded oscillatory solution $x_+(t)$ such that*

$$(2_+) \qquad x_+(t) = \lambda^{t/\tau} \omega_+(t) + o(\lambda^{t/\tau}) \quad \text{as} \quad t \to \infty.$$

Theorem 2. *Suppose that $\lambda > 1$ and that there exists a constant $a > 0$ such that*

$$\int^{\infty} t^{n-1} F(t, a\lambda^{g_1(t)/\tau}, \dots, a\lambda^{g_N(t)/\tau})\, dt < \infty. \tag{3}$$

Then

(i) *for any continuous periodic oscillatory function $\omega_-(t)$ with period τ, equation* (A_-) *has an unbounded oscillatory solution $x_-(t)$ such that*

$$x_-(t) = \lambda^{t/\tau}\omega_-(t) + o(1) \qquad \text{as} \quad t \to \infty, \tag{4_-}$$

(ii) *for any continuous oscillatory function $\omega_+(t)$ such that $\omega_+(t+\tau) = -\omega_+(t)$ for all t, equation* (A_+) *has an unbounded oscillatory solution $x_+(t)$ such that*

$$x_+(t) = \lambda^{t/\tau}\omega_+(t) + o(1) \qquad \text{as} \quad t \to \infty. \tag{4_+}$$

Theorem 3. *Suppose that $\lambda > 1$ and that there exist constants $\mu \in (1, \lambda)$ and $a > 0$ such that*

$$\int^{\infty} \mu^{-t/\tau} F(t, a\lambda^{g_1^*(t)/\tau}, \dots, a\lambda^{g_N^*(t)/\tau})\, dt < \infty, \tag{5}$$

where $g_i^*(t) = \max\{g_i(t), t\}$, $1 \le i \le N$.

Then

(i) *for any continuous periodic oscillatory function $\omega_-(t)$ with period τ, equation* (A_-) *has an unbounded oscillatory solution $x_-(t)$ such that*

$$x_-(t) = \lambda^{t/\tau}\omega_-(t) + o(\lambda^{t/\tau}) \qquad \text{as} \quad t \to \infty, \tag{6_-}$$

(ii) *for any continuous oscillatory function $\omega_+(t)$ such that $\omega_+(t+\tau) = -\omega_+(t)$ for all t, equation* (A_+) *has an unbounded oscillatory solution $x_+(t)$ such that*

$$x_+(t) = \lambda^{t/\tau}\omega_+(t) + o(\lambda^{t/\tau}) \qquad \text{as} \quad t \to \infty. \tag{6_+}$$

Remark. Each of the functions

$$a\cos(2m\pi t/\tau) + b\sin(2m\pi t/\tau), \qquad m = 1, 2, \dots,$$

and

$$a\cos((2m-1)\pi t/\tau) + b\sin((2m-1)\pi t/\tau), \qquad m = 1, 2, \dots,$$

a and b being constants with $a^2 + b^2 > 0$, can be used as $\omega_-(t)$ and $\omega_+(t)$, respectively, in the above theorems.

Proof of Theorem 1. Let $\omega_\pm(t)$ be fixed and put $\omega_\pm^* = \max|\omega_\pm(t)|$. Choose $c > 0$ and $T > t_0$ so that $(1+\omega_\pm^*)\lambda c/(\lambda-\mu) \le a$,

$$T_0 = \min\{T-\tau, \inf_{t\ge T} g_1(t), \dots, \inf_{t\ge T} g_N(t)\} \ge \max\{t_0,\tau\} \tag{7}$$

and

$$\int_T^\infty t^{n-1}\mu^{-t/\tau} F(t, a\lambda^{g_1(t)/\tau}, \dots, a\lambda^{g_N(t)/\tau})\, dt \le c. \tag{8}$$

