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C^1 WEAKLY CHAOTIC FUNCTIONS WITH ZERO TOPOLOGICAL ENTROPY AND NON–FLAT CRITICAL POINTS

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Abstract. It is proved that there exist C^1 unimodal functions analytic in a neighbourhood of their (only) critical point having simultaneously topological entropy zero, wandering intervals and a scrambled set of positive Lebesgue measure.

1. Introduction

Let $C^0(I,I)$ denote the set of continuous functions $f\colon I \to I$, where $I \subset \mathbb{R}$ is a compact interval. There exist several ways to measure how complicated is the dynamics of a function $f \in C^0(I,I)$, being two extensively studied approaches to the problem, **topological entropy** and **chaos in the sense of Li and Yorke**.

Definition 1.1. Let $f \in C^0(I,I)$ and $h(f)$ denote the topological entropy of f. Then f is said to be strongly chaotic if $h(f) > 0$.

See Alder, Konheim and McAndrew [**1**] for the definition of topological entropy. For piecewise monotone functions Misiurewicz and Szlenk [[**13**]] showed that $h(f) = \lim_{n\to\infty} \log \gamma_n$, being γ_n the number of pieces of monoticity of f^n, the n-th iterate of f.

Definition 1.2. Let $f \in C^0(I,I)$. Suppose that there exist $S \subset I$ with at least two elements and $\delta \geq 0$ such that for any $x, y \in S$, $x \neq y$, and any periodic point p of f

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \delta,$$
$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0,$$
$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| > \delta.$$

(Here a point $p \in I$ is said to be periodic of period r, $r \geq 1$, if $f^r(p) = p$ and $f^i(p) \neq p$ for any $i = 1, \dots, r-1$). Then f is said to be weakly chaotic and S is called a scrambled set or, when $\delta > 0$, a δ-scrambled set.

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This definition is equivalent (see Kuchta and Smítal [**9**]) to the original Li and Yorke's version [**10**]. In particular any weakly chaotic function possesses a Cantor type δ-scrambled set (see Janková and Smítal [**7**], Smítal [**17**]).

The relations between weak and strong chaos were analyzed in a number of papers. So it is well known (see for example Graw [**6**] or Janková and Smítal [**7**]) that any strongly chaotic function is weakly chaotic. The converse is not true: from a first example from Smítal [**17**] turns that there exist weakly chaotic functions with zero topological entropy and possessing a δ-scrambled set of positive Lebesgue measure (when one could say, in some sense, that chaos is 'empirically observable'). It happens that any weakly chaotic function with zero topological entropy must be of type 2^∞, that is, it has periodic points of periods 2^n, $n \geq 0$, but no other periods.

In a natural way the question of the equivalence between the notions of weak and strong chaos in more restricted class of functions arises. In order to situate adequately the problem we need two definitions.

Definition 1.3. Let $f \in C^0(I, I)$ and J be a subinterval of I. Then J is said to be a wandering interval (of f) if

(i) $f^n(J) \cap f^m(J) = \emptyset$ for any $n \neq m$;
(ii) there is not a periodic point p of f such that $\lim_{n\to\infty} |f^n(x) - f^n(p)| = 0$ for any $x \in J$.

Definition 1.4. Let $f \colon I \to I$ be a differentiable function and c be a critical point of f. Then c is said to be non-flat if there is a $k \geq 3$ such that f is of class C^k in a neighbourhood of c and $f^{(k-1)}(c) \neq 0$, where $f^{(l)}$ denotes the l-th derivative of f. Otherwise c is called flat.

Since a weakly chaotic function with zero topological entropy must have wandering intervals (see Smítal [**17**] or Balibrea Gallego and Jiménez López [**2**]), from Martens, de Melo and van Strien [**11**] follows that weak and strong chaos coincide in the class of C^2 functions with only non-flat critical points. On the other hand, examples of C^∞ weakly chaotic functions with flat critical points and zero topological entropy are known, see Misiurewicz and Smítal [**12**] and Jiménez López [**8**] (in the second one the scrambled set has positive measure).

Recall that $f \in C^0([a,b],[a,b])$ is said to be **unimodal** if there exists $c \in (a,b)$ such that f is strictly increasing in $[a,c)$ and strictly decreasing in $(c,b]$. The aim of this paper is to complete above program giving examples of C^1 unimodal functions analytic in a neighbourhood of their only critical point and having both zero topological entropy and a scrambled set of positive measure. This partially contradicts a Preiss and Smítal' conjecture [**15**] stating that any adequately smooth unimodal function with zero topological entropy can not be weakly chaotic.

More precisely, we are going to prove the following

Theorem 1.5. *For any $\alpha \in [0,1)$ there exists a unimodal C^1 function $f_\alpha\colon [0,1] \to [0,1]$, analytic in a neighbourhood of its only critical point and with zero topological entropy such that it has a δ-scrambled set of the Cantor type with Lebesgue measure α for an adequate $\delta > 0$.*

Remark. Note that from the comment above all these functions have wandering intervals. Until now, the known examples of C^1 functions with only non-flat critical points and wandering intervals had positive topological entropy (see [**4**]).

Remark. Since a C^1 function can not have a measurable scrambled set of full Lebesgue measure (see Balibrea Gallego and Jiménez López [**2**]), the restriction $\alpha < 1$ can not be removed.

The organization of the paper goes as follows. In Section 2 some necessary results about differentiability are given in order to ensure the condition of being of class C^1 for f_α. The properties of unimodal functions of type 2^∞ are revised in Section 3, while f_α is constructed in Section 4. Theorem is proved in Section 5. The whole argument combines ideas from Smítal [**16**] and the classical Denjoy's example [**5**] (see also [**14**]).

