Acta Math. Univ. Comenianae
Vol. LX, 2(1991), pp. 257–267

AMALGAMATIONS AND LINK GRAPHS OF CAYLEY GRAPHS

J. TOMANOVÁ

Abstract. The link of a vertex v in a graph G is the subgraph induced by all vertices adjacent to v. If all the links in G are isomorphic to the same graph L, then L is called the link graph of G. We consider the operation of an amalgamation of graphs. Using the construction of the free product of groups with amalgamated subgroups, we give a sufficient condition for a class of link graphs of Cayley graphs to be closed under amalgamations.

1. Introduction

The **link of a vertex** v of a graph G is the subgraph induced by all vertices adjacent to v; we denote it by link(v, G). If all the links in G are isomorphic to the same graph L, then we say that G **has a constant link** L and L is called the **link graph of** G**.** In 1963 Zykov [**6**] posed the problem of characterizing link graphs. It turned out that the problem is algorithmically unsolvable in the class of all (possibly infinite) graphs, see Bulitko [**2**]. However, the solution of Zykov's problem is known for certain classes of graphs (for survey see Hell [**4**] and Blass, Harary and Miller [**1**]). Consequently, it is natural to ask whether the class of link graphs is closed or not under standard binary operations, and how to modify the graph to be a link graph. These problems are treated in Hell [**4**].

In the present paper we shall discuss a similar question, namely, whether an amalgamation of link graphs results in a link graph or not. Using the construction of the free product of groups with amalgamated subgroups, we give a sufficient condition for a class of link graphs of Cayley graphs to be closed under amalgamations. Further, the class of m-treelike graphs is defined and some necessary and sufficient conditions for an m-treelike graph to be a link graph are derived.

Received July 9, 1991; revised September 20, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05C25; Secondary 05C75.

2. Preliminaries

2.1 Groups

We follow the standard terminology and notation of Lyndon and Shupp [**5**] and Blass, Harary and Miller [**1**].

All groups considered are finitely generated. Let H be a group. We use 1 to denote the identity element of H. $U \leq H$ means that U is a subgroup of H. $\langle X; R\rangle$ denotes the presentation with generators $x \in X$ and relators $r \in R$. Let

$$H_1 = \langle x_1, \dots, x_n; r_1, \dots, r_n\rangle \text{ and } H_2 = \langle y_1, \dots, y_m; s_1, \dots, s_m\rangle$$

be disjoint groups. Let $U_1 \leq H_1$ and $U_2 \leq H_2$ be subgroups, such that there exists an isomorphism $f\colon U_1 \to U_2$. Then the **free product of H_1 and H_2, amalgamating U_1 and U_2 by the isomorphism** f is the group

$$\langle x_1, \dots, x_n, y_1, \dots, y_m; r_1, \dots, r_n, s_1, \dots, s_m,\ u = f(u),\ u \in U_1\rangle\,.$$

In order to simplify the notation this group will be denoted by

$$\langle H_1 * H_2; u = f(u), u \in U_1\rangle\,.$$

2.2 Graphs

Let H be a group, and let $Z \subseteq H$ be a generating subset closed under inverses and not containing the identity. The **Cayley graph** $[H, Z]$ of H with respect to Z has H as its vertex set, with u and v adjacent if $u^{-1}v \in Z$.

Let G_1 and G_2 be graphs. Let $L_1 \leq G_1$ and $L_2 \leq G_2$ be subgraphs, and let $f\colon L_1 \to L_2$ be an isomorphism. The **graph with amalgamated subgraphs L_1 and L_2 by the isomorphism** f arises from the disjoint union of G_1 and G_2 by identifying every vertex $v \in V(L_1)$ with $f(v) \in V(L_2)$, and every edge of L_1 with the corresponding edge of L_2. The graph just defined will be denoted by (G_1, L_1, f, L_2, G_2).

3. Amalgamations of Link Graphs of Cayley Graphs

Consider the following problem. Let L_1 and L_2 be link graphs. Let $L_1' \leq L_1$ and $L_2' \leq L_2$ be subgraphs, such that there is an isomorphism $f\colon L_1' \to L_2'$. Does there exist a graph G with constant link $(L_1, L_1', f, L_2', L_2)$?

Note that the class of link graphs is not closed under amalgamations. Figure 1 shows the amalgamation of link graphs L_1 and L_2 by the isomorphism f which results in a non-link graph P_3.