Let X denote the set

$$X = \{x \in C[T_0,\infty) \colon |x(t)| \le c\mu^{t/\tau} \text{ for } t \ge T_0\},$$

which is a closed convex subset of Fréchet space $C[T_0,\infty)$ of continuous functions on $[T_0,\infty)$ with the usual metric topology. Motivated by Ruan [**6**], with each $x \in X$ we associate the functions $\hat{x}_\pm(t) \colon [T_0,\infty) \to \mathbb{R}$ defined by

$$\begin{aligned} &\hat{x}_\pm(t) = \frac{\lambda c}{\lambda-\mu}\lambda^{t/\tau}\omega_\pm(t) - \sum_{i=1}^\infty (\mp 1)^i \lambda^{-i} x(t+i\tau), \quad t \ge T-\tau, \\ &\hat{x}_\pm(t) = \hat{x}_\pm(t)(T-\tau), \quad T_0 \le t \le T-\tau. \end{aligned} \tag{9}$$

It is easily verified that, for each $x \in X$, $\hat{x}_\pm(t)$ are well defined and continuous on $[T_0,\infty)$ and satisfy

$$\hat{x}_\pm(t) \pm \lambda\hat{x}(t-\tau) = x(t), \quad t \ge T. \tag{10}$$

Since $x \in X$ implies

$$\sum_{i=1}^\infty \lambda^{-i}|x(t+i\tau)| \le \frac{\mu c}{\lambda-\mu}\mu^{t/\tau}, \quad t \ge T-\tau, \tag{11}$$

it follows from (9) that

$$|\hat{x}_\pm(t)| \le \frac{\lambda c\omega_\pm^*}{\lambda-\mu}\lambda^{t/\tau} + \frac{\mu c}{\lambda-\mu}\mu^{t/\tau} \le \frac{\lambda c(\omega_\pm^*+1)}{\lambda-\mu}\lambda^{t/\tau}, \quad t \ge T-\tau,$$

and hence, in view of the fact that $\lambda \le 1$,

$$|\hat{x}_\pm(t)(g_i(t))| \le \frac{\lambda c(\omega_\pm^*+1)}{\lambda-\mu}\lambda^{g_i(t)/\tau} \le a\lambda^{g_i(t)/\tau}, \quad t \ge T, \quad 1 \le i \le N. \tag{12}$$

Now we define the mappings $\mathcal{F}_\pm \colon X \to C[T_0,\infty)$, via (9), by

$$\begin{aligned} &\mathcal{F}_\pm x(t) = (-1)^{n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, \hat{x}_\pm(t)(g(s)))\, ds, \quad t \ge T, \\ &\mathcal{F}_\pm x(t) = \mathcal{F}_\pm x(T), \quad T_0 \le t \le T, \end{aligned} \tag{13}$$

where $f(s,\hat{x}_\pm(g(s))) \equiv f(s,\hat{x}_\pm(g_1(s)),\dots,\hat{x}_\pm(g_N(s)))$. It will be shown that $\mathcal{F}_\pm$ are continuous and map X into compact subsets of X, so that Schauder-Tychonoff fixed point theorem is applicable to $\mathcal{F}_\pm$. In fact, if $x \in X$, then, using (13), (12) and (8), we obtain

$$\begin{aligned} |\mathcal{F}_\pm x(t)| &\le \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}|f(s,\hat{x}_\pm(t)(g(s)))|\,ds \\ &\le \mu^{t/\tau}\int_t^\infty s^{n-1}\mu^{-s/\tau}F(s,|\hat{x}_\pm(t)(g_1(s))|,\dots,|\hat{x}_\pm(t)(g_N(s))|)\,ds \\ &\le \mu^{t/\tau}\int_T^\infty s^{n-1}\mu^{-s/\tau}F(s,a\lambda^{g_1(s)/\tau},\dots,a\lambda^{g_N(s)/\tau})\,ds \\ &\le c\mu^{t/\tau}, \quad t \ge T, \end{aligned}$$