Through all the paper, given I, J compact intervals $\psi(I;J)$ (respectively $\overline{\psi}(I;J)$) will denote the increasing (respectively decreasing) bijective linear function mapping I onto J. Also, if $a \neq b$ (not necessarily $a < b$) $\langle a,b\rangle$ will denote the compact interval having as endpoints a and b.

2. Some Analytic Tools

Lemma 2.1. *Let $[a,b],[c,d]$ be compact intervals, $\epsilon > 0$, $u,v > 0$ (resp. $u,v < 0$). Then there exists an increasing (resp. decreasing) C^1 diffeomorphism $f\colon [a,b] \to [c,d]$ such that*

(i) $f'(a) = u$, $f'(b) = v$;
(ii) $\min\{|u|,|v|,\frac{d-c}{b-a} - \epsilon\} < |f'(x)| < \max\{|u|,|v|,\frac{d-c}{b-a} + \epsilon\}$.

Proof. It is not restrictive to suppose $u,v > 0$, since the other possibility can be easily obtained from this one.

Put $a_0 = \min\{u,v,\frac{d-c}{b-a} - \epsilon\}$, $b_0 = \max\{u,v,\frac{d-c}{b-a} + \epsilon\}$. Consider a positive continuous function $g\colon [a,b] \to [a_0,b_0]$ satisfying $g(a) = u$, $g(b) = v$ and $g(a,b) \subset (a_0,b_0)$ and such that $\int_a^b g(t)dt = d-c$; the function $f(x) = c + \int_a^x g(t)dt$ does the job. □

Lemma 2.2. *Let $a < b$, $c < d$ be real numbers and m be a positive integer. Then*

$$a + \frac{i(c-a)}{m} < b + \frac{i(d-b)}{m}$$

for any $i = 0,1,\dots,m$.

We omit the simple proof.

The idea of the next lemma is to construct functions whose restrictions to certain intervals are linear through the composition of a sufficienty large number of 'almost ± 1 slope' functions. Since we get a result as general as possible, its formulation must necessarily be rather complicated.

Lemma 2.3. *Let $0 = \rho^0 < \gamma^1 < \rho^1 < \ldots < \gamma^r < \rho^r < \gamma^{r+1} = 1$ and $0 = \sigma^0 < \tau^1 < \sigma^1 < \ldots < \tau^r < \sigma^r < \tau^{r+1} = 1$ (resp. $1 = \sigma^0 > \tau^1 > \sigma^1 > \ldots > \tau^r > \sigma^r > \tau^{r+1} = 0$) be real numbers, $\epsilon > 0$. Then there exist a positive integer l and $\overline{\epsilon} > 0$ satisfying the following property:*

Take $m \geq l$ and let $\{[a_i, b_i]\}_{i=0}^m$ be an arbitrarily choosen collection of compact intervals such that

$$1 - \epsilon < \frac{b_{i+1} - a_{i+1}}{b_i - a_i} < 1 + \epsilon$$

for any $i = 0, \ldots, m-1$. Consider also a finite sequence of zeros and ones $(n(i))_{i=0}^{m-1}$ such that $\sum_{i=0}^{m-1} n(i)$ is even (resp. odd). Finally let $0 = \overline{\rho}^0 < \overline{\gamma}^1 < \overline{\rho}^1 < \ldots < \overline{\gamma}^r < \overline{\rho}^r < \overline{\gamma}^{r+1} = 1$ and $0 = \overline{\sigma}^0 < \overline{\tau}^1 < \overline{\sigma}^1 < \ldots < \overline{\tau}^r < \overline{\sigma}^r < \overline{\tau}^{r+1} = 1$ (resp. $1 = \overline{\sigma}^0 > \overline{\tau}^1 > \overline{\sigma}^1 > \ldots > \overline{\tau}^r > \overline{\sigma}^r > \overline{\tau}^{r+1} = 0$) holding

$$|\overline{\gamma}^j - \gamma^j| < \overline{\epsilon}, \ |\overline{\rho}^j - \rho^j| < \overline{\epsilon}, \ |\overline{\tau}^j - \tau^j| < \overline{\epsilon}, \ |\overline{\sigma}^j - \sigma^j| < \overline{\epsilon}$$

for any $j = 1, \ldots, r$. Define

$$a_0^j = \psi([0,1]; [a_0, b_0])(\overline{\gamma}^j) \quad \text{and} \quad a_m^j = \psi([0,1]; [a_m, b_m])(\overline{\tau}^j)$$

for any $j = 1, \ldots, r+1$,

$$b_0^j = \psi([0,1]; [a_0, b_0])(\overline{\rho}^j) \quad \text{and} \quad b_m^j = \psi([0,1]; [a_m, b_m])(\overline{\sigma}^j)$$

for any $j = 0, \ldots, r$. Then there exist a family $\{f_i : [a_i, b_i] \to [a_{i+1}, b_{i+1}]\}_{i=0}^{m-1}$ of C^1 diffeomorphisms, increasing if $n(i) = 0$ and decreasing if $n(i) = 1$ such that

(i) $|f_i'(a_i)| = |f_i'(b_i)| = 1$;
(ii) $1 - 2\epsilon < |f_i'(x)| < 1 + 2\epsilon$ *for any* $x \in [a_i, b_i]$;
(iii) *the composite function $g = f_{m-1} \circ \ldots \circ f_0$ satisfies*

$$g|_{[a_0^j, b_0^j]} = \psi([a_0^j, b_0^j]; [a_m^j, b_m^j])$$

$$\text{(resp. } g|_{[a_0^j, b_0^j]} = \overline{\psi}([a_0^j, b_0^j]; [a_m^j, b_m^j])) \text{ for any } j = 1, \ldots, r.$$

Proof. Throughout all the proof, $\sum_{s=0}^{-1} n(s)$ will mean zero.