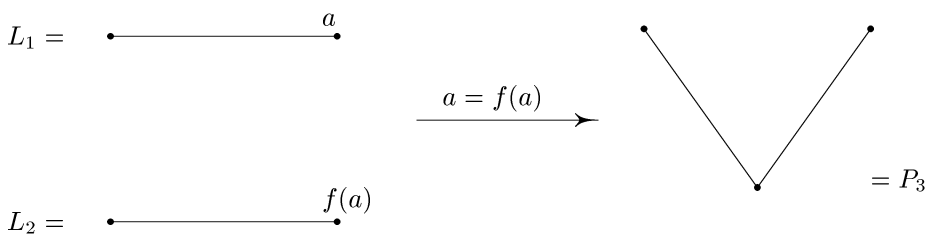


Figure 1.

Thus we are led to restrict the question to the special classes of graphs, namely, to the link graphs of Cayley graphs. (Recall that the Cayley graph $[H, Z]$ is vertex transitive. Hence $L = \mathrm{link}\,(1, [H, Z])$ is the constant link of $[H, Z]$.)

Set $I = \{1, 2\}$. Let H_i be a group with the generating subset Z_i, such that the identity element 1_i of H_i does not belong to Z_i and $Z_i = Z_i^{-1}$, $i \in I$. Consider the Cayley graph $[H_i, Z_i]$ with the constant link L_i, $i \in I$. Let $L'_1 \leq L_1$ and $L'_2 \leq L_2$ be subgraphs, and let $f\colon L'_1 \to L'_2$ be an isomorphism. As $\mathrm{link}\,(1_i, [H_i, Z_i]) \cong L_i$, the vertices in L_i can be considered as elements contained in Z_i, where $i \in I$. Thus the amalgamation of vertices in L'_1 and L'_2 by f induces an amalgamation of elements in Z_1 and Z_2.

Theorem 1. *Let $[H_i, Z_i]$, L_i and L'_1 with $i = 1, 2$ be the graphs defined above. Let $f\colon L'_1 \to L'_2$ be an isomorphism, and let $U_i \leq H_i$ be a subgroup such that $V(L'_i) \subseteq U_i$, for $i = 1, 2$. Suppose that the following properties hold.*
P1. *The isomorphism of graphs L'_1 and L'_2 can be extended to an isomorphism of subgroups U_1 and U_2.*
P2. $U_i \cap Z_i = V(L'_i)$ *for $i = 1, 2$.*
Then there is an infinite graph with constant link $(L_1, L'_1, f, L'_2, L_2)$.

Proof. Denote by $\overline{f}$ the isomorphism of subgroups U_1 and U_2 induced by f. Set

$$H = \left\langle H_1 * H_2;\ u = \overline{f}(u),\ u \in U_1 \right\rangle .$$

Since Z_i is closed under inverses and 1_i does not belong to Z_i for $i \in I$, the generating set of H (say Z) satisfies these conditions, too.

Let L denote the constant link of $[H, Z]$, i.e. $L \cong \mathrm{link}\,(1, [H, Z])$. From the properties P1 and P2 there follows that the elements from Z_1 and Z_2 identified by $\overline{f}$, correspond to the vertices in L_1 and L_2 identified by f, and vice versa. Next, suppose that $u \in Z_1 - V(L'_1)$ and $v \in Z_2 - V(L'_2)$. If $u^{-1}v = c$ and $c \in Z_1$ then $uc \in H_1$. However, it is in contradiction with the fact that $v \in Z_2 - V(L'_2)$. Using similar arguments one can show that $u^{-1}v$ does not belong to Z_2. It implies that L is isomorphic to the graph $(L_1, L'_1, f, L'_2, L_1)$, as required. $\square$

Obviously, the graph constructed by the procedure described above, is infinite. However, in the special cases "finitizing" relations for the group H can be found. For instance, consider a group H_1 generated by the set $Z_1 = \{BC, B, C, CB\}$, and with defining relations $B^2 = C^2 = 1_1$, $BC \neq CB$, $(BC)^3 = CB$. Let H_2 be an Abelian group generated by the set $Z_2 = \{D, A, DA\}$, and with $D^2 = A^2 = 1_2$. Let $V(L_1') = \{B\}$ and $V(L_2') = \{D\}$. Then $U_1 = \{1_1, B\}$ and $U_2 = \{1_2, D\}$. For the group $H = \langle H_1 * H_2; u = \overline{f}(u), u \in U_1 \rangle$ the finitizing relation can be specified as $AC = CA$. The Cayley graph of the group $\overline{H} = \langle H_1 * H_2;\ u = f(u),\ u \in U_1, AC = CA \rangle$ with generating set $\overline{Z} = \{BC, B, C, CB, A, BA\}$ is shown in Fig. 2.