which implies that $\mathcal{F}_\pm x \in X$. Thus, $\mathcal{F}_\pm(X) \subset X$. Let $\{x_k\}$ be a sequence of elements in X converging to an $x \in X$ in the topology of $C[T_0,\infty)$. Since

$$\sum_{i=1}^\infty (\pm 1)^i\lambda^{-i}x_k(t+i\tau) \to \sum_{i=1}^\infty (\pm 1)^i\lambda^{-i}x(t+i\tau) \quad \text{as } k \to \infty$$

uniformly on compact subintervals of $[T_0,\infty)$, the Lebesgue dominated convergence theorem shows that $\mathcal{F}_\pm x_k(t) \to \mathcal{F}_\pm x(t)$ uniformly on compact subintervals of $[T_0,\infty)$. This proves the continuity of $\mathcal{F}_\pm$. Finally, from the inequalities

$$|(\mathcal{F}_\pm x)'(t)| \le F(t,a\lambda^{g_1(t)/\tau},\dots,a\lambda^{g_N(t)/\tau}), t \ge T, \text{ for } n = 1$$

$$|(\mathcal{F}_\pm x)'(t)| \le \int_T^\infty \frac{(s-t)^{n-2}}{(n-2)!}F(s,a\lambda^{g_1(s)/\tau},\dots,a\lambda^{g_N(s)/\tau})\,ds, t \ge T, \text{ for } n \ge 2,$$

holding for all $x \in X$, we conclude via the Ascoli-Arzela theorem that the sets $\mathcal{F}_\pm(X)$ have compact closure in $C[T_0,\infty)$.

Therefore, by the Schauder-Tychonoff theorem, there exist fixed elements $\xi_\pm \in X$ of $\mathcal{F}_\pm$, which satisfy

$$\xi_\pm(t) = (-1)^{n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}f(s,\hat{\xi}_\pm(g)s)))\,ds, \quad t \ge T.$$

Combining the above equation with (10) (with $x = \xi$) yields

$$\hat{\xi}_\pm(t) \pm \lambda\hat{\xi}_\pm(t-\tau) = (-1)^{n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}f(s,\hat{\xi}_\pm(g(s)))\,ds, \quad t \ge T, \tag{14}$$

implying that $\hat{\xi}_\pm(t)$ solve the integral-difference equations ($\mathrm{B}_\pm$). Differentiation of (14) shows that $\hat{\xi}_\pm(t)$ are solutions of the neutral differential equations ($\mathrm{A}_\pm$) on $[T_0,\infty)$. From (9) and (11) we have

$$|\hat{\xi}_\pm(t) - \frac{\lambda c}{\lambda-\mu}\lambda^{t/\tau}\omega_\pm(t)| \le \frac{\mu c}{\lambda-\mu}\mu^{t/\tau}, \quad t \ge T-\tau,$$

which shows that $\hat{\xi}_{\pm}(t)$ satisfy the asymptotic relations $(2_{\pm})$. It is obvious that $\hat{\xi}_{\pm}(t)$ are oscillatory. $\square$

Proof of Theorem 2. The proof is essentially similar to that of Theorem 1. For fixed $\omega_{\pm}(t)$ define $\omega_{\pm}^{*} = \max|\omega_{\pm}(t)|$, and take $c > 0$ and $T > t_0$ so that $(1+\omega_{\pm}^{*})\lambda c/(\lambda-1) \le a$, (7) holds, and

$$\int_T^\infty t^{n-1}F(t, a\lambda^{g_1(t)/\tau}, \dots, a\lambda^{g_N(t)/\tau})\,dt \le c.$$