Take $\overline{\epsilon} > 0$ small enough such that

$$\min_{i \leq j \leq r} \{|\tau^j - \sigma^j|, \rho^j - \gamma^j\} > 3\overline{\epsilon}, \tag{1}$$

$$\min_{0 \leq j \leq r} \{|\sigma^j - \tau^{j+1}|, \gamma^{j+1} - \rho^j\} > 3\overline{\epsilon}. \tag{2}$$

Also consider l adequately large such that

$$1 - 2\epsilon < (1-\epsilon)(1 - \frac{1}{\overline{\epsilon} l}), \tag{3}$$

$$1 + 2\epsilon > (1+\epsilon)(1 + \frac{1}{\overline{\epsilon} l}). \tag{4}$$

We shall prove that statements in Lemma are fulfilled with these constants.

For example suppose $1 = \sigma^0 > \tau^1 > \sigma^1 > \ldots > \tau^r > \sigma^r > \tau^{r+1} = 0$. Consider $m \geq l$, $\{[a_i, b_i]\}_{i=0}^m$, $(n(i))_{i=0}^{m-1}$, $\overline{\gamma}^j$, $\overline{\tau}^j$, $j = 1, \ldots, r+1$, $\overline{\rho}^j, \overline{\sigma}^j$, $j = 0, \ldots, r$ as in the hypothesis. Then we can define for any $i = 0, \ldots, m$

$$\overline{\gamma}_i^j = \overline{\gamma}^j + \frac{i(1 - \overline{\tau}^j - \overline{\gamma}^j)}{m} \quad j = 1, \ldots, r+1$$
$$\overline{\rho}_i^j = \overline{\rho}^j + \frac{i(1 - \overline{\sigma}^j - \overline{\rho}^j)}{m} \quad j = 0, \ldots, r$$

if $\sum_{s=0}^{i-1} n(s)$ is even and

$$\overline{\gamma}_i^j = 1 - \overline{\gamma}^j - \frac{i(1 - \overline{\tau}^j - \overline{\gamma}^j)}{m} \quad j = 1, \ldots, r+1$$
$$\overline{\rho}_i^j = 1 - \overline{\rho}^j - \frac{i(1 - \overline{\sigma}^j - \overline{\rho}^j)}{m} \quad j = 0, \ldots, r$$

if $\sum_{s=0}^{i-1} n(s)$ is odd. Now put for any i

$$a_i^j = \psi([0,1]; [a_i, b_i])(\overline{\gamma}_i^j) \quad j = 1, \ldots, r+1$$
$$b_i^j = \psi([0,1]; [a_i, b_i])(\overline{\rho}_i^j) \quad j = 0, \ldots, r.$$

Note that by Lemma 2.2

$$a_i = b_i^0 < a_i^1 < b_i^1 < \ldots < a_i^r < b_i^r < a_i^{r+1} = b_i \tag{5}$$

if $\sum_{s=0}^{i-1} n(s)$ is even and

$$b_i = b_i^0 > a_i^1 > b_i^1 > \ldots > a_i^r > b_i^r > a_i^{r+1} = a_i \tag{6}$$

if $\sum_{s=0}^{i-1} n(s)$ is odd.

We claim

$$1 - 2\epsilon < \frac{|b_{i+1}^j - a_{i+1}^j|}{|b_i^j - a_i^j|} < 1 + 2\epsilon \tag{7}$$

for any $i = 0, \ldots, m-1$ and any $j = 1, \ldots, r$. In fact

$$\frac{|b_{i+1}^j - a_{i+1}^j|}{|b_i^j - a_i^j|} = \frac{b_{i+1} - a_{i+1}}{b_i - a_i} \, \frac{(i+1)(\overline{\tau}^j - \overline{\sigma}^j) + (m-i-1)(\overline{\rho}^j - \overline{\gamma}^j)}{i(\overline{\tau}^j - \overline{\sigma}^j) + (m-i)(\overline{\rho}^j - \overline{\gamma}^j)}$$
$$= \frac{b_{i+1} - a_{i+1}}{b_i - a_i} \left(1 \pm \frac{|(\overline{\tau}^j - \overline{\sigma}^j) - (\overline{\rho}^j - \overline{\gamma}^j)|}{i(\overline{\tau}^j - \overline{\sigma}^j) + (m-i)(\overline{\rho}^j - \overline{\gamma}^j)} \right)$$

On the other hand by (1) we have that

$$\frac{|(\overline{\tau}^j-\overline{\sigma}^j)-(\overline{\rho}^j-\overline{\gamma}^j)|}{i(\overline{\tau}^j-\overline{\sigma}^j)+(m-i)(\overline{\rho}^j-\overline{\gamma}^j)}<\frac{1}{m\min_{i\le j\le r}\{\overline{\tau}^j-\overline{\sigma}^j,\overline{\rho}^j-\overline{\gamma}^j\}}<\frac{1}{m\overline{\epsilon}}.$$

Then by the hypothesis

$$(1-\epsilon)(1-\frac{1}{m\overline{\epsilon}})<\frac{|b^j_{i+1}-a^j_{i+1}|}{|b^j_i-a^j_i|}<(1+\epsilon)(1+\frac{1}{m\overline{\epsilon}}).$$

From (3) and (4), (7) follows.

