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AN OMEGA THEOREM ON DIFFERENCES OF TWO SQUARES

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1. Introduction

Let $\rho(n)$ denote the number of pairs $(u, v) \in \mathbf{N} \times \mathbf{Z}$ with $u^2 - v^2 = n$. To study the average order of this function, we consider the Dirichlet summatory function

$$T(x) = \sum_{n \leq x} \rho(n)$$

where x is a large real variable. The functions $\rho(n)$ and $T(x)$ have been mentioned, e.g., by Krätzel [**6**], in the more general context that the square is replaced by a k-th power, $k \geq 2$. It is closely related to the classical divisor function $d(n)$. If $D(x)$ denotes the Dirichlet summatory function

$$D(x) = \sum_{n \leq x} d(n)$$

we have the relation (see Lemma 1 below)

$$T(x) = D(x) - 2D\left(\frac{x}{2}\right) + 2D\left(\frac{x}{4}\right). \tag{1}$$

It is known that

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x)$$

where γ denotes throughout this paper the Euler-Mascheroni constant and

$$\Delta(x) = O(x^{\frac{7}{22}+\epsilon}).$$

(See Iwaniec and Mozzochi [**4**] for this upper bound and the textbook of Krätzel [**5**] for an enlightening survey of the theory of Dirichlet's Divisor Problem and the definition of the O — and the Ω — symbols. We remark parenthetically that

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M. N. Huxley has recently announced an improvement of the exponent $\frac{7}{22}$ to $\frac{23}{73}$.)
From this it is clear that

$$T(x) = \frac{x}{2}\log x + (2\gamma - 1)\frac{x}{2} + \Theta(x)$$

with

$$\Theta(x) = O(x^{\frac{7}{22}+\epsilon}).$$

Concerning lower estimates, it is known that

$$\Delta(x) = \Omega_+(x^{\frac{1}{4}}(\log x)^{\frac{1}{4}}(\log\log x)^{\frac{3+\log 4}{4}}\exp(-A\sqrt{\log\log\log x}))$$

(see Hafner [**2**] for this comparatively recent refinement of Hardy's classical result [**3**]), and

$$\Delta(x) = \Omega_-(x^{\frac{1}{4}}\exp(c(\log\log x)^{\frac{1}{4}}(\log\log\log x)^{-\frac{3}{4}})) \qquad (c > 0)\,,$$

established by Corrádi and Kátai [**1**].

2. Subject and Result of This Paper

The aim of this note is a proof of a lower estimate for $\Theta(x)$ which is as sharp as Hafner's, on the basis of the main ideas of his method. We note that, in view of the alternating sign in (1) and the above — mentioned "asymmetry" of the Ω — estimates for $\Delta(x)$, it is not a priori clear that such a generalization is possible.

Theorem.

$$T(x) = \frac{x}{2}\log x + (2\gamma - 1)\frac{x}{2} + \Theta(x)$$

with

$$\Theta(x) = \Omega_+(x^{\frac{1}{4}}(\log x)^{\frac{1}{4}}(\log\log x)^{\frac{3+\log 4}{4}}\exp(-A\sqrt{\log\log\log x}))$$

where A is a positive absolute constant.

3. An Elementary Auxiliary Result

Lemma 1. *Define*

$$\rho(n) = \#\{(u,v) \in \mathbf{N}\times\mathbf{Z} : u^2 - v^2 = n\}, \quad \textit{then}$$
$$\rho(n) = d(n) - 2\,d\left(\frac{n}{2}\right) + 2\,d\left(\frac{n}{4}\right)$$

where $d(n)$ is the number of positive divisors of n and $d(x) = 0$ if $x \notin \mathbf{N}$.

Proof. Let

$$X(n) = \{(x,y) \in \mathbf{N}\times\mathbf{Z} : x^2 - y^2 = n\},$$
$$Y(n) = \{(u,v) \in \mathbf{N}\times\mathbf{N} : uv = n\}.$$

Obviously $|X(n)| = \rho(n)$ and $|Y(n)| = d(n)$.

Case 1: n is odd

$$(x,y) \in X(n) \leftrightarrow (u,v) = (x+y, x-y) \in Y(n)$$

defines a bijection between $X(n)$ and $Y(n)$ therefore $\rho(n) = d(n)$ and $d(\frac{n}{2}) = 0$, $d(\frac{n}{4}) = 0$.