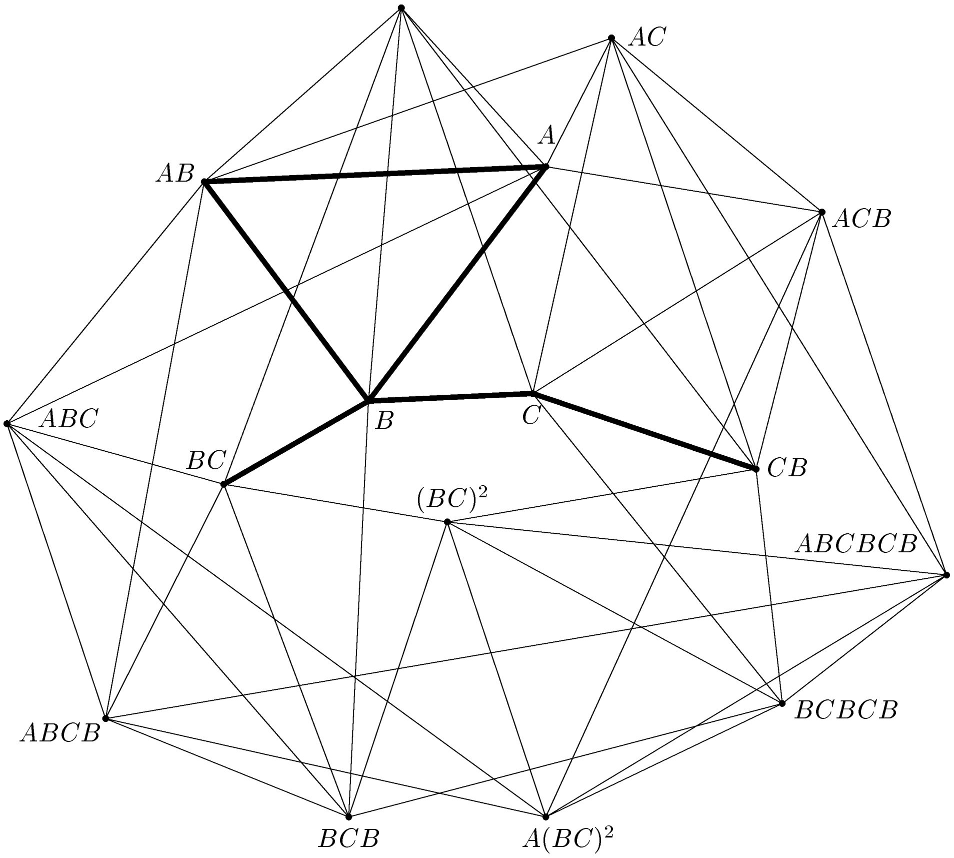


Figure 2.

Theorem 2. *Let $[H_i, Z_i]$ be a Cayley graph of an Abelian group H_i, such that* $\operatorname{link}(1_i, [H_i, Z_i]) \cong L_i$ *for $i = 1, 2$. Let $L_i' \leq L_i$, $i = 1, 2$ be a subgraph, and let $f\colon L_1' \to L_2'$ be an isomorphism. Consider the subgroups $U_1 \leq H_1$ and $U_2 \leq H_2$*

satisfying the assumptions of Theorem 1. *Then there exists a finite Cayley graph with constant link* $(L_1, L'_1, f, L'_2, L_2)$.

Proof. Let $\overline{f}$ be the isomorphism of U_1 and U_2 induced by f. Consider the group $H = <H_1 * H_2;\ u = \overline{f}(u),\ u \in U_1>$ and the relation

$$\text{(A)} \qquad uv = vu \quad \text{if} \quad u \in Z_1 \quad \text{and} \quad v \in Z_2.$$

We shall show that the Cayley graph of the group

$$\overline{H} = \langle H_1 * H_2;\ u = \overline{f}(u),\ u \in U_1, (A) \rangle$$

has the constant link $(L_1, L'_1, f, L'_2, L_2)$. In fact, we shall prove that
1. The generating set of $\overline{H}$ (say $\overline{Z}$) coincides with Z.
2. (A) preserves the edges and non-edges in $(L_1, L'_1, f, L'_2, L_2)$.