Analogously we prove now

$$1-2\epsilon<\frac{|a^{j+1}_{i+1}-b^j_{i+1}|}{|a^{j+1}_i-b^j_i|}<1+2\epsilon \tag{8}$$

for any $i=0,\dots,m-1$ and any $j=0,\dots,r$. Firstly

$$\begin{aligned}\frac{|a^{j+1}_{i+1}-b^j_{i+1}|}{|a^{j+1}_i-b^j_i|}&=\frac{b_{i+1}-a_{i+1}}{b_i-a_i}\cdot\\ &\cdot\left|\frac{m\overline{\gamma}^{j+1}+(i+1)(1-\overline{\tau}^{j+1}-\overline{\gamma}^{j+1})-m\overline{\rho}^j-(i+1)(1-\overline{\sigma}^j-\overline{\rho}^j)}{m\overline{\gamma}^{j+1}+i(1-\overline{\tau}^{j+1}-\overline{\gamma}^{j+1})-m\overline{\rho}^j-i(1-\overline{\sigma}^j-\overline{\rho}^j)}\right|\\ &=\frac{b_{i+1}-a_{i+1}}{b_i-a_i}\left(1\pm\frac{|(\overline{\sigma}^j-\overline{\tau}^{j+1})-(\overline{\gamma}^{j+1}-\overline{\rho}^j)|}{i(\overline{\sigma}^j-\overline{\tau}^{j+1})+(m-i)(\overline{\gamma}^{j+1}-\overline{\rho}^j)}\right)\end{aligned}$$

Using (2) and reasoning as before we have (8).

Now observe that from Lemma 2.1 and (5)–(8) one can construct for any $i=0,\dots,m-1$ a C^1 diffeomorphism $f_i\colon[a_i,b_i]\to[a_{i+1},b_{i+1}]$ such that $f_i(a^j_i)=a^j_{i+1}$ for $j=1,\dots,r+1$ and $f_i(b^j_i)=b^j_{i+1}$ for $j=0,\dots,r$, $f_i|_{\langle a^j_i,b^j_i\rangle}$ is linear and it fulfils the conditions (i), (ii) in Lemma.

This finishes the proof. □

3. On Unimodal Functions of Type 2^∞

Choose $\varphi\colon[0,1]\to[0,1]$ an unimodal function of type 2^∞ (for an example of such a function, see [**18**]). From now on we suppose φ fixed. Through a well known process of renormalization (cf. [**19**] for details; the unimodal functions $f:[0,1]\to[0,1]$ used in this paper verify $f(0)=f(1)=0$ but the conclusions can be immediately generalized) one can prove that if d is its turning point, the elements of the forward orbit $(d_n)_{n=0}^\infty$ of d, where $d_n=\varphi^n(d)$, are ordered in the interval as follows:

(i) $d_0<d_1$;

(ii) For a certain positive integer k, suppose that the points $\{d_n\}_{n=0}^{2^k-1}$ are ordered in the interval as $d_{l(0)}, d_{l(1)}, \dots, d_{l(2^k-2)}, d_{l(2^k-1)}$. Then for any $i \in \{0, \dots, 2^{k-1}-1\}$, consider the pair $d_{l(2i)}, d_{l(2i+1)}$. There are three possibilities:

(ii.1) if $l(2i) = 0$, then $d_{l(2i-1)} < d_{2^k} < d_0 < d_{2^k+l(2i+1)} < d_{l(2i+1)}$ (in the case $k = 1$ we clearly must remove the term $d_{l(2i-1)}$ of the list);

(ii.2) if $l(2i+1) = 0$, then $d_{l(2i)} < d_{2^k+l(2i)} < d_0 < d_{2^k} < d_{l(2i+2)}$;

(ii.3) if $l(2i) \neq 0 \neq l(2i+1)$ then $d_{l(2i)} < d_{2^k+l(2i)} < d_{2^k+l(2i+1)} < d_{l(2i+1)}$. Explicitly

$$\begin{gathered} d_0 < d_1 \\ d_2 < d_0 < d_3 < d_1 \\ d_2 < d_6 < d_0 < d_4 < d_3 < d_7 < d_5 < d_1 \\ \dots \end{gathered} \tag{9}$$

and so on.

On the other hand, for any positive integer n define the sequence $\beta^{-n} = (2^{i(n)+i} - n)_{i=0}^{\infty}$, where $i(n)$ is the first positive integer such that $2^{i(n)} - n$ is also positive. Note that for any n and i we have $\langle d_{\beta^{-n}(2i)}, d_{\beta^{-n}(2i+1)}\rangle \supset \langle d_{\beta^{-n}(2i+2)}, d_{\beta^{-n}(2i+3)}\rangle$. From this and (9) it is very easy to inductively construct a sequence $(d_{-n})_{n=1}^{\infty}$ such that $d_{-n} \in \bigcap_{i=0}^{\infty}\langle d_{\beta^{-n}(2i)}, d_{\beta^{-n}(2i+1)}\rangle$ and $\varphi(d_{-n}) = d_{-n+1}$ for any $n \geq 1$ (observe in particular that $\varphi(d_{-1}) = d$).

We need to construct unimodal chaotic functions of type 2^{∞}. Apart from above considerations about the turning point, recall that any chaotic function of type 2^{∞} must possess wandering intervals. All of this justifies the ensuing construction.

Define a total ordering $\prec$ in $\mathbb{Z}$ as follows: $i \prec j$ if and only if $d_i < d_j$. As usual, $i \preceq j$ will denote either $i \prec j$ or $i = j$. We already know some properties of this ordering. Additionally note also that two integers n, m are such that there is no j with $n \prec j \prec m$ or $m \prec j \prec n$ (we shall call them $\prec$-**consecutive**) if and only if they are in the form $3 \cdot 2^k - m$, $2^{k+2} - m$, with $k \geq 0$ and $m = 0, \dots, 2^k - 1$.

