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SELF–DUAL REGULAR MAPS FROM MEDIAL GRAPHS

D. ARCHDEACON, J. ŠIRÁŇ AND M. ŠKOVIERA

1. Introduction and Preliminaries

There are a number of papers dealing with symmetries of maps, i.e., of graphs embedded in closed surfaces. Many of them concentrate on the construction and characterization of regular maps. In this paper we investigate the relationship between the symmetries of a map and those of its medial map. The main issue of this study is a theorem saying that a map M (orientable or not) is regular and self-dual if and only if the medial map $m(M)$ is regular. This theorem combined with a lifting technique based on special voltage assignments allows us to construct new oriented regular self-dual maps from old ones. As a consequence, we obtain that if G is a d-valent graph admitting an oriented regular self-dual map, then the graph $G^{(k)}$ obtained from G by replacing each edge by k parallel edges also has such a map whenever g.c.d. $(d, k) = 1$. Similar results (but without self-duality) were obtained by S. Wilson [**7**] for some types of graphs. On the other hand, self-dual regular maps (with self-complementary underlying graphs) were investigated by A. T. White [**6**].

Regular maps can be introduced in several different ways. For our purposes it will be convenient to define them by means of an action of the map automorphism group on the set of corners. Let M be a map with underlying graph G. Then any ordered pair of two consecutive edges appearing on the boundary of a face of M determines a **corner**; the order in the pair determines the **orientation** of the corner. Let $C(M)$ be the set of all corners of M; obviously $\mid C(M) \mid = 4 \mid E(G) \mid$. The map M is said to be **regular** if the map automorphism group of M, $Aut\ M$, acts transitively on $C(M)$. This definition is easily seen to be equivalent to the one given in [**7**].

In the orientable case a slightly weaker definition is often convenient. Let the supporting surface S of M be orientable. If an orientation of S is specified then the map M is said to be **oriented**. The orientation of S then splits $C(M)$ into two disjoint subsets: $C^+(M)$, the set of corners whose orientation agrees with that

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of S, and $C^-(M)$, the complement of $C^+(M)$. Note that $| C^+(M) |= 2 | E(G) |$. An oriented map is said to be **orientably regular** if the group of orientation-preserving automorphisms of M, Aut^+M, acts transitively on $C^+(M)$. In [**7**] such maps are called rotary; Coxeter and Moser [**2**] and Jones and Singerman [**4**] call them regular.

Of course, an orientably regular map may admit an automorphism that reverses the orientation; such an automorphism is said to be a **reflection**. Orientably regular maps with a reflection are occasionally called reflexible.

2. Regularity, Duality, and Medial Maps

Let M be a map, that is, a 2-cell embedding of a graph G in an oriented surface S. To form the **medial map** of M, denoted by $m(M)$, first place a vertex v_e into the interior of every edge e. Then, for each face F of M, join v_e to v_f by an edge lying in F if and only if the edges e and f are consecutive on the boundary of F. The graph underlying the map $m(M)$ will be called the **medial graph** of G and denoted by $m(G)$. The medial graph is clearly 4-regular, as each face creates two adjacencies for each edge on its boundary.

Note that the faces of the medial map $m(M)$ split naturally into two types: faces containing vertices of the original map M (**vertex-faces**) and those corresponding to faces of M (**face-faces**). For the sake of convenience, we shall think of vertex-faces as coloured **black** and face-faces as coloured **white**. Obviously, this colouring is a proper face colouring of $m(M)$, and we shall always assume $m(M)$ to be coloured this way.

We explain the main ideas on oriented maps first, postponing the general case to the end of this section. Thus, let M be an oriented map. We pay particular attention to the orientation-preserving map automorphisms of $m(M)$ which preserve the two parts in the face bipartition; call them **colour-preserving automorphisms** of $m(M)$. We shall call the medial map $m(M)$ **orientably colour-regular** if for any two undirected edges x and y of $m(G)$ there is a colour-preserving automorphism of $m(M)$ which takes x to y.

