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ON DEGREES IN MULTIHOMOGENEOUS IDEAL THEORY

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The special case of an intersection of an algebraic variety with a hyperplane plays an important role in Bézout's theorem. In this area J. Stückrad and W. Vogel proved the so-called h_1-condition in 1971 (see [**4**]). We think this condition is also important in the multihomogeous case of Bézout's theorem and so we give here the extension of this condition for this case.

We want also look at the new investigations in [**3, 5**] of the problem of the converse to Bézout's theorem.

At the end of our paper we discuss a relation, which is used by D. W. Masser and G. Wüstholz in the proof of Lemma A1 in [**1**].

Here we confine ourselves to the bihomogeneous case, because we obtain all results in the multihomogeneous case analogously.

Let $R := K[x_0, \dots, x_k, y_0, \dots, y_m]$ be a polynomial ring over an infinite field K. Let a be a $0 \leq d$-dimensional multihomogeneous ideal of R. We let $H(s,t;a)$ denote the Hilbert function of a. For large s and t $H(s,t;a)$ is a polynomial in s and t:

$$H(s,t;a) = \sum \alpha_{ij}(a) \cdot \binom{s}{i} \cdot \binom{t}{j},$$

where the sum is taken over all $i, j \geq 0$ and $i + j \leq d$. The α_{ij} with $i + j = d$ are non-negative integers. They are called the degrees of the ideal a and will be denoted by δ_{ij}. In all other cases we set $\delta_{ij} = 0$. Furthermore, there exists at least one of the δ_{ij} with $i + j = d$ which is greater than zero. For these results see [**6**] Theorem 7 and 11.

In the following lemma we list some properties of the Hilbert function and the degrees:

Lemma 1. *Let a and b be multihomogeneous ideals and f a form of multidegree (σ, τ). Then*

1. $H(s,t;a+b) = H(s,t;a) + H(s,t;b) - H(s,t;a \cap b)$

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2. $H(s,t;(a,f)) = H(s,t;a) - H(s-\sigma,t-\tau;a:(f))$

   For the degrees we get the following results:

3. $\delta_{ij}((f)) = \begin{cases} \sigma, & \text{for } i = k-1, j = m \\ \tau, & \text{for } i = k, j = m-1 \\ 0, & \text{otherwise} \end{cases}$

   If a have dimension $d \geq 0$, then:

4. *If $i + j = d$ then $\delta_{ij}((a,f)) = \delta_{ij}(a) - \delta_{ij}(a:(f))$*
5. *If $i, j \geq 0$ and $i + j = d - 1$, then*

$$\alpha_{ij}((a,f)) = \sigma \cdot \delta_{i+1j}(a:(f)) + \tau \cdot \delta_{ij+1}(a:(f)) \\ + \alpha_{ij}(a) - \alpha_{ij}(a:(f))$$

6. *If a and b have the same dimension $d \geq 0$ and $a \supseteq b$ then $\delta_{ij}(a) \leq \delta_{ij}(b)$.*
7. *Let q be a multihomogeneous primary ideal belonging to the prime ideal p and let $l(q)$ be the length of q. Then $\delta_{ij}(q) = l(q) \cdot \delta_{ij}(p)$.*
8. *Let $a = q_1 \cap \cdots \cap q_\lambda$ be a primary decomposition of a with $\dim q_i = d$ for $1 \leq i \leq \mu$ and $\dim q_j < d$ for $\mu < j \leq \lambda$. Then*

$$\delta_{ij}(a) = \delta_{ij}(q_1) + \cdots + \delta_{ij}(q_\mu).$$

9. *Let a be an irrelevant ideal, i.e., there are nonnegative integers λ, μ, such that*

$$a \supseteq (x_1, \ldots, x_k)^\lambda \cdot (y_1, \ldots, y_m)^\mu.$$

   Then for large s and t $H(s,t;a) = 0$ and so $\delta_{ij}(a) = 0$ for all i, j. The dimension of a is -1.