Consider positive numbers μ_n for any $n \in \mathbb{Z}$ and $\mu_{i,j}$ for any pair $i < j$ $\prec$-consecutive, such that

$$\sum_n \mu_n + \sum_{i,j} \mu_{i,j} = 1. \tag{10}$$

Put $a_2 = 0$ and for any integer $n \neq 2$ define

$$a_n = \sum_{k \prec n} \mu_k + \sum_{i,j \preceq n} \mu_{i,j}.$$

Also let $b_n = a_n + \mu_n$ for any integer n. Finally consider the intervals $I_n = [a_n, b_n]$ for any n and $I_{i,j} = [b_i, a_j]$ for any $\prec$-consecutive integers $i < j$.

Let $A = (\bigcup_n I_n) \cup (\bigcup_{i,j} I_{i,j})$. Now suppose $\overline{f}\colon A \to [0,1]$ be any function satisfying the following conditions:

(i) $\overline{f}|_{I_n}: I_n \to I_{n+1}$ and it is an homeomorphism for any $n \neq 0$;
(ii) $\overline{f}|_{I_0}\colon I_0 \to I_1$ is continuous, $\overline{f}(a_0) = \overline{f}(b_0) = a_1$ and it has a unique turning point;
(iii) $\overline{f}|_{I_{3\cdot 2^k-m,2^{k+2}-m}}: I_{3\cdot 2^k-m,2^{k+2}-m} \to I_{3\cdot 2^k-m+1,2^{k+2}-m+1}$ and it is a homeomorphism for any $k \geq 0$ and $m = 1, \dots, 2^k - 1$;
(iv) $\overline{f}|_{I_{3\cdot 2^k,2^{k+2}}}: I_{3\cdot 2^k,2^{k+2}} \to [b_{3\cdot 2^k+1}, a_{2^{k+2}+1}]$ and it is a homeomorphism for any $k \geq 0$.
Note that for the properties of $\prec$ one can also get
(v) $\overline{f}|_{A\cap[0,a_0]}$ is strictly increasing, $\overline{f}|_{A\cap[b_0,1]}$ is strictly decreasing.

Then

Lemma 3.1. *Let* $\overline{f}\colon A \to [0,1]$ *satisfy* (i)–(v) *as above. Then it can be extended to a continuous unimodal function* $f\colon [0,1] \to [0,1]$.

Proof. Since by (10) A is dense in $[0,a_0]$ and $\overline{f}(A \cap [0,a_0])$ is dense in $[a_3,a_1]$, from (v) $\overline{f}$ can be extended in $[0,a_0]$ to a continuous strictly increasing function. In an analogous way we can extend $\overline{f}$ in the intervals $[b_0,a_{-1}]$ and $[b_{-1},1]$ to continuous strictly decreasing functions whose images respectively lie on $[b_0,a_1]$ and $[0,a_0]$. So we get a continuous function $f\colon [0,1] \to [0,1]$ such that $f|_A = \overline{f}$. Clearly f is unimodal. □

A function f constructed as in the above lemma will be called **appropriate** (with **associated parameters** $\mu_n, \mu_{i,j}$).

We intend to show that any appropriate function is of type 2^∞. For this purpose we need some basic results of kneading theory. A reader interested in the subject is referred to the already classic book of Collet and Eckmann [**3**].

Let Λ denote the set of the infinite sequences of $0,1,2$. If $\theta \in \Lambda$, its n-th element will be denoted as θ_n.

We introduce in Λ the **shift** operation $\mathcal{S}$ defined as follows. If $\theta = \theta_0\theta_1\theta_2\dots$, then $\mathcal{S}(\theta) = \theta_1\theta_2\theta_3\dots$. Denote by $\mathcal{S}^j$ the j-th iterate of $\mathcal{S}$. A sequence $\theta \in \Lambda$ will be called **periodic of period** r, $r \geq 1$, if $\mathcal{S}^r(\theta) = \theta$ and $\mathcal{S}^i(\theta) \neq \theta$ for any $i = 1, \dots, r-1$.

Also we will use the following total ordering in Λ. Let $\theta \neq \vartheta$ be two elements of Λ. Then we say that $\theta \lhd \vartheta$ if either

(i) there are an even number of 2's in $\theta_0\theta_1\dots\theta_{k-1} = \vartheta_0\vartheta_1\dots\vartheta_{k-1}$ and $\theta_k < \vartheta_k$
or
(ii) there are an odd number of 2's in $\theta_0\theta_1\dots\theta_{k-1} = \vartheta_0\vartheta_1\dots\vartheta_{k-1}$ and $\theta_k > \vartheta_k$.

Otherwise we say that $\vartheta \lhd \theta$. As usual $\trianglelefteq$ will denote $\lhd$ or $=$.

Now let $f\colon [0,1] \to [0,1]$ be an arbitrary unimodal function, c be its turning point and $x \in [0,1]$. We define the **extended itinerary** of x, $\iota_f(x) \in \Lambda$ as

$$\iota_f(x)_n = \begin{cases} 0 & \text{if } f^n(x) < c, \\ 1 & \text{if } f^n(x) = c, \\ 2 & \text{if } f^n(x) > c. \end{cases}$$

for any $n \geq 0$. If there is not possibility of confusion, we simply shall write $\iota(x)$.