Case 2: n is even, $n \equiv 0 \pmod 4$

$$(x,y) \in X(n) \leftrightarrow (u,v) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right) \in Y\left(\frac{n}{4}\right)$$

defines a bijection between $X(n)$ and $Y(\frac{n}{4})$, so $\rho(n) = d(\frac{n}{4})$.
Let $n = 2^k u$, u odd, so

$$d(n) - 2d\left(\frac{n}{2}\right) + 2d\left(\frac{n}{4}\right) = d(u)d(2^{k-2}) = d\left(\frac{n}{4}\right)$$

since d is multiplicative.

Case 3: n is even, $n \equiv 2 \pmod 4$. It is easily seen that in this case both sides of the equation are 0. □

4. Proof of the Theorem

It is well known that

$$\Delta(x) = \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}} \sum_{n=1}^{\infty} \frac{d(n)}{n^{\frac{3}{4}}} \cos\left(2\pi\sqrt{n\,x}\right) + O(1). \tag{2}$$

Following Hafner [**2**], we consider the integral

$$\int_{-1}^{1} \Delta(t+u)\, k_N(u)\, du$$

where $k_N(u)$ is a Fejér kernel. Since the series in (2) converges only conditionally, we prefer to use the classic result (see Landau [**7**])

$$\Delta_1(x) = \int_0^x (\Delta(y) - \frac{1}{4})\, dy = -\frac{1}{4\pi^4} \sum_{n=1}^{\infty} \frac{d(n)}{n^2} F(4\pi^2 nx),$$

where the sum converges absolutely and uniformly and $F'(w)$ has the asymptotic expansion

$$F'(w) = -\frac{\sqrt{\pi}}{2} w^{\frac{1}{4}} \cos(2\sqrt{w} - \frac{\pi}{4}) + O(w^{-\frac{1}{4}})$$

(see Landau [**7**] and Oberhettinger [**8**].) From this result and (1) it is easy to see that

$$\begin{aligned}\Theta_1(x) &:= \int_0^x (\Theta(y) - \frac{1}{4})\,dy \\ &= -\frac{1}{4\pi^4}\sum_{n=1}^{\infty}\frac{d(n)}{n^2}\left(F(4\pi^2nx) - 4F(2\pi^2nx) + 8F(\pi^2nx)\right).\end{aligned}$$

Let $g \in C^1[a,b]$. We use the identity

$$\int_b^a g'(t)\Theta_1(t)\,dt = g(t)\Theta_1(t)|_a^b - \int_b^a g(t)\Theta_1'(t)\,dt.$$

On the left side of this equation we insert our series for $\Theta_1(t)$. By partial integration and comparison we get:

$$\begin{aligned}\int_b^a g(t)(\Theta(t) - \frac{1}{4})\,dt = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{d(n)}{n}\int_b^a g(t)\Big(&F'(4\pi^2nt) \\ &- 2F'(2\pi^2nt) + 2F'(\pi^2nt)\Big)\,dt\ .\end{aligned} \tag{3}$$

We now substitute $t = u^2$, put $h(u) = g(u^2)u^{\frac{3}{2}}$, $\alpha = \sqrt{a}$, $\beta = \sqrt{b}$. Now let T and N be large real parameters, then we choose $\alpha = T-1$, $\beta = T+1$ and $h(u) = k_N(u-T)$ where

$$k_N(u) = \frac{\lambda_N}{2\pi}\left(\frac{\sin\lambda_N\frac{u}{2}}{\lambda_N\frac{u}{2}}\right)^2$$

is a Fejér kernel, and $\lambda_N = 4\pi\sqrt{N}$. It is well known (cf. e.g. [**7**, p. 19]) that (for $a > 0$)

$$\int_{-\infty}^{\infty} e^{iau}k_N(u)\,du = \begin{cases} 1 - \frac{a}{\lambda_N}, & 0 < a \le \lambda_N \\ 0, & \text{else.}\end{cases}$$

By observing that $k_N(\pm 1) = O(1)$, $k_N'(u) = O(u^{-2})$ for $|u| \ge 1$ (uniformly in $N \in \mathbf{N}$), and applying integration by parts, we conclude that

$$\int_{-1}^{1} e^{iau}k_N(u)\,du = \begin{cases} 1 - \frac{a}{\lambda_N} + O(\frac{1}{a}), & 0 < a \le \lambda_N \\ O(\frac{1}{a}), & \text{else.}\end{cases} \tag{4}$$