Define the set $Y \subset C[T_0,\infty)$ and the mappings $\mathcal{G}_{\pm}\colon Y \to C[T_0,\infty)$ as follows:

$$\begin{aligned}
&Y = \{y \in C[T_0,\infty)\colon |y(t)| \le c \quad \text{for} \quad t \ge T_0\};\\
&\mathcal{G}_{\pm}y(t) = (-1)^{n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, \tilde{y}_{\pm}(g(s)))\,ds, \quad t \ge T,\\
&\mathcal{G}_{\pm}y(t) = \mathcal{G}_{\pm}y(T), \quad T_0 \le t \le T,
\end{aligned}$$

where $f(s, \tilde{y}_{\pm}(g(s))) \equiv f(s, \tilde{y}_{\pm}(g_1(s)), \dots, \tilde{y}_{\pm}(g_N(s)))$ and $\tilde{y}_{\pm}\colon [T_0,\infty) \to \mathbb{R}$ denote the functions defined by

$$\begin{aligned}
\tilde{y}_{\pm}(t) &= \frac{\lambda c}{\lambda-1}\lambda^{t/\tau}\omega_{\pm}(t) - \sum_{i=1}^{\infty}(\mp 1)^i\lambda^{-i}y(t+i\tau), \quad t \ge T-\tau,\\
\tilde{y}_{\pm}(t) &= \frac{\lambda c}{\lambda-1}\lambda^{t/\tau}\omega_{\pm}(t) - \sum_{i=1}^{\infty}(\mp 1)^i\lambda^{-i}y(T+(i-1)\tau), \quad T_0 \le t \le T-\tau.
\end{aligned}$$

As is easily verified, $y \in Y$ implies that $\tilde{y}_{\pm} \in C[t_0,\infty)$ and

$$\tilde{y}_{\pm}(t) \pm \lambda\tilde{y}_{\pm}(t-\tau) = y(t), \quad t \ge T.$$

Furthermore, since $\sum_{i=1}^{\infty}\lambda^{-i}|y_{\pm}(t+i\tau)| \le c/(\lambda-1)$, $t \ge T-\tau$,

$$|\tilde{y}_{\pm}(t)| \le \frac{\lambda c\omega_{\pm}^{*}}{\lambda-1}\lambda^{t/\tau} + \frac{c}{\lambda-1} \le \frac{(\omega_{\pm}^{*}+1)\lambda c}{\lambda-1}\lambda^{t/\tau}, \quad t \ge T_0,$$

and so

$$|\tilde{y}_{\pm}(g_i(t))| \le a\lambda^{g_i(t)/\tau}, \quad t \ge T, \quad 1 \le i \le N.$$

Proceeding as in Theorem 1, we see that the Schauder-Tychonoff fixed point theorem is applicable to $\mathcal{G}_{\pm}$. Let $\eta_{\pm} \in X$ be fixed points of $\mathcal{G}_{\pm}$. Then, the functions $\hat{\eta}_{\pm}(t)$ satisfy the integral-difference equations $(\mathrm{B}_{\pm})$, so that they solve the neutral equations $(\mathrm{A}_{\pm})$. Since $\eta_{\pm}(t) \to 0$ as $t \to \infty$, the solutions $\hat{\eta}_{\pm}(t)$ have the asymptotic properties $(4_{\pm})$. Notice that $\lim_{t\to\infty}[\hat{\eta}_{\pm}(t) \pm \lambda\hat{\eta}_{\pm}(t-\tau)] = 0$. $\square$

Proof of Theorem 3. We intend to construct the desired oscillatory solutions as solutions of the integral-difference equations $(\mathrm{C}_{\pm})$. Let $\omega_{\pm}(t)$ be fixed and put

$\omega_\pm^* = \max|\omega_\pm(t)|$. Let $\nu \in (\mu, \lambda)$ be fixed and choose $c > 0$ and $T > t_0$ so that $(1 + \omega_\pm^*)\lambda c/(\lambda - \nu) \le a$, (7) holds,

$$t^{n-1}\mu^{t/\tau} \le \nu^{t/\tau} \quad \text{for} \quad t \ge T \tag{15}$$

and

$$\int_T^\infty \mu^{-t/\tau} F(t, a\lambda^{g_1^*(t)/\tau}, \dots, a\lambda^{g_N^*(t)/\tau})\, dt \le c. \tag{16}$$