Observe that if M is an orientably regular map then $m(M)$ is orientably colour-regular. Indeed, let $x = v_e v_f$ and $y = v_g v_h$ be two edges of $m(M)$. By regularity of M, there is an automorphism φ of M which maps the corner ef to the corner gh. Now, φ obviously induces an automorphism φ_m of $m(M)$. Since φ maps corners to corners, φ_m preserves vertex-faces and thereby colours of $m(M)$.

Conversely, assuming that $m(M)$ is orientably colour-regular we can show that M is regular. To see this, let ef and gh be two corners of M. The property of $m(M)$ implies that there is a colour-preserving automorphism ψ of $m(M)$ which sends $v_e v_f$ to $v_g v_h$. As ψ preserves colours, it induces the required automorphism of M taking ef to gh.

Thus we have proved:

Theorem 1. *Let M be an oriented map. Then M is orientably regular if and only if the medial map $m(M)$ is orientably colour-regular.*

Let M be an oriented map with underlying graph G and let M^* be the dual map of M; as usual, the graph underlying the map M^* will be denoted by G^*. For technical reasons we shall assume that M and M^* induce the **same** orientation of the ambient surface.

Adopting the standard notation, let $e \to e^*$ be the duality mapping of $E(G)$ onto $E(G^*)$. This mapping extends to a mapping of corners of M^* in the following way: to the corner ef of M there coresponds the dual corner f^*e^* (see Fig. 1).

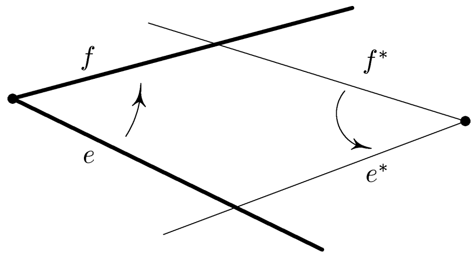

Figure 1.

If M and M^* is a pair of dual maps then we can assume that their medial graphs coincide. Thus, for their medial maps we have $m(M) = m(M^*)$ except for the face colourings: black faces of $m(M)$ become white faces of $m(M^*)$ and vice versa.

In Proposition 1 we have shown that the property of $m(M)$ being orientably colour-regular is equivalent to orientable regularity of M. This suggests the following question. Forgetting about the face colouring of $m(M)$, how can regularity of $m(M)$ be expressed in terms of properties of the original map M? We answer this question in the next theorem.

Theorem 2. *Let M be an oriented map. Then $m(M)$ is orientably regular if and only if M is orientably regular and self-dual.*

Proof. Let M be orientably regular and self-dual. Thus, there is an orientation-preserving map isomorphism $\theta\colon M \to M^*$. The mapping θ, being at the same time a self-homemorphism of the ambient surface, induces a map-isomorphism $\theta_m\colon m(M) \to m(M^*)$. Since $m(M) = m(M^*)$ and θ_m interchanges colours, θ_m is in fact a colour-reversing automorphism of $m(M)$.

The orientable regularity of M implies that $m(M)$ is orientably colour-regular (see Theorem 1). Let H be the set of colour-preserving automorphisms of $m(M)$. Consider the set $H \cup H\theta_m$. As θ_m interchanges colours, $H \cap H\theta_m$ is empty.

Therefore

$$| \, Aut \; m(M) \, | \geq | \, H \cup H\theta_m \, | = 2 \, | \, H \, | = 4 \, | \, E(M) \, | = 2 \, | \, E(m(M)) \, | \, .$$

This shows that $m(M)$ is orientably regular.