Take the elements $u \in Z_1 - U_1$ and $v \in Z_2 - U_2$. Set $uv = c$ and $vu = d$. If $c^{-1}d \in H_i$ for $i \in \{1, 2\}$ and $c^{-1}d \neq 1$ with respect to the defining relations in H_i, then (A) produces relations which are not valid in H_i, and consequently $Z \neq \overline{Z}$. Now, we shall show, it is not the case.

Since both H_1 and H_2 are Abelian groups, u and v can be written as $u = X_1Y_1$ and $v = X_2Y_2$ where $Y_i \in U_i$, $X_i \in Z_i - U_i$, and there is no element from U_i contained in X_i, $i = 1, 2$. As $Y_1X_2Y_2 \in H_2$ and $Y_2X_1Y_1 \in H_1$, we have $uv = X_1X_2Y_1Y_2$ and $vu = X_2X_1Y_2Y_1$. It implies that the equation $c^{-1}d = 1$ holds in H_i, $i = 1, 2$. Hence $Z = \overline{Z}$, and the applying of (A) does not result in the new edges in L_i, $i = 1, 2$. Similarly as in the proof of Theorem 1 one can derive that if $u \in Z_1 - U_1$ and $v \in Z_2 - U_2$ then $u^{-1}v \notin Z_i$ for $i = 1, 2$. This completes the proof. □

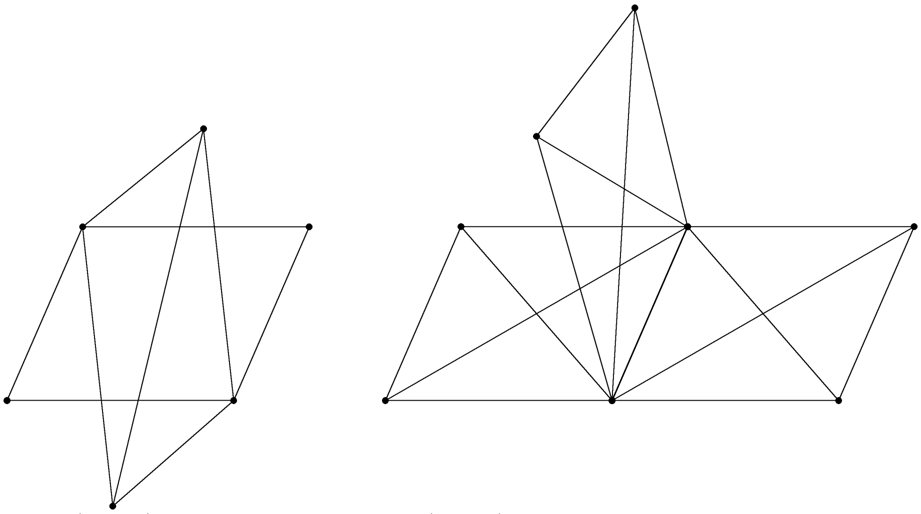
4. m-Treelike Graphs

In this section the operation of the amalgamation of groups will be used to construct graphs with constant link isomorphic to the so-called m-**treelike** graphs.

Definition 1. Let n and m be integers such that $n \geq 3$ and $m \geq 1$. A connected graph T is said to be m-treelike if
A. T does not contain any cycle of length greater than three as an induced subgraph.
B. The maximal cliques in T have the same size n. The intersection of any two maximal cliques is empty or is the complete graph on m vertices.

Note that the concept of m-**treelike** graph generalizes that of **treelike** graph introduced in Harary and Palmer [**3**].

An m-treelike graph is called m-**starlike** if all its maximal cliques have exactly m vertices in common. An m-starlike graph in which the number of maximal cliques is $k \geq 2$ will be denoted by $S(n, m, k)$, see Fig. 3.



$S(3,2,4)$ $S(4,2,3)$

Figure 3.

The next proposition gives a necessary and sufficient condition for an m-starlike graph to be a link graph. As an m-starlike graph has exactly m universal vertices, the assertion of our proposition follows also from Theorem 1 in Hell [**4**] (for the definition of an universal vertex see the same paper, [**4**]). However, the method we shall use to prove it allows us to construct Cayley graphs with constant link isomorphic to the prescribed m-starlike graphs.