Consider the set $Z \subset C[T_0, \infty)$ and the mappings $\mathcal{H}_\pm \colon Z \to C[T_0, \infty)$ defined by

$$Z = \{z \in C[T_0, \infty) \colon |z(t)| \le c\nu^{t/\tau} \quad \text{for} \quad t \ge T_0\}$$

and

$$\begin{aligned} &\mathcal{H}_\pm z(t) = -\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, \check{z}_\pm(g(s)))\, ds, \quad t \ge T, \\ &\mathcal{H}_\pm z(t) = 0, \quad T_0 \le t \le T, \end{aligned} \tag{17}$$

where $f(s, \check{z}_\pm(g(s))) \equiv f(s, \check{z}_\pm(g_1(s)), \dots, \check{z}_\pm(g_N(s)))$ and $\check{z}_\pm \colon [T_0, \infty) \to \mathbb{R}$ denote the functions constructed from $z \in Z$ according to the rule

$$\begin{aligned} &\check{z}_\pm(t) = \frac{\lambda c}{\lambda - \nu}\lambda^{t/\tau}\omega_\pm(t) - \sum_{i=1}^\infty (\mp 1)^i \lambda^{-i} z(t + i\tau), \quad t \ge T - \tau \\ &\check{z}_\pm(t) = \check{z}_\pm(T - \tau), \quad T_0 \le t \le T - \tau. \end{aligned} \tag{18}$$

For every $z \in Z$, $\check{z}_\pm \in C[T_0, \infty)$ and satisfy

$$\check{z}_\pm(t) \pm \lambda \check{z}_\pm(t - \tau) = z(t), \quad t \ge T. \tag{19}$$

Since

$$\sum_{i=1}^\infty \lambda^{-i}|z_\pm(t + i\tau)| \le [\nu c/(\lambda - \nu)]\nu^{t/\tau}, \quad t \ge T - \tau, \tag{20}$$

we have

$$|\check{z}_\pm(t)| \le \frac{\lambda c \omega_\pm^*}{\lambda - \nu}\lambda^{t/\tau} + \frac{\lambda c}{\lambda - \nu}\nu^{t/\tau} \le \frac{(\omega_\pm^* + 1)\lambda c}{\lambda - \nu}\lambda^{t/\tau}, \quad t \ge T - \tau,$$

which implies, in view of the fact that $\lambda > 1$, that

$$|\check{z}_\pm(g_i(t))| \le a\lambda^{g_i^*(t)/\tau}, \quad t \ge T, \quad 1 \le i \le N.$$

From (17), (15), (16) and the above inequality it follows that for $z \in Z$

$$\begin{aligned}|\mathcal{H}_\pm z(t)| &\le t^{n-1}\int_T^t |f(s,\check{z}_\pm(g(s)))|\,ds\\ &\le t^{n-1}\mu^{t/\tau}\int_T^t \mu^{-s/\tau}F(s,|\check{z}_\pm(g_1(s))|,\dots,|\check{z}_\pm(g_N(s))|)\,ds\\ &\le \nu^{t/\tau}\int_T^t \mu^{-s/\tau}F(s,a\lambda^{g_1^*(t)/\tau},\dots,a\lambda^{g_N^*(t)/\tau})\,ds\\ &\le c\nu^{t/\tau},\quad t\ge T.\end{aligned}$$

Thus, $\mathcal{H}_\pm$ map Z into itself. Since the continuity of $\mathcal{H}_\pm$ and the relative compactness of $\mathcal{H}_\pm(Z)$ can be verified easily, the Schauder-Tychonoff theorem guarantees the existence of the elements $\zeta_\pm \in Z$ such that $\zeta_\pm = \mathcal{H}_\pm\zeta_\pm$, i.e.,

$$\zeta_\pm(t) = -\int_T^t \frac{(t-s)^{n-1}}{(n-1)!}f(s,\check{\zeta}_\pm(g(s)))\,ds,\quad t\ge T.$$