Conversely, assume that $m(M)$ is orientably regular. By Theorem 1, M is orientably regular. It remains to prove that M is self-dual. Let σ be a colour-reversing automorphism of $m(M)$.(Such an automorphism clearly exists, for the number of colour-preserving automorphisms of $m(M)$ is $| \, E(m(M)) \, | < | \, Aut \; m(M) \, |$.) We can view σ as a map isomorphism $m(M) \to m(M^*)$ which interchanges colours. Again, regarding σ as a self-homeomorphism of the surface, we see that σ induces a bijection $\sigma_o \colon M \to M^*$. From the fact that σ is colour-reversing, it follows that σ_o maps corners of M to corners of M^*. Thus σ_o is a map isomorphism $M \to M^*$, which means that M is self-dual. □

As mentioned earlier, results analogous to Theorem 1 and Theorem 2 can be established without the assumption of orientability. In this case we define the map $m(M)$ to be **colour-regular** if for any two arcs (= edges with specified direction) x and y of $m(M)$ there is a colour-preserving automorphism of $m(M)$ which takes x to y. (The reader should compare this definition with the one given for the orientable case.) Since the arguments are similar to those presented above, we state the following summarizing theorem without proof.

Theorem 3. *Let M be a map. Then:*

(1) *M is regular if and only if $m(M)$ is colour-regular.*
(2) *M is orientably regular if and only if $m(M)$ is orientably colour-regular.*
(3) *M is regular and self-dual if and only if $m(M)$ is regular.*
(4) *M is orientably regular and self-dual if and only if $m(M)$ is orientably regular.*

3. Lifting Regular Medial Maps

Let M be an oriented map and let $m(M)$ be the face-coloured medial map of M. The chosen orientation of the surface induces an orientation of all white faces of $m(M)$. This enables to assign direction to every edge of $m(M)$ consistently with the white face it is adjacent to.

Let Γ be an Abelian group and g an arbitrary element of Γ. Let α be a voltage assignment α on $m(M)$ which to every edge of $m(M)$ directed as above assigns the element g (i.e., α is constant on edges with preferred direction); such a voltage assignment will be called **medial**. Note that a similar type of voltage assignment has already been considered by Archdeacon [**1**].

The notion of a medial voltage assignment can also be defined for non-orientable medial maps. Note, however, that if the supporting surface for M were non-orientable then the preferred directions of edges of $m(M)$ would not be well defined.

Therefore, in a medial voltage assignment on a non-orientable $m(M)$ the element $g \in \Gamma$ is always assumed to be of order 2.

In either case, a medial voltage assignment gives rise to a the derived graph and a derived embedding in the usual sense (see [**3**]). We shall restrict ourselves to the case when the derived embedding surface is connected. This is equivalent to assuming that Γ is a cyclic group generated by g. In particular, if $m(M)$ is non-orientable this means that $\Gamma \cong \mathbb{Z}_2$. Thus we can speak of the **derived map** which will be denoted by $m(M)^\alpha$.

The following theorem shows how to define map automorphisms of the derived map by means of automorphisms of the base map. As in Section 2 we explain the ideas on the orientable case. Let us first briefly recall the structure of the oriented derived map $m(M)^\alpha$ [**3**]. The vertex set of $m(M)^\alpha$ is $\{u_b;\ u$ a vertex of $m(M)$, $b \in \Gamma\}$. Each arc $x = uv$ of $m(M)$, emanating from a vertex u and terminating at v, lifts to $\mid \Gamma \mid$ arcs $x_b = u_b v_{b\alpha(x)}$, $b \in \Gamma$, of $m(M)^\alpha$. Moreover, if P and T are the rotation and the arc-reversing involution corresponding to $m(M)$, then the rotation P^α of the derived map $m(M)^\alpha$ is given by

$$P^\alpha(x_b) = (Px)_b$$

and the arc-reversing involution T^α of $m(M)^\alpha$ by

$$T^\alpha(x_b) = (Tx)_{b\alpha(x)}.$$

Theorem 4. *Let M be an oriented map and let α be a medial voltage assignment on $m(M)$. Let A be an automorphism of $m(M)$.*

(1) *If A preserves colours of faces and the preferred directions of edges of $m(M)$, then for each $a \in \Gamma$ the mapping A_a given by*

$$A_a(x_b) = (Ax)_{ab}$$

is an orientation-preserving automorphism of $m(M)^\alpha$.