Proposition 1. *An m-starlike graph $S(n,m,k)$ is the link graph if and only if $n+1=c(m+1)$ for an integer $c>1$.*

Proof of Proposition 1.

Sufficiency. Let S be an m-starlike graph with $k \geq 2$, and let $I=\{1,\dots,k\}$ be an index set. Since $n+1=c(m+1)$, each maximal clique in S (say C_i with $i \in I$) can be represented as the link graph of a Cayley graph defined in the following way. Let H_i be an Abelian group with the generating set

$$Z_i=\{x_i^h a_i^r \colon h \in \{0,\dots,m\},\ r \in \{0,\dots,c-1\},\ (h,r) \neq (0,0)\}$$

and with defining relations $x_i^{m+1}=a_i^c=1_i$ for $i \in I$.
Obviously, $\operatorname{link}(1_i,[H_i,Z_i]) \cong C_i$ for $i \in I$.

Consider the subgraph $C_i' \leq C_i$ induced by the vertices $x_i,\dots,x_i^m$, $i \in I$. Let U_i be a subgroup of H_i with $U_i=\{1_i,x_i,\dots,x_i^m\}$, and let $f_1 \colon C_1' \to C_2'$ be the mapping defined as $f_1(x_1^t)=x_2^t$ for $t=1,\dots,m$. Then f_1 is an isomorphism and moreover, it can be naturally extended to the isomorphism of U_1 and U_2. Set

$\overline{f}_1(x_1^t) = x_2^t$ for $t = 0, \ldots, m$. Then by Theorem 2, the Cayley graph of the group

$$P_1 = \langle H_1 * H_2;\ x_1 = \overline{f}_1(x_1),\ x_1 \in U_1,\ uv = vu,\ u \in Z_1 \text{ and } v \in Z_2 \rangle$$

has the constant link (say L_1) isomorphic to $S(n, m, 2)$.

If $k = 2$ then $L_1 \cong S$; otherwise consider the subgraph $L_1' \leq L_1$ induced by the vertices $x_1, \ldots, x_1^m$, and the subgraph $C_3' \leq C_3$ induced by the vertices $x_3, \ldots, x_3^m$. Let Z denote the generating set of P_1 and let 1 be its identity element. An isomorphism $f_2 \colon L_1' \to C_3'$ can be extended to the isomorphism (say $\overline{f}_2$) of groups $A_1 = \{1, x_1, \ldots, x_1^m\}$ and $U_3 = \{1_3, x_3, \ldots, x_3^m\}$. As $U_3 \cap Z_3 = C_3'$ and $A_1 \cap Z = L_1'$, the Cayley graph of the group

$$P_2 = \langle P_1 * H_3;\ x_1 = \overline{f}_2(x_1),\ x_1 \in A_1,\ uv = vu,\ u \in Z \text{ and } v \in Z_3 \rangle$$

has constant link (say L_2) isomorphic to $S(n, m, 3)$.

If $k = 3$ then $L_2 \cong S$; otherwise the construction described above will be used repeatedly (exactly $k-2$-times) to derive the Cayley graph with constant link S.

Necessity. In order to prove the necessary condition we shall need the next definition, given in [**1**]. Let u and v be adjacent vertices in a graph G. The number of vertices adjacent to both u and v is called the **relative degree,** and is denoted by $\alpha(u, v)$. If $\alpha(u, v) = q$ then we say that the edge (u, v) is **marked** q.

Suppose that an m-starlike graph S is the link graph of a graph G. If X denotes the centre of S then by the definition of the relative degree we obtain

$$\begin{aligned} &\alpha(x, y) = k(n-m) + (m-1) \qquad \text{for any } x, y \in X \text{ and} \\ &\alpha(a, x) = n - 1 \qquad \text{for any } x \in X \text{ and } a \in S - X. \end{aligned}$$

Let a be a vertex contained in $S - X$. Then there are m vertices in $S - X$ which belong to the centre of the link (a, G), say $a_1, \ldots, a_m$. Clearly, the vertices $a, a_1, \ldots, a_m$ belong to the same maximal clique in S, i.e.

$$\alpha(a, a_i) = \alpha(a_i, a_j) = k(n-m) + (m-1) \qquad \text{for } i, j = 1, \ldots, m.$$

Thus the edges **marked** $k(n-m) + (n-1)$ indicate a K_{m+1} factor of $S - X$, and the assertion follows. □

Let T be a graph of type $S(n, m, k)$. The set of all vertices in T with degree $k(n-m) + (m-1)$ will be denoted by X. Now, we shall introduce a new class of m-treelike graphs derived from a given graph $S(n, m, k)$.