In view of (19) the above can be rewritten as

$$\check{\zeta}_\pm(t)\pm\lambda\check{\zeta}_\pm(t-\tau) = -\int_T^t \frac{(t-s)^{n-1}}{(n-1)!}f(s,\check{\zeta}_\pm(g(s)))\,ds,\quad t\ge T, \tag{21}$$

which shows that $\check{\zeta}_\pm(t)$ solve the equations ($\mathrm{C}_\pm$) for $t \ge T$. Differentiating (21) n times, we see that $\check{\zeta}_\pm(t)$ are solutions of the neutral equations ($\mathrm{A}_\pm$) on $[T,\infty)$. That $\check{\zeta}_\pm(t)$ satisfy ($6_\pm$) follows readily from (18) and (20). Observe that the limits $\lim_{t\to\infty}[\check{\zeta}_\pm(t)\pm\lambda\check{\zeta}_\pm(t-\tau)]$ exist in the extended real line and are different from zero in general. This completes the proof. □

3. Examples

Examples illustrating the above results will now be given.

Example 1. Consider the equations

$$\frac{d^n}{dt^n}[x(t)\pm e^{-1}x(t-1)] = (1\pm e^{-2})e^{(1-\gamma\theta)t}|x(\theta t)|^\gamma \operatorname{sgn} x(\theta t) = 0, \tag{$22_\pm$}$$

where γ and θ are positive constants such that $\gamma\theta > 1$. These equations are special cases of ($\mathrm{A}_\pm$) in which $\lambda = e^{-1}$, $\tau = 1$, $g(t) = \theta t$ and the function $F(t,v)$ can be taken to be $F(t,v) = (1+e^{-2})e^{(1-\gamma\theta)t}v^\gamma$. Since

$$\begin{aligned}\int^\infty t^{n-1}\mu^{-t/\tau}F(t,a\lambda^{g(t)/\tau})\,dt &= a^\gamma(1+e^{-2})\int^\infty t^{n-1}\mu^{-t}e^{(1-2\gamma\theta)t}\,dt\\ &= a^\gamma(1+e^{-2})\int^\infty t^{n-1}(e^{1-2\gamma\theta}/\mu)^t\,dt < \infty\end{aligned}$$

for any μ such that $e^{1-2\gamma\theta} < \mu < e^{-1}$, condition (1) is satisfied for $(22_\pm)$, and so by Theorem 1, given any continuous periodic oscillatory function $\omega_-(t)$ of period 1 and any continuous oscillatory function $\omega_+(t)$ such that $\omega_+(t+1) = -\omega_+(t)$ for all t, there exist bounded oscillatory solutions $x_\pm(t)$ of $(22_\pm)$ satisfying

$$x_\pm(t) = e^{-t}\omega_\pm(t) + o(e^{-t}) \quad \text{as} \quad t \to \infty.$$

In particular, $(22_\pm)$ possess infinitely many bounded oscillatory solutions $x_m^\pm(t)$, $m = 1, 2, \dots$, such that

$$\begin{aligned} x_m^-(t) &= const \cdot e^{-t}\cos(2m\pi t) + o(e^{-t}) \quad \text{as} \quad t \to \infty, \\ x_m^+(t) &= const \cdot e^{-t}\cos((2m-1)\pi t) + o(e^{-t}) \quad \text{as} \quad t \to \infty. \end{aligned}$$