(2) *If A is a reflection of $m(M)$, then the same formula as in (1) defines reflections of $m(M)^\alpha$.*

(3) *If A reverses both the face colours as well as the preferred edge directions of $m(M)$, then for each $a \in \Gamma$ the mapping B_a given by*

$$B_a(x_b) = (Ax)_{ab^{-1}}$$

is an (orientation-preserving) automorphism of $m(M)^\alpha$.

Proof. Let A be an orientation-preserving automorphism of $m(M)$. This is equivalent to saying that A, regarded as a one-to-one mapping on the set of arcs of

$m(M)$, commutes with both the rotation P as well as the arc-reversing involution T; i.e., $AP = PA$ and $AT = TA$.
(1) To prove that A_a is an orientation-preserving automorphism of $m(M)^\alpha$ it is sufficient to verify that A_a commutes with both P^α and T^α. Employing the fact that A commutes with both P and T we successively obtain:

$$P^\alpha A_a(x_b) = P^\alpha((Ax)_{ab}) = (PAx)_{ab} = (APx)_{ab} = A_a(Px)_b = A_a P^\alpha(x_b).$$

To show that $T^\alpha A_a = A_a T^\alpha$ recall that A preserves the preferred directions of edges. Therefore, for every arc x of $m(M)$ we have $\alpha(Ax) = \alpha(x)$. Now,

$$\begin{aligned} T^\alpha A_a(x_b) &= T^\alpha((Ax)_{ab}) = (TAx)_{ab\alpha(Ax)} = (ATx)_{ab\alpha(x)} \\ &= A_a((Tx)_{b\alpha(x)}) = A_a T^\alpha(x_b). \end{aligned}$$

(2) The automorphism A is a reflection if and only if $AP = P^{-1}A$ and $AT = TA$. The proof of the fact that $A_a P^\alpha = (P^\alpha)^{-1} A_a$ and $A_a T^\alpha = T^\alpha A_a$ is similar to the above.
(3) In this case A is orientation-preserving, again. Since it is colour-reversing it follows that $\alpha(Ax) = \alpha(Tx)$ for every arc x of $m(M)$. The equality $P^\alpha B_a = B_a P^\alpha$ can be proved in the same way as in (1). In addition, using the fact that Γ is Abelian we obtain:

$$\begin{aligned} T^\alpha B_a(x_b) &= T^\alpha((Ax)_{ab^{-1}}) = (TAx)_{ab^{-1}\alpha(Ax)} \\ &= (ATx)_{a\alpha(Tx)b^{-1}} = (ATx)_{a(b\alpha(x))^{-1}} = B_a(Tx)_{b\alpha(x)} = B_a T^\alpha(x_b). \end{aligned}$$

This completes the proof. □

The above result has the following immediate consequence:

Theorem 5. *Let M be an oriented map and let α be a medial voltage assignment on $m(M)$. Then:*

(1) *If $m(M)$ is orientably colour-regular, then so is $m(M)^\alpha$.*
(2) *If $m(M)$ is orientably regular, then so is $m(M)^\alpha$.*
(3) *If $m(M)$ is regular, then so is $m(M)^\alpha$.*

Proof. It is sufficient to check that the automorphisms A_a and B_a introduced in Theorem 4 yield a group acting in an appropriate way on arcs of $m(M)$. □

A result similar to Theorem 5 can be proved in the general case as well. For non-orientable maps, however, the formalism employed in Theorem 4 has to be replaced by the technique of three involutions describing a map, see e.g. [**5**]. Since the difference is only technical and arguments are otherwise analogous, we state the theorem without proof.

