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# PERIODS OF PERIODIC POINTS FOR TRANSITIVE DEGREE ONE MAPS OF THE CIRCLE WITH A FIXED POINT

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ABSTRACT. A map of a circle is a continuous function from the circle to itself. Such a map is transitive if there is a point with a dense orbit. For degree one transitive maps of the circle with a fixed point, we give all possible sets of periods and the best lower bounds for topological entropy in terms of the set of periods.

## 0. INTRODUCTION

A map of a space $X$ is a continuous function $f: X \to X$. We say $f$ is transitive if there is a point with a dense orbit. We denote by $P(f)$ the set of periods of the periodic points under $f$, and by $\operatorname{ent}(f)$ its topological entropy.

Consider the following ordering of the set $\mathbb{N}$ of natural numbers: $3, 5, 7, \dots,$ $2\cdot 3, 2\cdot 5, \dots, \dots, 2^k\cdot 3, 2^k\cdot 5, \dots, \dots, 2^3, 2^2, 2, 1$. Let $S(n)$ be the set consisting of $n$ and all integers standing to the right of $n$ in the above order, and $S(2^\infty)$ the set of all powers of 2. In [**9**], Sarkovskii showed that for maps of the real line, the sets of periods of periodic points are of the form $S(n)$ for some $n \in \mathbb{N} \cup \{2^\infty\}$.

Block [**2**] proved the following result for degree one maps of the circle.

**Theorem 0.1.** [**2**]. *Let $f$ be a continuous degree one map of the circle with a fixed point. Then $P(f) = S(n) \cup \{j \in \mathbb{N} : j \ge k\}$ for some positive integer $k$ and some $n \in \mathbb{N} \cup \{2^\infty\}$. (Note: One of the sets may be empty).*

In this paper, we consider transitive maps of the circle and show how the above result changes when we impose this dynamical restriction. Our main result is:

**Theorem 3.1.** *Let $f$ be a transitive degree one map of the circle with a fixed point. Then $P(f) = \{1\} \cup \{j \in \mathbb{N} : j \ge k\}$ for some positive integer $k$ and* $\operatorname{ent}(f) \ge \log$ (largest zero of $x^{k+1} - x^k - x - 1$).

*Moreover, if $k = 2$ and there is a periodic point of periodic point of period two with rotation number zero, then* $\operatorname{ent}(f) > \log 2$.

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Note that there is strictly inequality in the $\log 2$ entropy bound, and that 2 is greater than the largest zero of $x^3 - x^2 - x - 1$. The converse to Theorem 3.1 is also true, i.e., every possible $P(f)$ is realizable, and the entropy bounds are sharp. In Section 4, we discuss some examples.

## 1. Background

For a map $f$ of a space $X$, and $n \geq 0$, $f^n$ is defined by: $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$. A point $x \in X$ is periodic of period $n$ if $f^n(x) = x$ and $n$ is the least integer for which this happens. $\mathrm{Orb}_f(x)$ (or $\mathrm{Orb}(x)$) denotes the orbit $\{f^n(x) : n \geq 0\}$ of the point $x$. A subset $E$ is $(f)$-**invariant** if $f(E) \subseteq E$. $\mathrm{Int}\,(E)$ and $\mathrm{cl}(E)$ denote the interior and closure, respectively, of a set $E$. We denote by $S^1$ the circle $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ and $\mathbb{Z}$ denote the real and integer numbers, respectively.

The ambient space is $S^1$. An interval $[a,b]$, $(a,b)$, $[a,b)$ or $(a,b]$ in $S^1$ is the closed, open or half-open arc, resp., from a counterclockwise to $b$. $e$ is the natural projection from $\mathbb{R}$ onto $S^1$ ($e(x) = \exp(2\pi i x)$). A **lift** $F$ of $f$ is map of the real line for which $f(e(x)) = e(F(x))$ for all $x \in \mathbb{R}$. There are countably many lifts of $f$ and any two differ by an integer. The **degree** of $f$, denoted $\deg\, f$, is the integer $n$ such that $F(x+1) = F(x) + n$ for all $x \in \mathbb{R}$ and for every lift $F$ of $f$. Note that $\deg\, f^k = (\deg\, f)^k$.