Example 2. Consider the equations

$$(23_\pm) \qquad \frac{d^n}{dt^n}[x(t) \pm ex(t-1)] + (-1)^{n+1}(1 \pm e^2)t^\gamma e^{-t}|x(\log t)|^\gamma \operatorname{sgn} x(\log t) = 0,$$

which are special cases of $(\mathrm{A}_\pm)$ with $\lambda = e$, $\tau = 1$, $g(t) = \log t$, and $F(t, v) = (1 + e^2)t^\gamma e^{-t} v^\gamma$. Suppose that $\gamma > 0$. Condition (3) is satisfied, since

$$\int^\infty t^{n-1} F(t, a\lambda^{g(t)/\tau})\, dt = a^\gamma(1 + e^2) \int^\infty t^{n+2\gamma-1} e^{-t}\, dt < \infty.$$

On the other hand, condition (5) is satisfied if $0 < \gamma < 2$, since in this case

$$\int^\infty \mu^{-t/\tau} F(t, a\lambda^{g^*(t)/\tau})\, dt = a^\gamma(1 + e^2) \int^\infty t^\gamma (e^{\gamma-1}/\mu)^t < \infty$$

for any μ such that $\max\{1, e^{\gamma-1}\} < \mu < e$. Theorem 2 implies that $(23_\pm)$ have unbounded oscillatory solutions $x_\pm(t)$ satisfying

$$x_\pm(t) = e^t\omega_\pm(t) + o(1) \quad \text{as} \quad t \to \infty \tag{24}$$

for any continuous oscillatory functions $\omega_\pm(t)$ such that $\omega_+(t+1) = -\omega_+(t)$ and $\omega_-(t+1) = \omega_-(t)$ for all t. According to Theorem 3, if $0 < \gamma < 2$, then $(23_\pm)$ have, in addition to these $x_\pm(t)$, unbounded oscillatory solutions $y_\pm(t)$ such that

$$y_\pm(t) = e^t\omega_\pm(t) + o(e^t) \quad \text{as} \quad t \to \infty$$

for the same functions $\omega_\pm(t)$ as above. In particular, if $0 < \gamma < 2$, there exist for every $m = 1, 2, \dots$ two different kinds of oscillatory solutions $x_m^\pm(t)$ and $y_m^\pm(t)$ with the properties

$$\begin{aligned} x_m^-(t) &= const \cdot e^t \sin(2m\pi t) + o(1) \quad \text{as} \quad t \to \infty, \\ y_m^-(t) &= const \cdot e^t \sin(2m\pi t) + o(e^t) \quad \text{as} \quad t \to \infty; \\ x_m^+(t) &= const \cdot e^t \sin((2m-1)\pi t) + o(1) \quad \text{as} \quad t \to \infty, \\ y_m^+(t) &= const \cdot e^t \sin((2m-1)\pi t) + o(e^t) \quad \text{as} \quad t \to \infty. \end{aligned}$$

References

1. Bainov D. D., Myshkis A. D. and Zahariev A. I., *Sufficient conditions for the existence of bounded non-oscillating solutions of functional differential equations of neutral type*, Rend. Mat. **7** (1987), 353–359.

2. Gopalsamy K., *Oscillation and nonoscillation in neutral differential equations with variable parameters*, J. Math. Phys. Sci. **21** (1987), 593–611.

3. Jaroš J. and Kusano T., *Oscillation theory of higher order linear functional differential equations of neutral type*, Hiroshima Math. J. **18** (1988), 509–531.

4. ______, *Asymptotic behavior of nonoscillatory solutions of nonlinear functional differential equations of neutral type*, Funkcial. Ekvac. **32** (1989), 251–263.

5. ______, *Nonoscillation theorems for a class of neutral functional differential equations of arbitrary order*, Proc. Seventh Czechoslovak Conference on Differential Equations and Their Applications, Prague, August 21-25, 1989 (to appear).

6. Ruan J., *Types and criteria of nonoscillatory solutions for second order linear neutral differential equations*, Chinese Ann. Math. Ser. A **8** (1987), 114–124. (Chinese)

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