Lemma 3.2. *Let $f\colon [0,1] \to [0,1]$ be a unimodal function and $x, y \in [0,1]$. If $x < y$ then $\iota(x) \lhd \iota(y)$.*

Proof. See Lemma II.1.3 from [**3**]. □

Lemma 3.3. *Let $f\colon [0,1] \to [0,1]$ be a unimodal function.*

(i) *Suppose that $p \in [0,1]$ is periodic of period r. Then $\iota(p)$ is also periodic of period r.*
(ii) *Suppose that there exists $x \in [0,1]$ such that $\iota(x)$ is periodic of period r. Then there exists $p \in [0,1]$ periodic of period r.*

Proof. (i) is trivial; for (ii) see Lemma II.3.4 from [**3**]. □

Lemma 3.4. *Let $f\colon [0,1] \to [0,1]$ be an unimodal function and c be its turning point. Suppose that $\iota(c)$ is not periodic. Let $\theta \in \Lambda$ such that $\iota(0) \lhd \theta$ and $\mathcal{S}^j(\theta) \lhd \iota(f(c))$ for any $j \geq 0$. Then there exists $x \in [0,1]$ such that $\iota(x) = \theta$.*

Proof. See Theorem II.3.8 from [**3**]. □

Remark. The unimodal functions considered in [**3**] verify $f(c) = 1$, but the generalization of above results to arbitrary unimodal functions is immediate.

Now we already can prove

Lemma 3.5. *Let $f\colon [0,1] \to [0,1]$ be an appropriate function. Then it is of type 2^∞.*

Proof. Let us return to φ introduced above. Suppose p be a periodic point of φ of period r. Since $\iota_\varphi(d)$ is not periodic, by Lemmas 3.2 and 3.3 the sequence $\vartheta = \iota_\varphi(p)$ is in the conditions of Lemma 3.4. Let c be the turning point of f. From $\iota_f(c) = \iota_\varphi(d)$ there exist a point $x \in [0,1]$ such that $\iota_f(x) = \iota_\varphi(p)$. Now from Lemma 3.3 there exists a periodic point q of f of period r.

Analogously, if there exists a periodic point of f of period r then also there exists a periodic point of φ with the same period. This finishes the proof. □

4 Construction of f_α

In this section we shall construct f_α. Until the end of paper, $0 \leq \alpha < 1$ will be fixed.

We intend f_α to be an appropriate function. So we begin describing its associated parameters. Firstly, put $\overline{\mu}_n = \frac{1}{(n-1)^2}$ for any integer $n \neq 1$ and $\overline{\mu}_1 = 1$. Moreover let

$$\xi = \frac{22+\pi^2}{16}$$

and take for every $k > 0$

$$\overline{\mu}_{3\cdot 2^k - m, 2^{k+2}-m} = \frac{1}{4^k}$$

if $k \leq m \leq 2^k - 1$ and

$$\overline{\mu}_{3\cdot 2^k - m, 2^{k+2}-m} = \frac{\sqrt{[k]}\xi^{k-m}}{4^k} \tag{11}$$

if $0 \leq m < k$. Finally choose ζ such that $\mu_n = \zeta\overline{\mu}_n$, $\mu_{i,j} = \zeta\overline{\mu}_{i,j}$ satisfy

$$\sum_n \mu_n + \sum_{i>3, j>4} \mu_{i,j} < 1 - \alpha$$

and define $\mu_{3,4}$ in such a way that (10) holds. Until the end of proof these constants and the corresponding $I_n, a_n, b_n, I_{i,j}$ as in the above Lemma 3.1 will remain fixed.

Lemma 4.1. *Let $f\colon [0,1] \to [0,1]$ an appropriate function with associated parameters $\mu_n, \mu_{i,j}$. Then*

(i) $\displaystyle\lim_{n\to\pm\infty} \frac{\mu_{n+1}}{\mu_n} = 1;$

(ii) $\displaystyle\lim_{(i,j)\to(\infty,\infty)} \frac{\mu_{i+1,j+1}}{\mu_{i,j}} = 1;$

(iii) $\displaystyle\lim_{k\to\infty} \frac{a_{2^{k+2}+1} - b_{3\cdot 2^k+1}}{\mu_{3\cdot 2^k, 2^{k+2}}} = 1;$

(iv) $\displaystyle\lim_{k\to\infty} \frac{a_{2^{k+2}+1} - b_{3\cdot 2^k+1}}{\mu_{3\cdot 2^k - 2^k+1, 2^{k+2}-2^k+1}} = \xi.$

Proof. (i) and (ii) are obvious and (iv) easily follows from (iii). So it only rests to prove (iii). In order to simplify the notation we call $I_{3\cdot 2^k - m, 2^{k+2}-m}$, with $k \geq 0$ and $m = 0, \ldots, 2^k - 1$, an interval of **type** k. Note that $[b_{3\cdot 2^k+1}, a_{2^{k+2}+1}]$ exactly contains one interval of each type k and $k+1$, and 2^j intervals of type $k+2+j$ for $j \geq 0$. In addition to this, observe that it contains no intervals of type l with length larger than $\frac{1}{4^l}$ for any $l < 2^{k+1}$. Moreover, the intervals I_n included in $[b_{3\cdot 2^k+1}, a_{2^{k+2}+1}]$ are precisely these with $n = 1 \pm 2^{k+1}(2r+1)$, $r \geq 0$. Since

$$\frac{1}{4^k} + \frac{1}{4^{k+1}} + \sum_{j=0}^{\infty} \frac{2^j}{4^{k+2+j}} + 2\sum_{r=0}^{\infty} \frac{1}{[2^{k+1}(2r+1)]^2} = \frac{\xi}{4^k},$$

we have from (10) and (11) that

$$\left| \frac{a_{2^{k+2}+1} - b_{3\cdot 2^k+1}}{\mu_{3\cdot 2^k, 2^{k+2}}} - 1 \right| < \frac{1}{16\xi} \sum_{j=2^{k+1}-k-2}^{\infty} \frac{1}{2^j} = \frac{1}{2^{2^{k+1}-k+3}\xi}.$$