Proof. See van der Waerden [**6**] or the appendix of [**1**]. □

Remark. We always use the terminus primary decomposition for a normal decomposition (i.e., this is an irredundant one in which the prime ideals belonging to various primary components are all different.)

By $\dim a$ we always mean the bihomogeneous dimension of the bihomogeneous ideal a, that is the Krull-dimension minus 2.

Notation. If a is a multihomogeneous ideal with $\dim a > 0$, then we fix a decomposition of a:

$$a = a_d \cap a_{d-1} \cap c,$$

where a_d is the intersection of all d-dimensionaly primary components of a, a_{d-1} is the intersection of all $(d-1)$-dimensional primary components of a, and c is the intersection of all other primary components of a.

Now we formulate the multihomogeneous condition corresponding to the h_1-condition and give an elementary proof of it (see [**4**]).

Proposition 1. *Let a be an multihomogeneous ideal with* $\dim a = d > 0$ *and f be a form with the multidegree (σ, τ) and let* $\dim(a, f) = d-1$. *Then the following conditions are equivalent:*

(i) *for all $i, j \geq 0$ and $i + j = d-1$ $\alpha_{ij}(a) = \alpha_{ij}(a : (f))$*
(ii) *f is not contained in any $(d-1)$-dimensional prime ideal belonging to a.*

Proof. (ii) $\Longrightarrow$ (i): We get by computation of the Hilbert function $H(s, t; a) = H(s, t; a_d \cap a_{d-1} \cap c)$ according to Lemma 1.1, that the α_{ij} with $i + j = d-1$ only depend on $a_d \cap a_{d-1}$. Therefore we obtain (i).

(i) $\Longrightarrow$ (ii): Also by Lemma 1.1 we get for $i, j \geq 0$ and $i + j = d-1$:

$$\alpha_{ij}(a) = \alpha_{ij}(a_d \cap a_{d-1}) = \alpha_{ij}(a_d) + \delta_{ij}(a_{d-1}) - \delta_{ij}(a_d + a_{d-1})$$

and in accordance with $\dim(a, f) = d-1$ (and hence $a_d : (f) = a_d$)

$$\alpha_{ij}(a : (f)) = \alpha_{ij}(a_d) + \delta_{ij}(a_{d-1} : (f)) - \delta_{ij}(a_d + a_{d-1} : (f)).$$

Hence by Lemma 1.4 and (i) we obtain

$$(1) \quad \delta_{ij}(a_{d-1}, f) = \delta_{ij}(a_{d-1}) - \delta_{ij}(a_{d-1} : (f)) = \delta_{ij}(a_d, a_{d-1}) - \delta_{ij}(a_d, a_{d-1} : (f)).$$

Now we distinguish two cases:

1. $\dim(a_{d-1}, f) < d-1$.

Then it follows that f is not contained in any $(d-1)$-dimensional prime ideal of a_{d-1}. This is (ii).

2. $\dim(a_{d-1}, f) = d-1$.

Then there are integers $i, j \geq 0$ and $i + j = d-1$, such that

$$\delta_{ij}(a_{d-1}, f) > 0,$$

and we get for such i, j by (1) and Lemma 1.6:

$$(2) \qquad \delta_{ij}(a_{d-1}, a_d \cdot (f)) \geq \delta_{ij}(a_{d-1}, a_d) \geq \delta_{ij}(a_{d-1}, f) > 0.$$

In particular this yields

$$(2') \qquad \dim(a_{d-1}, a_d \cdot (f)) = \dim(a_{d-1}, a_d) = d-1.$$

Furthemore by Lemma 1.4 and (1), (2) and the equality

$$(a_{d-1} : (f), a_d) = (a_{d-1}, a_d \cdot (f)) : (f) \qquad \text{(see [2] p .63)}$$

we get the following chain of inequalities

$$\begin{aligned}\delta_{ij}(a_{d-1}, f) &= \delta_{ij}(a_{d-1}, a_d) - \delta_{ij}(a_{d-1} : (f), a_d)\\ &\le \delta_{ij}(a_{d-1}, a_d \cdot (f)) - \delta_{ij}(a_{d-1} : (f), a_d)\\ &= \delta_{ij}(a_{d-1}, a_d \cdot (f)) - \delta_{ij}((a_{d-1}, a_d \cdot (f)) : (f))\\ &= \delta_{ij}(a_{d-1}, a_d \cdot (f), f)\\ &= \delta_{ij}(a_{d-1}, f).\end{aligned}$$

Hence we get $\delta_{ij}(a_{d-1}, a_d) = \delta_{ij}(a_{d-1}, a_d \cdot (f))$ for all i, j and therefore

$$a_d \subseteq (a_{d-1}, a_d \cdot (f))_{d-1}, \tag{3}$$

where $(\dots)_{d-1}$ denotes according to the "Notation" the intersection of all $(d-1)$-primary components of $(\dots)$.
(This one can see as follows. If there is a form $g \in a_d$ and $g \notin (a_{d-1}, a_d \cdot (f))_{d-1}$. Then we look at:

$$\begin{aligned}\delta_{ij}(a_{d-1}, a_d) &= \delta_{ij}(a_{d-1}, a_d \cdot (f), a_d)\\ &\le \delta_{ij}(a_{d-1}, a_d \cdot (f), g) \qquad \text{(by (2') and Lemma 1.6)}\\ &= \delta_{ij}(a_{d-1}, a_d \cdot (f) - \delta_{ij}((a_{d-1}, a_d \cdot (f)) : (g)) \qquad \text{(Lemma 1.4)}\\ &\le \delta_{ij}(a_{d-1}, a_d \cdot (f))\end{aligned}$$

and we get

$$\dim((a_{d-1}, a_d \cdot (f)) : (g)) = d - 1, \text{because } g \notin (a_{d-1}, a_d \cdot (f))_{d-1}$$

hence there is a pair $i, j \geq 0$ and $i + j = d - 1$, such that

$$\delta_{ij}((a_{d-1}, a_d \cdot (f)) : (g)) > 0.$$

With the inequality above we get a contradiction!)

By $(a_{d-1}, a_d \cdot (f)) \subseteq (a_{d-1}, f)$ and (2') there exists one $(d-1)$-dimensional prime ideal p belonging to (a_{d-1}, f), which belongs also to $(a_{d-1}, a_d \cdot (f))_{d-1}$.

We look at the local ring R_p. Then we get

$$a_d \cdot R_p \subseteq (a_{d-1}, a_d \cdot (f))_{d-1} \cdot R_p = (a_{d-1}, a_d \cdot (f)) \cdot R_p.$$

By multiplication with $(f) \cdot R_p$ and addition of $a_{d-1} \cdot R_p$ we obtain

$$(a_{d-1}, a_d \cdot (f)) \cdot R_p = (a_{d-1}, a_d \cdot (f^2)) \cdot R_p.$$

Successively we get for all integers $n > 0$

$$(a_{d-1}, a_d \cdot (f)) \cdot R_p = (a_{d-1}, a_d \cdot (f^n)) \cdot R_p.$$

Hence it follows that

$$\begin{aligned} a_{d-1} \cdot R_p \subseteq (a_{d-1}, a_d \cdot (f)) \cdot R_p &= \bigcap_{n>0} (a_{d-1}, a_d \cdot (f^n)) \cdot R_p \\ &\subseteq \bigcap_{n>0} (a_{d-1}, a_d \cdot p^n) \cdot R_p = a_{d-1} \cdot R_p, \text{i.e.} \\ a_d \cdot a_{d-1} R_p \subseteq R_p = (a_{d-1}, a_d \cdot (f)) \cdot R_p, \end{aligned}$$

and therefore

$$a_d \subseteq a_d \cdot R_p \cap R \subseteq a_{d-1} \cdot R_p \cap R = q,$$

where q is a p-primary ideal belonging to a. This q is redundant in the primary decomposition of a. Hence only case 1 occurs and this proves the proposition. $\square$

Lemma 2. *Let a be a multihomogeneous ideal of dimension $d > 0$ and f be a form of R of multidegree (σ, τ) with $\sigma, \tau \geq 1$. If for all $i, j \geq 0$ and $i + j = d - 1$*

(i) $\alpha_{ij}((a, f)) = \sigma \cdot \delta_{i+1j}(a) + \tau \cdot \delta_{ij+1}(a)$, *and*
(ii) $\alpha_{ij}(a) = \alpha_{ij}(a : (f))$.

Then $\dim(a, f) = d - 1$.

Proof. By Lemma 1.5 we obtain for all $i, j \geq 0$ and $i + j = d - 1$

$$\begin{aligned} \alpha_{ij}((a, f)) &= \sigma \cdot \delta_{i+1j}(a : (f)) + \tau \cdot \delta_{ij+1}(a : (f)) + \alpha_{ij}(a) - \alpha_{ij}(a : (f)) \\ &= \sigma \cdot \delta_{i+1j}(a : (f)) + \tau \cdot \delta_{ij+1}(a : (f)) \qquad \text{by (ii)} \\ &= \sigma \cdot \delta_{i+1j}(a) + \tau \cdot \delta_{ij+1}(a). \qquad \text{by (i)} \end{aligned}$$

By $\sigma, \tau \geq 1$ for all $i, j \geq 0$ and $i + j = d$ we get

$$\delta_{ij}(a : (f)) = \delta_{ij}(a).$$

Lemma 1.7 and 1.8 yield, for all d-dimensional prime and their coresponding primary ideals of a,

$$l(q : (f)) \cdot \delta_{ij}(p) = \delta_{ij}(q : (f)) = \delta_{ij}(q) = l(q) \cdot \delta_{ij}(p).$$

There is at least one pair (i, j), such that $\delta_{ij}(p) > 0$. Then it follows that $l(q : (f)) = l(q)$, and by $q \subseteq q : (f)$ yields $q = q : (f)$. Therefore f is not contained in any d-dimensional prime ideal of a. $\square$

Combining of Prop. 1 and Lemma 2 we obtain a numerical condition to check if a form f is contained in one of the d- or $(d-1)$-dimensional prime ideals of a:

Proposition 2. *Let a be a multihomogeneous ideal of dimension $d > 0$ and let f be a form of R with multidegree (σ, τ) and $\sigma, \tau > 0$. Then the following conditions are equivalent:*

(i) *For all $i, j \geq 0$ and $i + j = d - 1$*

$$\alpha_{ij}((a, f)) = \sigma \cdot \delta_{i+1j}(a) + \tau \cdot \delta_{ij+1}(a), \text{ and}$$
$$\alpha_{ij}(a) = \alpha_{ij}(a : (f))$$

(ii) *f is not contained in one of the d- or $(d-1)$-dimesional prime ideals.*

Proof. (i) $\Longrightarrow$ (ii): By combining Prop. 1 and Lemma 2.

(ii) $\Longrightarrow$ (i): With the notations of Prop. 1 $a_d = a_d : (f)$ and $a_{d-1} = a_{d-1} : (f)$. In accordance with the proof of Prop. 1 we know that α_{ij} and δ_{ij} with $i, j \geq 0$ and $i + j = d - 1$ and $i + j = d$, respectively, only depend on these components, therefore so (i) follows. □

Let be $R := K[x_0, \dots, x_m; y_0, \dots, y_n]$ and $f_1, \dots, f_r$ $(r \geq 1)$ be forms of R with the multidegrees $(\sigma_{i1}, \sigma_{i2})$ and $\sigma_{i1}, \sigma_{i2} > 0$. We regard $b := (f_1, \dots, f_r)$.

If $\dim b = m + n - r$, then we can compute the degrees of b according by Lemma 1.3 and 1.5 and we obtain for $0 \leq k \leq m$ and $0 \leq l \leq n$ and $k + l = r$

$$\delta_{m-k,n-l}(b) = \sum \sigma_{1j_1} \cdot \ldots \cdot \sigma_{rj_r},$$

where the sum is taken over all $j_1, \dots, j_r \in \{1, 2\}$ with $j_1 + \dots + j_r = r + 1$. In all other cases $\delta_{ij}(b) = 0$.

Extending Prop. 1 to the intersection of r hyperplanes, we obtain the following

Theorem. *Let a be a multihomogeneous ideal of dimension $d \geq r \geq 1$ and $f_1, \dots, f_r$ be forms with multidegrees $(\sigma_{i1}, \sigma_{i2})$ and $\sigma_{i1}, \sigma_{i2} > 0$. Furthermore let $b = (f_1, \dots, f_r)$ with $\dim b = m + n - r$. Then the following conditions are equivalent:*

(i) $\delta_{ij}(a + b) = \sum_{k+l=r} \delta_{i+k,j+l}(a) \cdot \delta_{m-k,n-l}(b)$ *for all $i, j \geq 0$.*

(ii) $\dim(a + b) = d - r$ *and for all $i, j \geq 0$ with $i + j = d - s$*

$$\alpha_{ij}(a, f_1, \dots, f_s) = \alpha_{ij}((a, f_1, \dots, f_s) : f_{s+1}).$$

(iii) *f_{s+1} is not contained in the highest and second highest dimensional prime ideals of $(a, f_1, \dots, f_s)$ for $s = 0, \dots, r - 1$.*

Proof. (ii) $\Longrightarrow$ (iii): follows from Prop. 1.

(iii) $\Longrightarrow$ (i): will proved by induction on r:
Let $r = 1$. Then we obtain (i) from Lemma 1.3 and 1.5.

Let $r > 1$. We set $a' = (a, f_1, \dots, f_{r-1}), f = f_r, \sigma = \sigma_{r1}, \tau = \sigma_{r2}$ and $b' := (f_1, \dots, f_{r-1})$. Then we get by Lemma 1.5 and (iii) for all $i, j \geq 0$

$$\begin{aligned}
\delta_{ij}(a+b) &= \delta_{ij}(a', f) = \sigma \cdot \delta_{i+1j}(a') + \tau \cdot \delta_{ij+1}(a') \\
&= \sigma \cdot \sum_{k+l=r-1} \delta_{i+k+1,j+l}(a) \cdot \delta_{m-k,n-l}(b') \\
&\qquad + \tau \cdot \sum_{k+l=r-1} \delta_{i+k,j+l+1}(a) \cdot \delta_{m-k,n-l}(b') \\
&= \sigma \cdot \delta_{i+r,j}(a) \cdot \delta_{m-r+1,n}(b') + \tau \cdot \delta_{i+r,j}(a) \cdot \delta_{m-r,n+1}(b') \\
&\qquad + \sum_{k=0}^{r-2} (\sigma \cdot \delta_{i+k+1,j+r-k-1}(a) \cdot \delta_{m-k,n-r+k+1}(b')) \\
&\qquad + \sum_{k=1}^{r-1} (\tau \cdot \delta_{i+k,j+r-k}(a) \cdot \delta_{m-k,n-r+k+1}(b')) \\
&\qquad + \sigma \cdot \delta_{i,j+r}(a) \cdot \delta_{m,n-r+1}(b') + \tau \cdot \delta_{i,j+r}(a) \cdot \delta_{m+1,n-r}(b') \\
&= \sum_{k=0}^{r} \Big(\sigma \cdot \delta_{i+k,j+r-k}(a) \cdot \delta_{m-k+1,n-r+k}(b') \\
&\qquad + \tau \cdot \delta_{i+k,j+r-k}(a) \cdot \delta_{m-k,n-r+k+1}(b') \Big) \\
&= \sum_{k=0}^{r} (\delta_{i+k,j+r-k}(a) \cdot \delta_{m-k,n-r+k}(b)) = \sum_{k+l=r} \delta_{i+k,j+l}(a) \cdot \delta_{m-k,n-l}(b).
\end{aligned}$$

(i) $\Longrightarrow$ (ii): Since $\delta_{ij}(a+b)$ is greater than zero only for such $i, j \geq 0$ for which at least one of the $\delta_{i+k,j+l}(a)$ is greater than zero with $k, l \geq 0$ and $k + l = r$, we obtain

$$\dim(a+b) = i + j = (i+k) + (j+l) - r = \dim a - r,$$

so the intersection is proper.

Let $a' := (a, f_1, \dots, f_s)$ and $f = f_{s+1}$ and $\sigma = \sigma_{s+1,1}, \tau = \sigma_{s+1,2}$ and $\sigma, \tau > 0$. Then $\dim(a', f) = \dim a' - 1$ and we can use the computation of the proof of Prop. 1 for the $\alpha_{ij}(a' : (f))$ with $i + j = \dim(a, f)$:

$$\begin{aligned}
\alpha_{ij}(a') - \alpha_{ij}(a' : (f)) &= \delta_{ij}(a'_{d-1}) - \delta_{ij}(a'_{d-1} : (f)) \\
&\quad - \big(\delta_{ij}(a'_d + a'_{d-1}) - \delta_{ij}(a'_d + a'_{d-1} : (f))\big) \\
&= \delta_{ij}(a'_{d-1}, f) - \big(\delta_{ij}(a'_d + a'_{d-1}) - \delta_{ij}(a'_d + a'_{d-1} : (f))\big) \\
&\geq \delta_{ij}(a'_{d-1}, f) - \big(\delta_{ij}(a'_d \cdot (f), a'_{d-1}) - \delta_{ij}((a'_d \cdot (f), a'_{d-1}) : (f))\big) \\
&= \delta_{ij}(a'_{d-1}, f) - \delta_{ij}(a'_{d-1}, a'_d \cdot (f), f) \\
&= \delta_{ij}(a'_{d-1}, f) - \delta_{ij}(a'_{d-1}, f) = 0.
\end{aligned}$$

I.e., $\alpha_{ij}(a') - \alpha_{ij}(a' : (f)) \geq 0$.

Assume $\alpha_{ij}(a') \neq \alpha_{ij}(a' : (f))$ for one pair i, j. Then

$$\begin{aligned}\delta_{ij}(a', f) &= \sigma \cdot \delta_{i+1j}(a') + \tau \cdot \delta_{ij+1}(a') + \alpha_{ij}(a') - \alpha_{ij}(a' : (f)) \\ &> \sigma \cdot \delta_{i+1j}(a') + \tau \cdot \delta_{ij+1}(a').\end{aligned}$$

Continuing this process in accordance with (iii) $\implies$ (i), we obtain

$$\delta_{ij}(a+b) > \sum_{k+l=r} \delta_{i+k,j+l}(a) \cdot \delta_{m-k,n-l}(b)$$

for at least one pair $i, j \geq 0$ and $i + j = \dim a - r$. This is a contradiction to (i).□

At the end of this paper we want to comment on the proof of positivity of one of the degrees of a relevant (i.e. not irrelevant) multihomogeneous ideal of Lemma A1 of [**1**] p. 264. This statement was proved there as follows.

At first they take

$$H(r, a) = \sum H(s, t; a),$$

where the sum is taken over all $i, j \geq 0$ and $s + t = r$. Then the polynomial form of the Hilbert function is substituted and it follows that

$$h_0(a) = \sum \delta_{ij}(a), \tag{*}$$

where the sum is taken over all $i, j \geq 0$ and $i + j = d$.

But only for large s and t it is allowed to take the Hilbert polynomial for the Hilbert function. Indeed this relation does not generally holds, as the following example shows:

$$a = (x_1 y_0, x_0) = (x_0, x_1) \cap (x_0, y_0) \subseteq K[x_0, x_1; y_0, y_1].$$

The bihomogeneous dimension is zero, and we obtain by Lemma 1.5 and 1.9

$$\delta_{00}(a) = \delta_{00}((x_0, y_0)) = 1, \text{but } h_0(a) = 2.$$

Van der Waerden shows in [**7**] this relation (*) for irreducible varieties of the multiple projective space and their corresponding varieties in the projective space. In our language we get (*) for relevant multihomogeneous prime ideals. By Lemma 1.7 and 1.8 we get (*) for arbitrary multihomogeneous ideals for which all highest dimensionally homogeneous primary components are relevant.

Our example shows the importance of the last assumption.

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