We next substitute $u - T \to u$. Hence we get from (3) and the asymptotic expansion for $F'(w)$

$$\begin{aligned}J(T) := \pi\int_{-1}^{1} k_N(u)G(u)\,du = \sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{3}{4}}}\int_{-1}^{1} k_N(u)(c_1\cos{(\lambda_n(u+T) - \frac{\pi}{4})} \\ + \cos{(\mu_n(u+T) - \frac{\pi}{4})} - c_2\cos{(\theta_n(u+T) - \frac{\pi}{4})})\,du + O(1)\end{aligned}$$

where

$$G(u) = \left(\Theta((u+T)^2) - \frac{1}{4}\right)(u+T)^{-\frac{1}{2}}$$

$\lambda_n = 4\pi\sqrt{n}$, $\mu_n = 2\pi\sqrt{n}$, $\theta_n = 2\sqrt{2}\pi\sqrt{n}$ and $c_1 = \frac{\sqrt{2}}{2}$, $c_2 = \sqrt[4]{2}$. Using the real part of (4), we obtain from this

$$J(T) = c_1 \sum_{n \le N} a_n \left(1 - \frac{\lambda_n}{\lambda_N}\right) \cos(\lambda_n T - \frac{\pi}{4}) + \sum_{n \le 4N} a_n \left(1 - \frac{\mu_n}{\lambda_N}\right) \cos(\mu_n T - \frac{\pi}{4})$$

$$- \sum_{n \le 2N} a_n \left(1 - \frac{\theta_n}{\lambda_N}\right) \cos(\theta_n T - \frac{\pi}{4}) + O(1),$$

where $a_n = d(n)n^{-3/4}$ for short, throughout the rest of the paper. The key step is now the application of Dirichlet's approximation principle to the first and the second sum. In Hardy's theorem ([**3**]), this principle is applied to all the terms in the sum. Hafner instead applies this principle only to those terms in the sums which yield the main contribution. Let

$$P_1(N) = \left\{n \le N : \omega(n) \ge 2\log\log N - A\sqrt{\log\log N}\right\},$$

$N_1 = |P_1|$, where A is a positive constant and $\omega(n)$ is the number of distinct prime divisors of n.

Lemma 2 (See [**2**]).

a) $\displaystyle\sum_{\substack{n \notin P_1 \\ n \le N}} \frac{d(n)}{n^{\frac{3}{4}}} \le \frac{C}{A^2} N^{\frac{1}{4}} \log N$

b) $N_1 \ll N(\log N)^{1-\log 4} \exp\left(A \log 2\sqrt{\log\log N}\right)$

where C is an absolute constant.

Remark. From a) it follows that $N_1 \to \infty$ for $N \to \infty$ (provided that A is sufficiently large).

Lemma 3 (Dirichlet's approximation principle).
Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_s) \in \mathbf{R}^s$, $t_0 \in \mathbf{R}^+$, then there exist $\underline{p} \in \mathbf{Z}^s$ and $t \in \mathbf{R}$ with $\|t\underline{\alpha} - \underline{p}\|_\infty < \frac{1}{16}$ and $t_0 < t < t_0 16^s$.

We decompose the representation for $J(T)$ (with $P_2 = P_1(4N)$, $N_2 = |P_2|$, for short):

$$J(T) = c_1 \sum_{n \in P_1} a_n \left(1 - \frac{\lambda_n}{\lambda_N}\right) \cos\left(\lambda_n T - \frac{\pi}{4}\right) + c_1 \sum_{\substack{n \notin P_1 \\ n \le N}} a_n \left(1 - \frac{\lambda_n}{\lambda_N}\right) \cos\left(\lambda_n T - \frac{\pi}{4}\right)$$

$$+\sum_{n\in P_2} a_n\left(1-\frac{\mu_n}{\lambda_N}\right)\cos(\mu_n T-\frac{\pi}{4})+\sum_{\substack{n\notin P_2\\ n\le 4N}} a_n\left(1-\frac{\mu_n}{\lambda_N}\right)\cos(\mu_n T-\frac{\pi}{4})$$
$$-c_2\sum_{n\le 2N} a_n\left(1-\frac{\theta_n}{\lambda_N}\right)\cos(\theta_n T-\frac{\pi}{4})+O(1).$$