From here, (iii) follows. □

Lemma 4.2. *Let for any* $k \geq 0$

$$\sigma_k = \frac{a_{3\cdot 2^k - 2^k + 1} - b_{3\cdot 2^k + 1}}{a_{2^{k+2}+1} - b_{3\cdot 2^k+1}},$$
$$\tau_k = \frac{b_{2^{k+3}-2^{k+1}+1} - b_{3\cdot 2^k+1}}{a_{2^{k+2}+1} - b_{3\cdot 2^k+1}}.$$

Then

$$\lim_{k\to\infty} \sigma_k = \frac{1}{\xi}, \ \lim_{k\to\infty} \tau_k = 1 - \frac{1}{4\xi}.$$

Proof. Note that

$$3\cdot 2^k + 1 = 2^{k+2} - 2^k + 1, \ 2^{k+2} + 1 = 3\cdot 2^{k+1} - 2^{k+1} + 1.$$

Since $a_{3\cdot 2^k - 2^k+1} - b_{3\cdot 2^k+1} = \frac{1}{4^k}$, $a_{2^{k+2}+1} - b_{2^{k+3}-2^{k+1}+1} = \frac{1}{4^{k+1}}$, Lemma immediately follows from Lemma 4.1 (iv). □

To finish with the construction of f_α we need additional notation. First choose adequate constants $0 < \gamma_\alpha < \rho_\alpha < 1$ in such a way that $(\rho_\alpha - \gamma_\alpha)(a_3 - b_4) > \alpha$. Let $\bar{a} = b_{2^{k+2}-2^k+1}$, $\bar{b} = a_{3\cdot 2^k - 2^k + 1}$ and construct

$$J_k = [\bar{a} + \gamma_\alpha(\bar{b} - \bar{a}), \bar{a} + \rho_\alpha(\bar{b} - \bar{a})]$$

for any $k \geq 0$. Also define $g_k = \overline{\psi}(J_k; J_{k+1})$.

Now let J be an open subinterval of J_0 and $k \geq 0$. To simplify the notation put $J_k = [a, b]$, $J_{k+1} = [c, d]$, $(u, v) = \psi(J_0; J_k)(J)$, $u_0 = u + \frac{1}{5}(v-u)$, $u_1 = u + \frac{2}{5}(v-u)$, $v_0 = b + \frac{1}{3}(\bar{b} - b)$, $v_1 = b + \frac{2}{3}(\bar{b} - b)$, $z = g_k(u)$, $w = g_k(v)$. Suppose $a < u < v < b$. Then we construct

$$g_{k,J} : [a, u] \cup [u_0, u_1] \cup [v, b] \cup [v_0, v_1] \to J_{k+1}$$

as follows:

$$g_{k,J}|_{[a,u]} = \overline{\psi}([a, u]; [z, d]);$$
$$g_{k,J}|_{[u_0,u_1]} = \overline{\psi}([u_0, u_1]; [c, w]);$$
$$g_{k,J}|_{[v,b]} = \overline{\psi}([v, b]; [v_0, v_1]);$$
$$g_{k,J}|_{[v_0,v_1]} = \overline{\psi}([v_0, v_1]; [u_0, u_1]).$$

Moreover, put $u_2 = u + \frac{3}{5}(v - u)$, $u_3 = u + \frac{4}{5}(v - u)$ and define

$$\bar{g}_{k,J} : [a, u] \cup [u_0, u_1] \cup [u_2, u_3] \cup [v, b] \cup [v_0, v_1] \to J_{k+1}$$

such that

$$\bar{g}_{k,J}|_{[a,u]\cup[u_0,u_1]\cup[v_0,v_1]} = g_{k,J}|_{[a,u]\cup[u_0,u_1]\cup[v_0,v_1]};$$
$$\bar{g}_{k,J}|_{[u_2,u_3]} = \overline{\psi}([u_2,u_3];[v_0,v_1]);$$
$$\bar{g}_{k,J}|_{[v,b]} = \overline{\psi}([v,b];[u_2,u_3]).$$

Finally let $S = J_0 \setminus \bigcup_{m=0}^{\infty} U_m$ be a Cantor type set of measure α, where U_m are open intervals pairwise disjoint and included in J_0. Let $(V_l)_{l=1}^{\infty}$ be a sequence containing each interval U_m infinitely times. A_l and B_l will respectively denote the left and right connected component of $J_0 \setminus V_l$ (we suppose S choosen in such a way that A_l and B_l are always non-degenerate) and $W_l = \overline{\psi}(J_0;J_0)(V_l)$.