Theorem 6. *Let M be a map and let α be a medial voltage assignment on $m(M)$. Then:*

(1) *If $m(M)$ is colour-regular, then so is $m(M)^\alpha$.*
(2) *If $m(M)$ is regular, then so is $m(M)^\alpha$.*

4. Applications

Let M be a map and let α be a medial voltage assignment on $m(M)$. Clearly, the face colouring of $m(M)$ lifts to a proper face 2-colouring of the derived map $m(M)^\alpha$. Since the derived map is 4-valent, $m(M)^\alpha$ is the medial map of some map L. Now assume that the original map M is regular and self-dual. By Theorem 3, $m(M)$ is regular; by Theorem 6 the same holds for $m(M)^\alpha$. Applying Theorem 3 again we see that the new map L is regular and self-dual (and thus has the same property as the original map M). Therefore, our theory provides a convenient tool for constructing new regular self-dual maps. We shall demonstrate this by establishing the following "orientable" result first.

For a graph G, let $G^{(k)}$ denote the graph obtained from G by replacing each edge with k parallel edges having the same end-vertices.

Theorem 7. *Let G be ad-valent graph of order n admitting a regular (or only orientably regular) self-dual map on an orientable surface of genus g. If k is a positive integer such that g.c.d. $(d,k)=1$, then $G^{(k)}$ admits a regular (orientably regular) self-dual map of genus $kg+(k-1)(n-1)$.*

Proof. We prove only the regular case; the other case follows analogously. Let M be a regular self-dual map on the surface of genus g such that the underlying graph of M is G. By Theorem 2, the medial map $m(M)$ is regular. Let H be the graph underlying $m(M)$ whose edges are directed as described in the beginning of Section 3. Consider the voltage assignment α on H which assigns to the preferred orientation of every edge the element 1 in the cyclic group $\mathbb{Z}_k$; this is clearly a medial voltage assignment. Observe that each face of $m(M)$ has length d and its boundary carries the net voltage d in $\mathbb{Z}_k$. Since g.c.d. $(d,k)=1$, each face F of $m(M)$ lifts to a single face F^α of length kd in the derived map $m(M)^\alpha$. As already noted, $m(M)^\alpha$ is again a medial map. By Theorem 6, $m(M)^\alpha$ is a regular map and by Theorem 3 it corresponds to a self-dual map $\tilde{M}$ with underlying graph, say, $\tilde{G}$. Let F_0 and F_1 be two vertex-faces of $m(M)$ having a common vertex v. Then it is easy to see that every vertex from the fibre over v is a common vertex of the lifted faces F_0^α and F_1^α. Thus an adjacency of two vertex-faces F_0 and F_1 in $m(M)$ gives rise to a k-fold adjacency of the lifted vertex-faces F_0^α and F_1^α. Consequently, $\tilde{G}$ coincides with $G^{(k)}$.

Finally, using Euler's formula it is readily computed that the derived surface has genus $kg+(k-1)(n-1)$. The proof is complete. □

Using the same method, a similar (but weaker) result can be proved also in the non-orientable case. In a regular non-orientable medial map the group acts transitively on the directed edges. Since each edge receives inverse assignments under its two directions, the constant assignment on this orbit must be an element of order 2. That is, we are forced to have $k=2$ for non-orientable maps.

Theorem 8. *Let G be a graph with all vertices of odd valency. Assume that G admits a non-orientable regular self-dual map on a surface with crosscap number $h \geq 1$. Then $G^{(2)}$ has a regular self-dual map on an orientable surface of genus $h-1$.*

Note that while the process in Theorem 7 can be iterated, one cannot use the last result repeatedly. There are two reasons for this: the graph $G^{(2)}$ has even valency, and the resulting derived surface (obtained by $\mathbb{Z}_2$-voltage assignment on the medial map) is necessarily orientable.

Finally, observe that the coverings $m(M)^\alpha \to m(M)$ resulting from Theorem 6 are all cyclic, i.e., their covering transformation group is cyclic. This raises the question of the existence of a non-cyclic covering map $m(M)^\alpha$ which would still be regular.

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D. Archdeacon, Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05405, U.S.A.; *e*-mail: archdeac@uvm.edu

J. Širáň, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia; *e*-mail: siran@mff.uniba.cs

M. Škoviera, Department of Computer Science, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia; *e*-mail: skoviera@mff.uniba.cs