Definition 2. Let k and l be integers such that $k \geq 2$ and $l \geq 1$. Define the class $S(n, m, k, l)$ of m-treelike graphs with $n \geq 2m$ as follows.
1. $S(n, m, k, 1) = \{S(n, m, k)\}$.
2. A. Let $k \geq 3$ and $l \geq 2$. Consider a graph $T \in S(n, m, k, l-1)$, and all its

maximal cliques such that each of them contains a vertex at distance $l-1$ from X; the maximal cliques with the above property will be denoted by $C_1, \ldots, C_j$, where $j \leq k$. Let $C_i' \leq C_i$ be complete subgraph on m vertices, such that if u and v belong to $V(C_i')$ then $\deg(u,T) = \deg(v,T) = n-1$, $i = 1, \ldots, j$. Further, let $L_1, \ldots, L_t$ be complete graphs on n vertices with $t \leq j$, and $L_i' \leq L_i$ be the complete subgraph on m vertices, $i = 1, \ldots, t$. Consider an isomorphism $f_i\colon C_i' \to L_i'$ where $i = 1, \ldots, t$. We define $H_t^j(T)$ to be the graph derived from the disjoint union of $T, L_1, \ldots, L_t$ by identifying every vertex $v \in V(C_i')$ with $f_i(v) \in V(L_i')$, and every edge of C_i' with the corresponding edge of L_i', $i = 1, \ldots, t$.

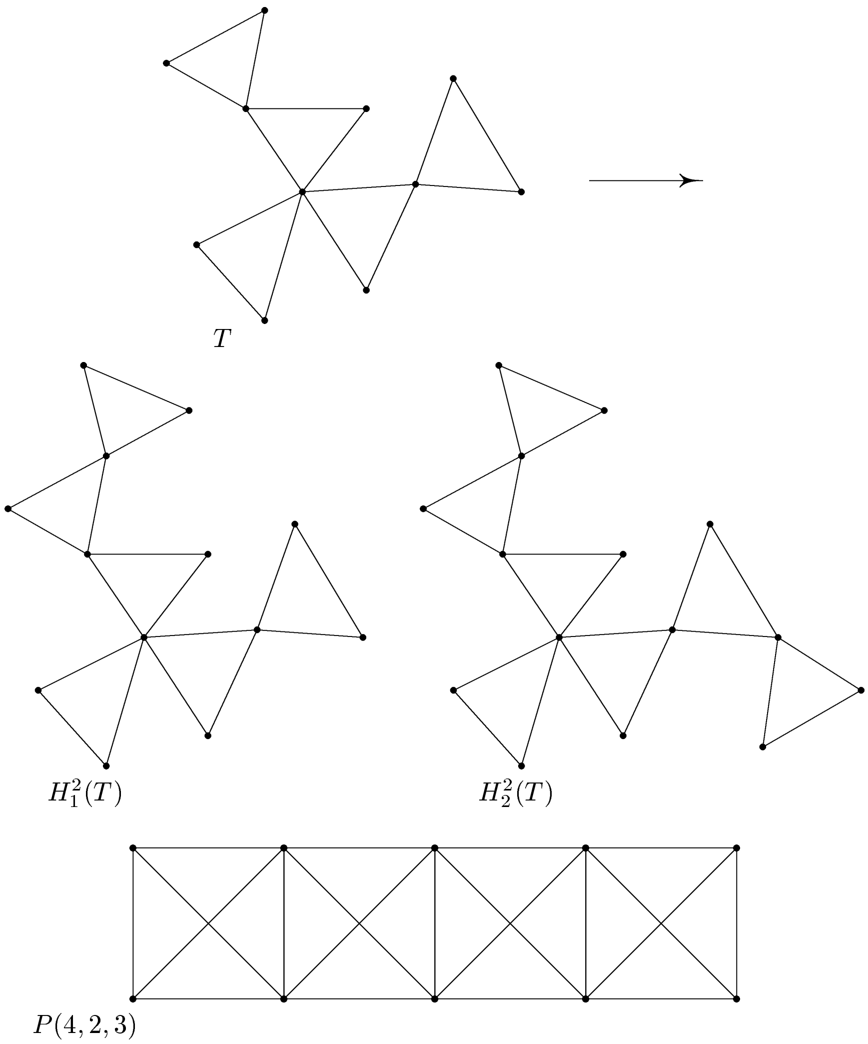


Figure 4.