In our proofs, we adopt the notion of $f$-covers from [**5**]. Let $J$ and $K$ be nondegenerate proper closed intervals. We say $J$ $f$-**covers** $K$ ($n$ times) if there exist subintervals $\{L_i : i \leq i \leq n\}$ of $J$, with pairwise disjoint interiors, such that, for each $i$, $f(L_i) = K$. Note that if $F$ is a lift of $f$, and $J'$ and $K'$ are interval lifts to $R$ of $J$ and $K$, resp., then $J$ $F$-covers $K$ if and only if $F(J')$ contains some integer translate $K' + m$ of $K'$.

**Lemma 1.1.** [**2**]. *Let $I = [a,b]$ be a proper closed interval of $S^1$. If $f(a) = c$ and $F(b) = d$ and $c \neq d$, then $I$ $f$-covers either $[c,d]$ or $[d,c]$.*

$f$-covers can be used to infer the existence of certain periods and obtain estimates on topological entropy, a topological conjugacy invariant of continuous maps. More specifically, if $P$ is a finite (but not necessarily invariant) subset of $S^1$, label the points in $P$ $x_1, x_2, \ldots, x_n$ so that the intervals $I_1 = [x_1, x_2]$, $\ldots$, $I_n = [x_n, x_1]$, we call $P$-**intervals**, have pairwise disjoint interiors.

The $P$-**graph** of $f$ is the directed graph having as vertices the $P$-intervals, and with $k$ arrows from $I_i$ to $I_j$ if and only if $I_i$ $f$-covers $I_j$ exactly $k$ times. The following lemma states that closed walks in the $P$-**graph** force the existence of periodic orbits that move in the same order.

**Lemma 1.2.** [**5**]. *Let $f$ be a map of the interval or the circle. If $J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$ is a loop in the $P$-graph of $f$, then there exists a fixed point $x$ of $f^n$ such that $f^i(x) \in J_i$ for $i = 0, 1, \ldots, n-1$.*

The $P$-**matrix** of $f$ is defined by $(A)_{ij}$ = (no. of arrows from $I_i$ to $I_j$). The Perron-Frobenius Theorem [**8**] guarantees the existence of a non-negative eigenvalue, denoted $r(A)$, of maximum modulus. If $A'$ is any proper submatrix of $A$, then $r(A) \geq r(A')$, with strict inequality if $A > 0$. In the remainder of this paper, we shall abuse notation and set $\log 0 = 0$.

**Lemma 1.3.** [**5, 6**]. *Let $P$ be a finite subset of $S^1$, and $A$ the $P$-matrix of $f$. Then* $\mathrm{ent}\,(f) \geq \log r(A)$, *with equality if $P$ is invariant, and $f$ is monotone between adjacent points of $P$.*

In the proof of Theorem 3.1 we show the existence of a subset called an $n$-horseshoe (for $f$), i.e., a collection $J_1, \ldots, J_n$ of closed subintervals with pairwise disjoint interiors, such that for $1 \leq i \leq n$, $J_i$ $f$-covers $J_1, \ldots, J_n$. When such a collection exists, the set of endpoints of the $J_i$'s yields a $P$-matrix with a proper sub-matrix whose entries are all $\geq 1$. Lemma 1.3 and standard Perron-Frobenius arguments imply $\mathrm{ent}\,(f) \geq \log n$.

**Lemma 1.4.** [**3,** Lemma 2]. *Let $f\colon S^1 \to S^1$ have an $n$-horseshoe ($n \geq 2$). Then $f$ has periodic points of all periods, and* $\mathrm{ent}\,(f) \geq \log n$.

## 2. Transitivity

In this section, we prove the following analogue of a result of Barge and Martin [**1,** Theorem 13] on transitive maps of the real line.