For any $T_0>1$, we apply Dirichlet's approximation principle to $\underline{\alpha}=(\lambda'_1,\dots,\lambda'_{N_1},\mu'_1,\dots,\mu'_{N_2})\in\mathbf{R}^{N_1+N_2}$ where $\lambda_n=4\pi\sqrt{n}=2\pi\lambda'_n$ and $\mu_n=2\pi\sqrt{n}=2\pi\mu'_n$, to find a T in the interval

$$T_0<T<T_0(16)^{N_1+N_2} \tag{5}$$

with $\cos(\lambda_n T-\frac{\pi}{4})\ge c$ for all $n\in P_1$ and $\cos(\mu_n T-\frac{\pi}{4})\ge c$ for all $n\in P_2$ ($c=\cos\frac{\pi}{8}=0.92388$). We estimate the other sums trivially by $-1\le\cos(.)\le 1$, and get

$$J(T)\ge cc_1\sum_{n\le N} a_n\left(1-\frac{\lambda_n}{\lambda_N}\right)+c\sum_{n\le 4N} a_n\left(1-\frac{\mu_n}{\lambda_N}\right)-c_2\sum_{n\le 2N} a_n\left(1-\frac{\theta_n}{\lambda_N}\right)$$
$$-(c_1+cc_1)\sum_{\substack{n\notin P_1\\ n\le N}}\frac{d(n)}{n^{\frac{3}{4}}}-(1+c)\sum_{\substack{n\notin P_2\\ n\le 4N}}\frac{d(n)}{n^{\frac{3}{4}}}\;+O(1)\,.$$

For the last two sums we have an estimate by Lemma 2a). To estimate the other sums, we consider first the second and the third sum. It is quite simple to show that

$$c\,a_n\left(1-\frac{\mu_n}{\lambda_N}\right)-c_2\;a_n\left(1-\frac{\theta_n}{\lambda_N}\right)\ge 0\quad\text{if}\quad n>\frac{N}{2}.$$

Therefore

$$J(T)\ge cc_1\sum_{n\le N} a_n\left(1-\frac{\lambda_n}{\lambda_N}\right)+c\sum_{n\le\frac{N}{2}} a_n\left(1-\frac{\mu_n}{\lambda_N}\right)-c_2\sum_{n\le\frac{N}{2}} a_n\left(1-\frac{\theta_n}{\lambda_N}\right)$$
$$-\frac{K_1}{A^2}N^{\frac{1}{4}}\log N$$

with a constant K_1. We further observe that

$$c\,c_1\,a_n\left(1-\frac{\lambda_n}{\lambda_N}\right)+c\,a_n\left(1-\frac{\mu_n}{\lambda_N}\right)-c_2\,a_n\left(1-\frac{\theta_n}{\lambda_N}\right)\ge K\,a_n$$

for $n\le\frac{N}{2}$, with $K=0.1939>0$.

Hence we get

$$J(T) \geq C\, N^{\frac{1}{4}} \log N \tag{6}$$

with a suitable constant $C > 0$, for N sufficiently large.

We now make our choice of the parameters involved: Let T_0 be fixed and sufficiently large, then we choose for N the least positive integer such that $N_1 + N_2$ exceeds $\log T_0$. (This is well-defined by the Remark after Lemma 2.) We pick T as above (cf. Lemma 3), and deduce from (5) and Lemma 2 (b) that

$$\log T \ll N_1 + N_2 \ll N(\log N)^{1-\log 4} \exp\left(A \log 2 \sqrt{\log\log N}\right).$$

By a short calculation, we derive from (6)

$$J(T) \gg (\log T)^{\frac{1}{4}} (\log\log T)^{\frac{3+\log 4}{4}} \exp\left(-A' \sqrt{\log\log\log T}\right)$$

with $A' = A \log 2$. Since $k_N(u)$ is positive and

$$0 < b_1 < \int_{-1}^{1} k_N(u)\, du < b_2 < 1$$

with absolute constants b_1, b_2, uniformly in $N \geq 1$, we may conclude that there exists a value v with $T - 1 \leq v \leq T + 1$ for which

$$G(v - T) \geq C_1 (\log v)^{\frac{1}{4}} (\log\log v)^{\frac{3+\log 4}{4}} \exp\left(-A' \sqrt{\log\log\log v}\right)$$

with a suitable constant C_1. Since $v > T_0 - 1$, and T_0 can be chosen arbitrarily large, this completes the proof of the theorem. □

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