We already are in conditions to construct f_α. If we choose a strictly increasing sequence $(k(l))_{l=1}^{\infty}$ of multiples of 4 with each $k(l)$ adequately large, by virtue of Lemmas 2.1, 2.3, 4.1 and 4.2 it is clear that there exists a function $f_\alpha\colon [0,1] \to [0,1]$ verifying the following properties:

(P1) it is appropriate with associated parameters μ_n, $\mu_{i,j}$ (see the beginning of this section);

(P2) $f_\alpha^{2^k}|_{I_{3\cdot 2^k-2^k+1,2^{k+2}-2^k+1}}$ extends to g_k if $k \neq k(l), k(l)+1, k(l)+2, k(l)+3$ for any l, extends to $\bar{g}_{k,V_l}$ if $k = k(l)$ or $k = k(l)+2$ and extends to g_{k,W_l} if $k = k(l)+1$ or $k = k(l)+3$;

(P3) $f_\alpha|_{I_n}$, $f_\alpha|_{I_{i,j}}$ are of class C^1 for any $n \neq 0$ and i,j and analytic for $n = 0$,with $|f'_\alpha| = 1$ on the endpoints of all these intervals. Moreover

$$\lim_{n\to\pm\infty} \max_{x\in I_n}\{|f'_\alpha|(x)\} = \lim_{n\to\pm\infty} \min_{x\in I_n}\{|f'_\alpha|(x)\} = 1,$$
$$\lim_{(i,j)\to(\infty,\infty)} \max_{x\in I_{i,j}}\{|f'_\alpha|(x)\} = \lim_{(i,j)\to(\infty,\infty)} \min_{x\in I_{i,j}}\{|f'_\alpha|(x)\} = 1.$$

5. Proof of Theorem

In this last section we shall check that f_α is the function that we need. Note that by (P1) and Lemmas 3.1 and 3.5 it is unimodal of type 2^∞.

Lemma 5.1. *f_α is a function of class C^1.*

Proof. Let $x \in [0,1]$ with $x \notin I_n$, $x \notin I_{i,j}$, for any n and i,j and suppose for example y belonging to the interior of a certain $I_{i,j}$, with z the endpoint of $I_{i,j}$ located between x and y. Note that by (10),

$$|y-x| = \sum_{n:\ I_n\subset\langle x,y\rangle} \mu_n + \sum_{i,j:\ I_{i,j}\subset\langle x,y\rangle} \mu_{i,j} + |y-z|.$$

and similarly for $|f_\alpha(y) - f_\alpha(x)|$. Thus if x and y are sufficiently near we have that from Lemma 4.1 and (P3)

$$1-\epsilon < \left|\frac{f_\alpha(y)-f_\alpha(x)}{y-x}\right| < 1+\epsilon$$

for any $\epsilon > 0$. A similar argument runs for y verifying each of the other possibilities and so we conclude that f_α is differentiable in x being $|f'_\alpha(x)| = 1$. Reasoning in the same way we also prove that $|f'_\alpha(x)| = 1$ when x is an endpoint of the intervals I_n, $I_{i,j}$.

From this and (P3) it clearly follows that $f_\alpha(x)$ is a C^1 function. □

Lemma 5.2. *S is a δ-scrambled set of f_α for any $\delta < b_0 - a_0$.*

Proof. Take $k \geq 0$. Note that if $k \neq k(l), k(l)+1, k(l)+2, k(l)+3$ for any l then from (P2) it follows

$$f_\alpha^{2^k}|_{J_k} = \overline{\psi}(J_k; J_{k+1}). \tag{12}$$

Now consider the possibility $k = k(l)$. Put $C^i = \psi(J_0; J_{k+i})(A_k)$ and $D^i = \psi(J_0; J_{k+i})(B_k)$ if $i = 0, 2, 4$, $C^i = \overline{\psi}(J_0; J_{k+i})(A_k)$ and $D^i = \overline{\psi}(J_0; J_{k+i})(B_k)$ if $i = 1, 3$. First of all, from (P2) we have

$$f_\alpha^{3\cdot 2^k - 1}(x) < a_0 \text{ for any } x \in C^0 \text{ and } f_\alpha^{3\cdot 2^k - 1}(y) > b_0 \text{ for any } y \in D^0. \tag{13}$$

Moreover

$$f_\alpha^{2^k}|_{C^0} = \overline{\psi}(C^0; C^1),\ f_\alpha^{3\cdot 2^k}|_{D^0} = \psi(D^0; D^1),$$
$$f_\alpha^{2^{k+2}}|_{C^1} = \overline{\psi}(C^1; C^2),\ f_\alpha^{2^{k+1}}|_{D^1} = \overline{\psi}(D^1; D^2).$$

From this,

$$f_\alpha^{5\cdot 2^k}|_{C^0} = \psi(C^0; C^2),\ f_\alpha^{5\cdot 2^k}|_{D^0} = \overline{\psi}(D^0; D^2).$$

Repeating this process for C^i, D^i, $i = 3, 4$, we finally obtain

$$f_\alpha^{5(2^k + 2^{k+2})}|_{C^0} = \psi(C^0; C^4),\ f_\alpha^{5(2^k + 2^{k+2})}|_{D^0} = \psi(D^0; D^4). \tag{14}$$

Now from (12)–(14) and the fact that the length of the intervals J_k converges to zero, easily follows that for any $x \neq y$ points of S,

$$\limsup_{n\to\infty} |f_\alpha^n(x) - f_\alpha^n(y)| \geq b_0 - a_0,$$
$$\liminf_{n\to\infty} |f_\alpha^n(x) - f_\alpha^n(y)| = 0.$$

Finally from (12)–(14) again we obtain that given $x \in S$ and $m \geq 0$ it is possible to choose $r > 0$ and a strictly increasing sequence $(r(n))_n$ of multiples of 2^m such that $f_\alpha^{r(n)+r}(x) < a_0$ if n is odd and $f_\alpha^{r(n)+r}(x) > b_0$ if n is even. Thus for any periodic point p of f we have

$$\limsup_{n\to\infty} |f_\alpha^n(x) - f_\alpha^n(p)| \geq b_0 - a_0.$$

□

With these last two lemmas, the proof of Theorem is finished.

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