**Theorem 2.1.** *Let $f$ be a transitive map of the circle with a fixed point. If $f^2$ is transitive, then $f$ has a periodic point of odd period $k > 1$.*

**Remark.** This result is new only for $|\deg f| \leq 1$ [**5**].

The proof of the theorem is based on the following results of Coven and Mulvey [**7**]:

**Theorem 2.2.** [**7,** Corollary 3.4]. *For a transitive map of the circle with periodic points, the set of periodic points is dense.*

**Lemma 2.3.** [**7,** Lemma 5.2]. *If $f$ has periodic points and $f^n$ is transitive for every $n > 0$, then for every non-degenerate interval $E$ in $S^1$, $\bigcup_{n\geq 0} f^n(E)$ misses at most one point, which must then be a fixed point.*

**Theorem 2.4.** [**7,** Theorem C]. *Let $f$ be a continuous map of the circle. Then the following statements are equivalent:*
1) *There is an $m$ such that $f^{2m}$ is transitive, and $f^m$ has a fixed point.*
2) *$f^n$ is transitive for every $n > 0$, and $f$ has periodic points.*
3) *$f$ is topologically mixed (i.e., for every pair $U$, $V$ of non-empty open sets, there is an $N$ such that $f^n(U) \cap V \neq \emptyset$ for every $n \geq N$).*

We also use the following equivalent definitions of transitivity:
1) There is a point with a dense orbit.
2) The only closed invariant set $K$ with $\mathrm{int}\,(K) \neq \emptyset$ is the whole space
3) If $\mathrm{int}\,(E) \neq \emptyset$, then $\mathrm{cl}(\bigcup_{n\geq 0} f^n(E))$ is the whole space.

*Proof of Theorem* 2.1. **Notation:** If $U = [a, b]$, $V = [c, d]$ are non-overlapping closed intervals, we denote by $\langle U, V\rangle$ the open interval $(b, c)$.

Since $f$ is transitive, there is a point with a dense orbit. For this $x$, we have $x$, $f(x)$ and $f^2(x)$ distinct. Then by continuity of $f$, there is an interval $J$ about $x$ with $J$, $f(J)$ and $f^2(J)$ pairwise disjoint. By shrinking $J$, if necessary, we may assume that $f^2(J)$ contains no fixed point. Start at $J$ and label the other two intervals as $J'$ and $J''$ in the counterclockwise direction.

By Theorem 2.4, $f^n$ is transitive for every $n > 0$. We then choose a periodic orbit $\mathrm{Orb}(c)$ in the following way. By Lemma 2.3, either
(1) $\bigcup_{n\geq 0} f^{mn}(J) = S^1$ for every $m \geq 1$, or
(2) $\bigcup_{n\geq 0} f^{mn}(J) = S^1 - \{p\}$ for some $m \geq 1$ and some $p$ fixed by $f^m$.

If (1) holds, use Theorem 2.2 to choose a periodic point $c$ with period $t \geq 2$, such that $\mathrm{Orb}(c)$ meets both $\langle J, J'\rangle$ and $\langle J'', J\rangle$. By shrinking $J$, we may assume that $\mathrm{Orb}(c)$ does not meet $J$. Call $c_1$ and $c_2$ the points in $\mathrm{Orb}(c)$ such that $J$ lies in $(c_1, c_2)$ and $f(J) \cup f^2(J) \cup \mathrm{Orb}(c)$ lies in $[c_2, c_1]$.

If (2) holds, we may assume that $p$ is not in $J \cup f(J) \cup f^2(J)$. Since $g = f^m$ is transitive, use Theorem 2.2 to choose $c$ with $g$-period $t \geq 2$ so that for some points $c_1$, $c_2$ in $\mathrm{Orb}_f(c)$, $p \in (c_2, c_1)$, and $J \cup f(J) \cup f^2(J) \cup \mathrm{Orb}_f(c)$ lies in $[c_1, c_2]$. In either case, there exists $M > 0$ such that for all $n \geq M$, $\mathrm{Orb}(c) \subseteq f^n(J)$. (For example, in (1) $\mathrm{Orb}(c) \subseteq \bigcup_{n\geq 0} f^{tn}(J)$ and$f^t$ fixed every point in $\mathrm{Orb}(c)$. Let $M = t(k_1 + \cdots + k_t)$, where $c_i \in f^{tk_i}(J)$ for $c_i \in \mathrm{Orb}(c)$. Make a similar determination in (2), since $\mathrm{Orb}(c) \subset \bigcup_{n\geq 0} g^{tn}(J) = \bigcup_{n\geq 0} f^{mtn}(J)$ and $f^{mt}$ fixes every point in $\mathrm{Orb}(c)$.)