Let G be an m-treelike graph. If there is a graph $T \in S(n,m,k,l-1)$ such that $H_t^j(T)$ is isomorphic to G for some t, then G belongs to the class $S(n,m,k,l)$.

B. If $k=2$ then the class $S(n,m,2,l)$ contains a single element, and we denote it by $P(n,m,l)$, see Fig. 4.

Figure 4 shows the graph $T \in S(3,1,3,2)$, and graphs $H_1^2(T)$ and $H_2^2(T)$ derived from T.

To simplify the statement of the next proposition we give the following definition. Let T be a graph in $S(n,m,k,l)$ with $k \geq 3$. We say that the **branch** B_i **of** T **at** X **has length** l_i if there is a maximal clique C in B_i, such that C contains a vertex v at distance l_i from X, $i \in \{1,\dots,k\}$.
The branches of T at X will be simply called the branches of T.
Note that if $T \in S(n,m,k,l)$ then $1 \leq l_i \leq l$ for $i=1,\dots,k$, and there is at least one branch in T of length equal to l.

Proposition 2. *Let T be a graph in $S(n,m,k,l)$ with $k \geq 3$. Suppose that the length of each branch in T is greater than or equal to* 2. *If T is a* **link graph then** $n+1 \geq (m+1)^2$**.**

Proof. Let G be a graph with constant link T. Take a vertex $v \in G$, and consider the link(v,G) and the corresponding set $X = \{x\colon x \in \operatorname{link}(v,G)$ such that $\deg(x, \operatorname{link}(v,G)) = k(n-m)+(m-1)\}$.
By the definition of the relative degree we have

$$\alpha(x,y) = k(n-m)+(m-1) \qquad \text{for any } x,\, y \in X.$$

Let $I = \{1,\dots,k\}$. To each $i \in I$ there corresponds a branch B_i in T containing a maximal clique C_i so that $X \leq C_i$. As $n \geq 2m$ there are m vertices in C_i (say $s_{i,1},\dots,s_{i,m}$) $i \in I$, with degrees equal to $r = 2(n-m)+(m-1)$.
Since

$$\alpha(v, s_{i,j}) = r \qquad \text{for } j=1,\dots,m,$$

we obtain

$$\alpha(s_{i,p}, s_{i,j}) = r \qquad \text{for } p,\, j=1,\dots,m.$$

The last equation follows from the following fact: if

$$\alpha(s_{i,p}, s_{i,j}) = k(n-n)+(m-1) \qquad \text{for } p,j \in \{1,\dots,m\}$$

then by the definition of m-treelike graph we have

$$\deg(v, \operatorname{link}(s_{i,j},G)) = k(n-m)+(m-1), \qquad \text{for } j=1,\dots,m.$$

However, it is a contradiction with

$$\alpha(v, s_{i,j}) = r, \qquad \text{for } j=1,\dots,m.$$

Using similar arguments one can derive the inequality

$$\alpha(x, s_{i,j}) \neq r \qquad \text{for any } x \in X, \text{ and } j = 1, \dots, m.$$

Next, consider link (x, T) where $x \in X$, and the maximal clique $C'_i = C_i - \{x\} \cup \{v\}$ where $i \in I$. As $\alpha(x, s_{i,j}) \neq r$ for $j = 1, \dots, m$, there are m vertices in C'_i, say $s'_{i,1}, \dots, s'_{i,m}$, such that $\alpha(x, s'_{i,j}) = \alpha(s'_{i,j}, s'_{i,p})_p = r$ for p, $j = 1, \dots, m$, and $i \in I$. Hence, $n - 2m \geq m^2$. □

Theorem 3. *Let T be a graph from the class $S(n, m, k, l)$ with $k \geq 2$ and $l \geq 2$. If $n + 1 = c(m+1)^2$ for an integer $c \geq 1$ then T is the link graph.*

Proof. First we construct the Cayley graph with constant link $S(n, m, k)$. Let $I = \{1, \dots, k\}$ be the index set.