By Lemma 1.1, for each $n \geq M$, $J$ $f^n$-covers either $[c_1, c_2]$ or $[c_2, c_1]$.

If (1) holds, choose an odd $n > M$. If $J$ $F^n$-covers $[c_1, c_2]$, then $J$ $F^n$-covers itself. By Lemma 1.2, $J$ contains a periodic point of period a divisor of $n$. Since $J$ does not contain a fixed point, this point is of some odd period $> 1$. If $J$ $f^n$-covers $[c_2, c_1]$, then $J$ $f^n$-covers $f^2(J)$. Therefore, $f^2(J)$ $f^{n-2}$-covers itself, and again, since $f^2(J)$ has no fixed point, it has a periodic point of add period $> 1$.

If (2) holds, choose $n \geq M$ such that $n$ is also a multiple of $m$. Then since $p \in [c_2, c_1]$, $J$ $f^n$-covers $[c_1, c_2]$, hence $f^n$-covers $J$, $f(J)$ and $f^2(J)$. If $n$ is odd,then $J$ has a point of odd period $> 1$. If $n$ is even, then since $f(J)$ $f^{n-1}$ covers itself, $f(J)$ has a point of odd period $> 1$. $\square$

## 3. Degree One Maps

Let deg $f = 1$. If $F$ is a lift of $f$ and $x$ is $f$-periodic of period $n$ with $e(y) = x$, then $F^n(y) = y + k$ for some integer $k$. The number $k/n$, denoted $\rho_F(x)$, is the **rotation number** of $X$. This is independent of the choice of $y$, and if $F' = F+m$, then $\rho_{F'}(x) = \rho_F(x) + m$.

In this section , we prove our main result.

**Theorem 3.1.** *Let $f$ be a transitive degree one map of the circle with a fixed point. Then $P(f) = \{1\} \cup \{j \in \mathbb{N} : j \geq k\}$ for some positive integer $k$ and* $\mathrm{ent}\,(f) \geq \log$ (largest zero of $x^{k+1} - x^k - x - 1$).

*Moreover, if $k = 2$ and there is a periodic point of period two with rotation number zero, then* $\mathrm{ent}\,(f) > \log 2$.

**Lemma 3.2** [**5**]. *Let $f$ be a continuous degree one map of the circle with a fixed point. Then if $f$ has a fixed point and a periodic point of period $n > 1$ having different rotation numbers, then $f$ has periodic points of all periods larger than $n$, and* $\mathrm{ent}\,(f) \geq \log$ (largest zero of $x^{n+1} - x^n - x - 1$).

**Lemma 3.3.** *If $f$ is a transitive, degree one map of the circle with a fixed point, then $f^n$ is transitive for every $n > 0$ (hence is topologically mixing).*

*Proof.* By Theorem 2.4, it is enough to show that $f^2$ is transitive. Suppose $f^2$ is not transitive.

Let $p$ be a fixed point of $f$.

**Claim:** There is a nondegenerate proper closed interval $K$ such that:
(i) $f^2(K) = K$
(ii) $K \cup f(K) = S^1$
(iii) $\mathrm{int}\,[K \cap f(K)] = \emptyset$.

By [**7,** Lemma 2.1], (i)–(iii) hold for some closed proper subset $K$ with nonempty interior. To see that $K$ is an interval, let $L$ be a nondegenerate component of $K$, and $L^* = \mathrm{cl}\left[\bigcup_{n\geq 0} f^{2n}(L)\right]$. [**7**] shows that $L^* \subseteq K$ has finitely many components, each with non-empty interior, and they are permuted by $f^2$. Since $f^2|K$ is transitive and $L^*$ is nondegenerate, closed and $f^2$-invariant, $L^* = K$. So $K$ and $f(K)$ each has finitely many components, alternating on $S^1$.

Thus, $p$ must be a common endpoint of components $K_1$ of $K$ and $K_2$ of $f(K)$, and $f$ must permute $K_1$ and $K_2$. Since $K_1 \cup K_2$ is closed, $f$-invariant and has non-empty interior, it must be the whole circle, $f$-invariant and has non-empty interior, it must be the whole circle, and $K = K_1$.

Now let $p'$ be the lift of $p$ to $[0, 1]$, $F$ the lift of $f$ that fixed $p'$ (hence, also $p' + 1$). Then a left of $K$ to $[p', p' + 1]$ has either $p'$ or $p' + 1$ as an endpoint. A consideration of cases shows that no such lift can exist. ☐

**Proposition 3.4.** *Let $f$ be a topologically mixing map of the circle, with a lift $F$ such that for some $0 < x < y < 1$, $F(0) = F(y) = 0$ and $F(x) = y$. Then* $\mathrm{ent}\,(f) > \log 2$.

(*The same conclusion holds if instead $F(1) = F(x) = 1$ and $F(y) = x$.*)

*Proof.* We will prove the proposition for when $F(x) = y$ and $F(Y) = 0$. (The second case can be handled in a similar way.) To simplify notation, we will also call $0$, $x$ and $y$ their respective projections in $S^1$.

By Theorem 2.4, $f^n$ is transitive for every $n > 0$. Since $[0, y] \subseteq f([0, y])$, but $[0, y]$ cannot be $f$-invariant, there exists a non-degenerate interval $J$ in $[y, 0]$ that is adjacent to $[0, y]$ (i.e., has $y$ or $0$ as an endpoint) such that $J \subseteq f[0, y]$.

Suppose for the moment that $J = [y, w]$. Let $P = \{0, x, y, w\}$. Since $f$ is mixing, $J$ meets $f^n(J)$ for all large enough $n$. Since $x$ is not a fixed point, $x \in f^n(J)$ for infinitely many $n$. (If $x \notin f^n(J)$ for $n \geq N$, let $K = f^N(J)$. Then $x \notin \bigcup_{n\geq 0} f^n(K)$, hence by Lemma 2.3 must be fixed.)

Therefore, $\{x, 0\} \subseteq f^n(J)$ for some large enough $n$. Thus, by Lemma 1.1, $J$ has to $f^n$-cover either $[0, x]$ or $[x, y]$.

With $F$ as given, it is easy to see that in the $P$-graph of $f^n$ there are at least $2^{n-1}$ arrows each from $[0, x]$ to itself and to $[x, y]$, and from $[x, y]$ to itself and to $[0, x]$; there is at least one arrow from either $[0, x]$ or $[x, y]$ to $J$ (depending on which one covers $J$), and at least one arrow from $J$ to either $[0, x]$ or $[x, y]$. In any case, the submatrix $B$ corresponding to this subgraph is irreducible (i.e., $B^m > 0$ for some $m > 0$) and so by Perron-Frobenius arguments, $r(B^m) > 2^{mn}$. Since $k \cdot \mathrm{ent}\,(f) = \mathrm{ent}\,(f^k)$ for any $k \geq 0$, and the corresponding entries in the $P$-matrix of $f^{mn}$ are greater than or equal to that in $B^m$, by Lemma 1.3, $mn \cdot \mathrm{ent}\,(f) = \mathrm{ent}\,(f^{mn}) \geq \log r(B^m) > \log 2^{mn}$, i.e., $\mathrm{ent}\,(f) > \log 2$.

The same argument is valid if $J = [w, 0]$. (Here $\min F \mid [0, y] < 0$, and we look at the lift to $[-1, 0]$ of $J$.) □

*Proof of Theorem* 3.1. If $f$ has no point of period 2, then by Theorem 0.1,

$$P(f) = \{1\} \cup \{j \in \mathbb{N} : j \geq k\}$$

and $\{j \in \mathbb{N} : j \geq k\} \neq \emptyset$ by Lemma 3.3 and Theorem 2.1. Let $k$ be the smallest period greater than 1 in $P(f)$. Suppose that $x$ has $f$-period $k$. Then $\rho(x) \neq 0$; otherwise, for some lift $F$ of $f$, the lifts $e^{-1}\{x\}$ of $x$ are all $F$-periodic of period $k$. By [**9**], $2 \in P(F)$. But a point $z \in \mathbb{R}$ of $F$-period two either projects to a fixed point of $f$ or to a point of $f$-period two. Since $f$ has no point of period two, $z$ project to an $f$-fixed point, and so $F(z) = z + j$, $j \in \mathbb{Z} - \{0\}$. Since $\deg f = 1$, $z = F^2(z) = F(z + j) = z + 2j$, a contradiction. Thus $\rho(x) \neq 0$, and by Lemma 3.2, $\mathrm{ent}\,(f) \geq \log$ (largest zero of $x^{k+1} - x^k - x - 1$).

If there is a point of period two with nonzero rotation number, then Lemma 3.2 again implies that $f$ has points of all periods and $\mathrm{ent}\,(f) \geq \log$ (largest zero of $x^3 - x^2 - x - 1$).

We will show that if there is a point of period two with rotation number zero, then $f$ has a 2-horseshoe. By Lemma 1.4 $f$ has points of all periods, and $\operatorname{ent}(f) \geq \log 2$. Proposition 3.4 will be used to show strict inequality.

Let $a < b$ in $(0,1)$ be a lift of a period two orbit having rotation number zero. Then $F(a) = b + n$, $F(b) = a - n$ for some integer $n$. If $n > 0$, or if $n < -1$, then the intervals $[0, a]$, $[a, b]$, $[b, 1]$ indicate a 3-horseshoe for $f$ and we are done.

So assume that either:

(i) $F(a) = b - 1$, $F(b) = a + 1$; or

(ii) $F(a) = b$, $F(b) = a$.

In either case, there is a fixed point $q$ of $F$ in $[a, b]$. We need look only at case (ii) since if (i) holds then $b < a + 1$ and both lie in $[q, q + 1]$. Since $F(b) = a + 1$, $F(a + 1) = b$, using $q$ in place of 0, $b$ in place of $a$, and $a + 1$ in place of $b$ puts us in (ii).

It is clear that $f$ has a 2-horseshoe if for some $s$, $t$, $u$, $0 \leq s < t < u \leq 1$, either

$$\begin{array}{lll} & (*) & F(s), F(u) \leq s \ \text{ and } \ F(t) \geq u; \\ \text{or} & (**) & F(s), F(u) \geq u \ \text{ and } \ F(t) \leq s. \end{array}$$

Let $a_0 = a$, $b_0 = b$. Assume that $F$ has no 2-horseshoe in $[a_0, b_0]$. Since $[a, b]$ cannot be $F$-invariant and $[a, b] \subseteq F[a, b] \subseteq \ldots$, we have $F[a, b] = [a_1, b_1]$, where $a_1 \leq a_0$ and $b_0 \leq b_1$. Notice that if $a_1$ is attained in $[a_0, q]$ then $(**)$ holds for $\{a_0, z, q\}$ where $a_0 < z < q$ and $F(z) = a_1$. Similarly, $(*)$ holds if $b_1$ is attained in $[q, b_0]$. So $[a_0, q]$ must $F$-cover $[b_0, b_1]$ and $[q, b_0]$ must $F$-cover $[a_1, a_0]$, and at least one of these intervals is nondegenerate.

Now suppose $a_1 \leq 0$ (resp., $b_1 \geq 1$). Then $(*)$ (resp., $(**)$) holds for $\{0, a_0, x\}$ (resp., $\{y, b_0, 1\}$) where $q < x < b_0$ and $F(x) = 0$ (resp., $a_0 < y < q$ and $F(y) = 1$). So we may suppose $a_1 > 0$, $b_1 < 1$.

Suppose there exist $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ such that:

$$(1)\ \ F[a_{k-1}, b_{k-1}] = [a_k, b_k] = [a_k, a_{k-1}] \cup [a_{k-1}, b_{k-1}] \cup [b_{k-1}, b_k] \qquad (1 \leq k \leq n)$$

where at least one outside interval is nondegenerate;

(2) Neither $(*)$ nor $(**)$ holds in $[a_k, b_k]$, $0 \leq k \leq n - 1$.

(3) $0 < a_n \leq \cdots \leq a_1 \leq a_0$; $b_0 \leq b_1 \leq \cdots \leq b_n < 1$.

Then $b_k$ is only attained in $[a_{k-1}, a_{k-2}]$, and $a_k$ is only attained in $[b_{k-2}, b_{k-1}]$ for all $k > 1$, (resp., in $[a_0, q]$ and $[q, b + 0]$ if $k = 1$). If this process can go on forever then by transitivity of $f$, $\operatorname{cl}(\bigcup_{n \geq 0} [a_n, b_n]) = [0, 1]$. Evidently, there are $\{c_n\}_{n \geq 0}$ and $\{d_n\}_{n \geq 0}$ in $[0, 1]$ with $\lim c_n = 0$, $\lim d_n = 1$, $\lim F(c_n) = 1$ and $\lim F(d_n) = 0$. Since 0 and 1 are $F$-fixed, this is impossible. Thus for some $n \geq 0$, $(*)$ or $(**)$ must hold in one of $[0, b_n]$, $[a_n, b_n]$, or $[a_n, 1]$.

To see that ent $(f) > \log 2$, note that if $(*)$ or $(**)$ holds then we may assume $F$ satisfies the conditions of Proposition 3.4 by looking, if necessary, at another interval $[z, z+1]$ in place of $[0,1]$. (For example, if $a_1$ is attained in $(a_0, q)$, use $[q-1, q]$.) $\square$

## 4. Examples

Block's examples in [**2**] of degree one maps $f_k$ $(k \geq 2)$ have $P(f_k) = \{1\} \cup \{j \in \mathbb{N} : j \geq k\}$, and ent $(f_k)$ equal to the bound of Theorem 3.1. By [**4,** Theorem 3.1] the irreducibility of each $P_k$-matrix implies transitivity of $f_k$. Notice that $f_1$ has no point of period two with rotation number zero.

We show that the $\log 2$ bound is also sharp by the following sequence of degree one transitive maps each one having a fixed point and a period-two point with rotation number zero.

Define the lift $F_0$ by $F_0(0) = 0$, $F_0(1/6) = -1/3$, $F_0(1/3) = 0$, $F_0(1/2) = 2/3$, $F_0(2/3) = 1/3$, $F_0(1) = 1$, and linear between these points, and let $P_0$ be (the projection of) $\{0, 1/3, 2/3\}$.

For $n \geq 1$, define $F_n$ by $F_n(0) = 0$, $F_n(1/(2^{n+1} \cdot 3)) = -1/3$, $F_n(1/(2^n \cdot 3) = 0$, $F_n(1/(2^{n-1} \cdot 3)) = 1/(2^n \cdot 3)$, $\dots$ , $F_n(1/3) = 1/(2 \cdot 3)$, $F_n(1/2) = 2/3$, $F_n(2/3) = 1/3$, $F_n(1) = 1$, and linear between these points, and let $P_n$ be (the projection of) $\{0, 1/(2^n \cdot 3), 1/(2^{n-1} \cdot 3), \dots, 1/3, 2/3\}$.

Application of [**4,** Theorem 3.1] again implies that all the the $f_n$'s are transitive. The induced $P_n$-graphs indicate a point of period two with rotation number zero for $f_n$. By Lemma 1.3, ent $(f_n) = \log r_n$, where $r_n$ is the largest zero of the characteristic polynomial $p_n(x) = x^{n+1} \cdot (x-1) \cdot (x-2) - 2$, $(n \geq 0)$ of the $P_n$-matrix. It is an elementary argument to show that $r_1 > r_2 > \cdots > 2$, and $\lim_{n\to\infty} r_n = 2$.

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