Consider an Abelian group H with the generating set

$$Z = \{x^h y_i^q a_i^p \colon h, q \in \{0, \dots, m\},\ p \in \{0, \dots, c-1\},\ (h, q, p) \neq (0,0,0),\ i \in I\}$$

and with $x^{m+1} = y_i^{m+1} = a_i^c = 1$ for $i \in I$.
For each $i \in I$ the elements from the set

$$Z_i = \{x^h y_i^q a_i^p \colon h, q \in \{0, \dots, m\},\ p \in \{0, \dots, c-1\},\ (h, q, p) \neq (0,0,0)\}$$

correspond to the vertices of the complete subgraph on n vertices in L, where L denotes the constant link of $[H, Z]$. Since the elements $a_i^p a_j^r$, $y_i^q y_j^s$, $y_i^q a_j^p$ do not belong to Z if $i \neq j$, p, $r \in \{1, \dots, c-1\}$, and q, $s \in \{1, \dots, m\}$, L is isomorphic to $S(n, m, k)$.

Let G be a graph which belongs to the class $S(n, m, k, 2)$. Now, the graph with constant link isomorphic to G will be derived from $[H, Z]$. Let H'_1 be an Abelian group generated by the set

$$Z'_1 = \{r_1^h s_1^q b_1^p \colon h, q \in \{0, \dots, m\},\ p \in \{0, \dots, c-1\},\ (h, q, p) \neq (0,0,0)\}$$

with $r_1^{m+1} = s_1^{n+1} = b_1^c = 1_1$.
Then link $(1_1, [H'_1, Z'_1])$ is isomorphic to the complete graph on n vertices, say L_1. Let $L'_1 \leq L_1$ be the subgraph induced by the vertices $r_1, \dots, r_1^m$, and let $L' \leq L$ be the subgraph induced by the vertices $y_1, \dots, y_1^m$. Then the mapping $f \colon L'_1 \to L'$ defined as $f(r_1^h) = y_1^h$ for $h = 1, \dots, m$ is an isomorphism, and it can be extended to the isomorphism of the groups $U'_1 = \{1_1, r_1, \dots, r_1^m\}$ and $U = \{1, y_1, \dots, y_1^m\}$, say $\overline{f}$. According to Theorem 2 we obtain, that the Cayley graph of the group

$$H' = \left\langle H * H'_1;\ r_1 = \overline{f}(r_1),\ r_1 \in U'_1,\ uv = vu,\ u \in Z \text{ and } v \in Z'_1 \right\rangle$$

has constant link isomorphic to a graph (say G_1) from $S(n, m, k, 2)$. Let Z' denote the generating set of H'.

If $k = 2$ then $G \cong G_1$, and the above construction gives the graph $[H', Z']$ with the constant link from $S(n, m, 2, 2)$. Since link $(1, [H', Z'])$ contains the subgraph induced by the vertices $s_1, \dots, s_1^m$ and H' contains the subgroup $\{1, s_1, \dots, s_1^m\}$, the operation of the amalgamation can be used repeatedly. In such a way we can construct the Cayley graph with the constant link isomorphic to a graph from $S(n, m, k, l)$ with $k = 2$ and $l \geq 2$.

Suppose that $k \geq 3$. Then G has j $(1 \leq j \leq k)$ branches of length two, and link $(1, [H', Z'])$ has exactly one branch of length 2. However, link $(1, [H', Z'])$ contains the subgraph induced by the set $Y_i = \{y_i, \dots, y_i^m\}$, such that $\{1\} \cup Y_i$ is the subgroup of H' for $i = 1, \dots, j$. It means that each of j branches in link $(1, [H', Z'])$ can be prolonged by the analogous procedure as we have prolonged the branch containing the subgraph Y_1. Hence, there exists a Cayley graph with the constant link isomorphic to G. As link $(1, [H', Z'])$ contains the subgraph induced by the set $s_1, \dots, s_1^m$, and H' contains the subgroup $\{1, s_1, \dots, s_1^m\}$, the operation of the amalgamation can be used to determine a Cayley graph with the constant link isomorphic to a graph from $S(n, m, k, 3)$. Hence, the proof of theorem follows by induction on l. $\square$

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J. Tomanová, Department of Algebra and Number Theory, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia