Acta Math. Univ. Comenianae
Vol. LXI, 1(1992), pp. 101–115

DIVISOR PROBLEMS IN SPECIAL SETS OF POSITIVE INTEGERS

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1. Introduction

For infinite sets of natural numbers S_1, S_2, we define the arithmetic function

$$\tau_{S_1,S_2}(n) = \#\{(m_1,m_2) \in S_1 \times S_2 : m_1 m_2 = n\} \qquad (n \in \mathbb{N}).$$

To study its average order, it is usual to consider the corresponding Dirichlet's summatory function

$$\sum_{n \le x} \tau_{S_1,S_2}(n)$$

where x is a large real variable. For $S_1 = S_2 = \mathbb{N}$, this is just the classical Dirichlet divisor problem: See Krätzel [**7**] for a survey of its history and Huxley [**4, 5**] for the hitherto sharpest results. In recent times, Smith and Subbarao [**19**], the author [**13**], and Varbanec and Zarzycki [**20**] investigated the case $S_1 = \mathbb{N}$, $S_2 = \mathcal{A}$, where $\mathcal{A}$ denotes throughout the sequel an arithmetic progression

$$\mathcal{A} = \mathcal{A}(a,q) = \{m \in \mathbb{N} : m \equiv a \pmod{q}\} \qquad (1 \le a \le q).$$

Articles by Mercier and the author [**10, 11**] discuss the situation that S_1, S_2 are the images of $\mathbb{N}$ under certain (monotonic) polynomial functions p_1, p_2 with integer coefficients.

In the present paper, we will consider (in fact in a more general context) the case that one or both of S_1, S_2 is equal to the set $\mathbf{B} = \mathbf{B}_{\mathbf{Q}(i)}$ consisting of those natural numbers which can be written as a sum of two integer squares.

For a given natural number n, there arise two questions in a natural way:

(i) *How many divisors of n belong to the set $\mathbf{B}$?*

(ii) *In how many ways can n be written as a product of two elements of $\mathbf{B}$?*

Question (i) leads to the arithmetic function $\tau_{\mathbf{B},\mathbb{N}}(n)$. A result on this is contained in a quite recent paper of Varbanec [**21**] who actually considered the more general

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Received November 14, 1991.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 11N37, 11N69, 11R47.
*This article is part of a research project supported by the *Austrian Science Foundation* (Nr. P8488-PHY).

function $\tau_{\mathbf{B},\mathcal{A}}(n)$, obtaining an estimate uniform in a and q. Our first aim is to improve his result up to an error term which can be called "final" on the basis of our present knowledge about zero-free regions of the Riemann and Dedekind zeta-functions (Theorem 1).

Since $\mathbf{B}$ forms a semigroup with respect to multiplication, question (ii) can also be viewed as a "Dirichlet divisor problem in the set $\mathbf{B}$". We will establish an asymptotic formula for $\sum_{n\le x}\tau_{\mathbf{B},\mathbf{B}}(n)$ with an order term corresponding to the hitherto sharpest one in the Prime Number Theorem (Theorem 2).

2. Statement of Results

Theorem 1. *For an algebraic number field K which is a Galois extension of the rationals of degree $[K:\mathbb{Q}]=r\ge 2$, let $\mathcal{O}_K$ denote the set of integer ideals in the ring of algebraic integers in K, and define $\mathbf{B}=\mathbf{B}_K$ as the set of all positive integers n for which there exists at least one ideal $\mathcal{I}\in\mathcal{O}_K$ with norm equal to n. Let $\mathcal{A}=\mathcal{A}(a,q)$ be an arithmetic progression $(1\le a\le q)$, then the asymptotic formula*

$$\sum_{n\le x}\tau_{\mathbf{B},\mathcal{A}}(n)=\frac{x}{a}\sum_{k=0}^{M(\frac{x}{a})}A_k^{(1)}(\log\frac{x}{a})^{-k-1+1/r}+\frac{x}{q}\sum_{k=0}^{M(\frac{x}{q})}A_k^{(2)}(\log\frac{x}{q})^{-k+1/r}$$

$$+O(\frac{x}{a}\exp(-c(\log(\frac{3x}{a}))^{3/5}(\log\log(\frac{3x}{a}))^{-1/5}))$$

holds uniformly in $1\le a\le q\le x$, where

$$M(w)\overset{\text{def}}{=}[c'(\log 3w)^{3/5}(\log\log 3w)^{-6/5}]\,, \tag{2.1}$$

$c>0$, $c'>0$ and the O-constant depend at most on the field K but not on a and q. The coefficients $A_k^{(1)}$ and $A_k^{(2)}$ are computable and satisfy

$$A_k^{(i)}\le(b_*k)^k \tag{2.2}$$

(for $k\ge 1$) with some constant $b_*>0$ independent of $\mathcal{A}(a,q)$.

Theorem 2. *Let $K=\mathbb{Q}(\sqrt{D})$ be a quadratic number field with discriminant D, and define $\mathbf{B}=\mathbf{B}_K$ as before, then we have the asymptotic formula*

$$\sum_{n\le x}\tau_{\mathbf{B},\mathbf{B}}(n)=A^*x+x^{\frac12}\sum_{k=0}^{M(x)}A_k(\log x)^{-\frac12-k}$$
$$+O(x^{\frac12}\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}))$$

where $M(x)$ is defined in (2.1), the A_k's are computable and satisfy (2.2). The leading coefficient A^* can be given explicitly as

$$A^* = \rho \prod_{p|D} (1 - \frac{1}{p})^{-1} \prod_{p \in \mathbb{P}_2} (1 - \frac{1}{p^2})^{-1}$$

where ρ is the residue of the Dedekind zeta-function $\zeta_K(s)$ at $s = 1$ and $\mathbb{P}_2$ denotes the set of all rational primes p such that (p) is a prime ideal in $\mathcal{O}_K$.

Remarks.

1. The bound (2.2) for the coefficients $A_k^{(i)}$ ensures that later terms in our expansions cannot exceed the size of the first terms. Furthermore, it shows that, for every $N \le M(x)$, we could break up the expansion in Theorem 2 after the term with $(\log x)^{-\frac{1}{2}-N}$, obtaining an order term $O(x^{1/2}(\log x)^{-3/2-N})$. Of course, the corresponding assertion holds for the two expansions in Theorem 1; in particular, the upper limit $M(\frac{x}{q})$ in the second sum can be replaced by $M(\frac{x}{a})$, without getting a new error term.

2. It should be pointed out that the restriction on the quadratic case in Theorem 2 is natural and necessary: As we can see from the proof below, the generating function of $\tau_{\mathbf{B},\mathbf{B}}(n)$ contains a factor $(\zeta_K(s))^{2/r}$. For $r = [K : \mathbb{Q}] = 2$, this has a simple pole at $s = 1$ which can be "isolated" in a way that we obtain the leading term A^*x and an expansion in terms which are $o(x^{1/2})$. If $r > 2$, the point $s = 1$ would be a branch point of the generating function: We do not see a way to get a better error term than $O(x \exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}))$ in this case.

3. Our proofs are based on a well-established method of analytic number theory. This can be traced back to a classic paper of Selberg [**17**], and articles by Rieger [**16**], Kolesnik and Straus [**6**] and others. An enlightening account on the theory can be found in the book of De Koninck and Ivić [**2**].

3. Preliminaries

Throughout the paper, b and c (also with a subscript or a dash) denote positive constants which may depend on the field K but not on the progression $\mathcal{A}(a, q)$. (This applies to all O- and $\ll$-constants as well, throughout the paper.)

Let $H(s)$ be any analytic function without zeros on a certain simply connected domain S of $\mathbb{C}$ which contains the real line to the right of $s = \sigma_0$ where $\sigma_0 = 1$ or $\frac{1}{2}$. Suppose that $H(s) \in \mathbb{R}^+$ for real $s > \sigma_0$, and let $\alpha \in \mathbb{R}$ arbitrary. Then we define $(H(s))^\alpha$ on S by

$$(H(s))^\alpha = \exp\bigl(\alpha(\log(H(2)) + \int_2^s \frac{H'(z)}{H(z)} dz)\bigr) ,$$

the path of integration being completely contained in S but otherwise arbitrary.

In our analysis, S will usually be a domain symmetric with respect to the real line, with a "cut" along $L = \{s \in \mathbb{R} : s \leq \sigma_0\}$ (such that $S \cap L = \emptyset$). We will join in the common abuse of terminology to think of an "upper" and "lower edge" of $L \cap \partial S$, on which $(H(s))^\alpha$ are attributed two different values, depending on whether L is approached from above or from below.

In our first Lemma, we summarize the present state of art about zero-free regions of Dedekind zeta-functions.

Lemma 1 (See T. Mitsui [**12**]). *Let $\zeta_K(s)$ denote the Dedekind zeta-function of an arbitrary algebraic number field K. Define for short*

$$\psi(t) = (\log t)^{2/3}(\log\log t)^{1/3} \qquad (t \geq 3)$$

and, for positive constants $b_1 \geq 3$ and b_2,

$$\lambda(t) = \begin{cases} 1 - b_0 \stackrel{\text{def}}{=} 1 - \frac{b_2}{\psi(b_1)}, & \text{for } |t| \leq b_1\,, \\ 1 - \frac{b_2}{\psi(|t|)}, & \text{for } |t| \geq b_1\,. \end{cases}$$

Then there exist values of b_1, b_2, b_3 such that for all $s = \sigma + it$ with

$$\sigma \geq \lambda(t)\,, \qquad |s-1| \geq \varepsilon\,, \qquad (0 < \varepsilon < 1)$$

it is true that

$$\zeta_K(s) \neq 0\,, \qquad \frac{\zeta_K'(s)}{\zeta_K(s)} \ll \psi(|t|+3) + \frac{1}{\varepsilon}$$

and

$$(\zeta_K(s))^{\pm 1} \ll (\log(2+|t|))^{b_3} + \frac{1}{\varepsilon}\,.$$

Proof. This is essentially Lemma 11 of Mitsui [**12**]. The very last assertion is readily derived on classical lines; see e.g. Prachar [**15,** p. 71.] □

Our next auxiliary result provides an asymptotic expansion for a certain contour integral which is essential in the type of problem under consideration.

Lemma 2. *Let $H(s)$ be a holomorphic function on the disk*

$$\{s \in \mathbb{C} : |s-1| < 2b_0\} \qquad (b_0 > 0 \text{ fixed})$$

and let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Let $\mathcal{C}_0$ denote the circle $|s-1| = b_0$, with positive orientation, starting and ending at $1 - b_0$. For a large real variable w, it follows that

$$\frac{1}{2\pi i}\int_{\mathcal{C}_0} (s-1)^{-\alpha} H(s) w^s\, ds = w \sum_{k=0}^{M(w)} \frac{\beta_k}{\Gamma(\alpha-k)} (\log w)^{\alpha-k-1}$$
$$+ O(w \exp(-c''(\log w)^{3/5}(\log\log w)^{-1/5})) \qquad (c'' > 0)$$

where $M(w)$ is defined as in (2.1), β_k are the coefficients in the Taylor expansion of $H(s)$ at $s = 1$. By Cauchy's estimates and standard results on the Gamma-function, they satisfy

$$\frac{\beta_k}{\Gamma(\alpha - k)} \ll b_0^{-k}\Gamma(1-\alpha+k) \max_{|s-1|=b_0} |H(s)| \ll (b_0^{-1}k)^k \max_{|s-1|=b_0} |H(s)| . \tag{3.1}$$

The constant c'' and the O- and $\ll$-constants depend only on α.

Proof. Results of this type are essentially well-known to experts. The details of the argument for the present statement may be found (in a special context, w.l.o.g.) in [**14**], formula (3.5) and sequel. □

Our next lemma summarizes what is known about the density of the sets $\mathbf{B}_K$ in $\mathbb{N}$.

Lemma 3. *For an algebraic number field K which is a Galois extension of the rationals of degree $[K : \mathbb{Q}] = r \geq 2$, and large real x,*

$$\begin{aligned} B(x) \overset{\text{def}}{=} \#\{n \in \mathbf{B} :\ n \leq x\} = \frac{1}{2\pi i}\int_{\mathcal{C}_0} (s-1)^{-1/r} H(s) x^s \, ds \\ + O(x \exp(-c^*(\log x)^{3/5}(\log\log x)^{-1/5})) \qquad (c^* > 0) \end{aligned}$$

where $\mathcal{C}_0$ is defined as in Lemma 2 (with $b_0 > 0$ suitable), and $H(s)$ is holomorphic in a neighbourhood of $s = 1$.

Proof. Although this assertion does not contain too much of novelty either, at least for $K = \mathbb{Q}(i)$ (see Landau [**8**] and Shanks [**18**]), we sketch the argument for convenience of the reader.

Let us denote by $\mathbf{i}_S(\cdot)$ the indicator function of any set $S \subset \mathbb{N}$. It follows from the decomposition laws in $\mathcal{O}_K$ (cf. Hecke [**3**]) that, for all rational primes p which do not divide the discriminant D of K, $\mathbf{i}_{\mathbf{B}}(p) = 1$ if and only if p splits into r distinct prime ideals in $\mathcal{O}_K$. Consequently, for $\operatorname{Re} s > 1$,

$$F_0(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \mathbf{i}_{\mathbf{B}}(n)\, n^{-s} = (\zeta_K(s))^{1/r} H_1(s) , \tag{3.2}$$

where $H_1(s)$ has a Dirichlet series absolutely convergent for $\operatorname{Re} s > \frac{1}{2}$. In view of Lemma 1, $F_0(s)$ possesses thus an analytic continuation into a certain simply connected domain part of which is to the left of the line $\operatorname{Re}\, s = 1$. By the truncated Perron's formula (see e.g. Prachar [**15**], p. 376 f, in particular formula (3.5)), we obtain for large x, $1 \leq T \leq x$ and $\omega = 1 + \frac{1}{\log x}$,

$$B(x) = \frac{1}{2\pi i}\int_{\omega - iT}^{\omega + iT} F_0(s) x^s \frac{ds}{s} + O(\frac{x}{T}\log x) .$$

Now let $\mathcal{C}_1$ denote the path from $\lambda(T) - iT$ to $1 - b_0$ along $\sigma = \lambda(t)$ (b_0 and $\lambda(.)$ as defined in Lemma 1), and let $\mathcal{C}_2$ lead from $1 - b_0$ to $\lambda(T) + iT$, again along $\sigma = \lambda(t)$. By Lemma 1, it is clear that

$$\int_{\lambda(T)\pm iT}^{\omega\pm iT} F_0(s) x^s \frac{ds}{s} \ll \frac{x}{T} (\log T)^{b_4} ,$$

and, for $j = 1, 2$, and T sufficiently large,

$$\int_{\mathcal{C}_j} F_0(s) x^s \frac{ds}{s} \ll x^{\lambda(T)} (\log T)^{1+b_4} .$$

For positive constants $c_1, c_2, \ldots$, we define for short

$$\delta_j(x) \stackrel{\text{def}}{=} \exp(-c_j (\log x)^{3/5} (\log\log x)^{-1/5})$$

(to be used throughout the sequel), and choose $T = (\delta_1(x))^{-1}$ (with suitable c_1). We thus obtain

$$B(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_0} F_0(s) x^s \frac{ds}{s} + O(x\delta_2(x)) ,$$

which together with (3.2) gives the assertion of Lemma 3. $\square$

Lemma 4. *Let $d^*(a, q; n)$ denote the number of (positive) divisors of $n \in \mathbb{N}$ which lie in the arithmetic progression $\mathcal{A}(a, q)$ and are greater than q. For a large real variable x,*

$$\sum_{n\le x} d^*(a, q; n) = \frac{x}{q} \log \frac{x}{q} + \gamma^*(\frac{a}{q}) \frac{x}{q} + O((\frac{x}{q})^{1/3})$$

uniformly in $1 \le a \le q \le x$, where $\gamma^*(\cdot)$ is continuous on the compact unit interval.

Proof. This follows by a short and simple computation (using the Euler summation formula) from the author's result in [**13**]. $\square$

4. Proof of Theorem 1

In order to obtain an estimate uniform in a and q, it is important to isolate the contribution of the possibly "small" divisor a to $\sum_{n\le x} \tau_{\mathbf{B},\mathcal{A}}(n)$. We put $\mathcal{A}^* = \mathcal{A} \setminus \{a\}$ and $\tau^*(n) = \tau_{\mathbf{B},\mathcal{A}^*}(n)$, then it is clear that

$$\sum_{n\le x} \tau_{\mathbf{B},\mathcal{A}}(n) = B(\frac{x}{a}) + T^*(x) , \qquad T^*(x) \stackrel{\text{def}}{=} \sum_{n\le x} \tau^*(n) . \tag{4.1}$$

For Re $s > 1$, it is clear that

$$\sum_{n=1}^{\infty} \tau^*(n)\, n^{-s} = F_0(s) \zeta^*(s, \frac{a}{q}) q^{-s} ,$$

where $F_0(s)$ has been defined in (3.2) and $\zeta^*(s,\xi) = \zeta(s,\xi) - \xi^{-s}$, $\zeta(s,\xi)$ the Hurwitz zeta function for $0 < \xi \le 1$. For later reference we note that $\zeta^*(s,\xi)$ can be represented by

$$\zeta^*(s,\xi) = (1+\xi)^{-s} + \frac{(1+\xi)^{1-s}}{s-1} - s\int_1^\infty \{u\}(u+\xi)^{-s-1}\,du\,, \tag{4.2}$$

in the halfplane Re $s > 0$ with the exception of $s = 1$. (Here $\{\cdot\}$ denotes the fractional part. Cf. Apostol [**1,** p. 269].) By a version of Perron's formula,

$$T_1^*(x) \overset{\text{def}}{=} \int_0^x T^*(qu)\,du = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F_0(s)\zeta^*(s,\frac{a}{q})\,x^{s+1}\,\frac{ds}{s(s+1)}\,. \tag{4.3}$$

Now let $\mathcal{C}_1^*$ denote the path from $1-i\infty$ to $1-b_0$, $\mathcal{C}_2^*$ the path from $1-b_0$ to $1+i\infty$, both along $\sigma = \lambda(t)$, and put $\mathcal{C} = \mathcal{C}_1^* \cup \mathcal{C}_0 \cup \mathcal{C}_2^*$. ($b_0$, $\lambda(t)$ and $\mathcal{C}_0$ are defined as in section 3.) We observe that, for $1-b_0 \le \operatorname{Re} s \le 2$,

$$F_0(s)\zeta^*(s,\frac{a}{q}) \ll (1+|\operatorname{Im} s|)^{2b_0}\,, \tag{4.4}$$

uniformly in a and q. (This is an immediate consequence of Lemma 1 and (3.2), as far as the factor $F_0(s)$ is concerned. For $\zeta^*(s,\frac{a}{q})$, the necessary bound can be found in Apostol [**1,** p. 270].) Consequently,

$$T_1^*(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} F_0(s)\zeta^*(s,\frac{a}{q})\,x^{s+1}\,\frac{ds}{s(s+1)}\,.$$

Furthermore, defining $T = (\delta_3(x))^{-1}$ and appealing to (4.4) again, we see that (for $j = 1,2$)

$$\begin{aligned}\int_{\mathcal{C}_j^*} F_0(s)\zeta^*(s,\frac{a}{q})\,x^{s+1}\,\frac{ds}{s(s+1)} &= \int_{|\operatorname{Im} s|\ge T} + \int_{|\operatorname{Im} s|\le T}\\ &\ll x^2\,T^{2b_0-1} + x^{1+\lambda(T)} \ll x^2\delta_4(x)\,,\end{aligned}$$

hence

$$T_1^*(x) = I_*(x) + O(x^2\delta_4(x))\,, \tag{4.5}$$

where

$$I_*(x) \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{\mathcal{C}_0} F_0(s)\zeta^*(s,\frac{a}{q})\,x^{s+1}\,\frac{ds}{s(s+1)} \tag{4.6}$$

Employing a technique due to Rieger [**16**], we now put, for $u \ge 1$,

$$f(u) = T^*(qu) - I_*'(u)\,,$$

then (4.5) implies that

$$\int_1^x f(u)du \ll x^2\delta_4(x)\,. \tag{4.7}$$

(Note that $T^*(qu) = 0$ for $u < 1$.) In order to estimate the difference $f(x) - f(y)$, for $1 \le y \le x$, we first observe that

(4.8)
$$I'_*(x) - I'_*(y) = \int_y^x I''_*(u)\,du = \int_y^x \Big(\frac{1}{2\pi i}\int_{\mathcal{C}_0} F_0(s)\zeta^*(s,\frac{a}{q})u^{s-1}\,ds\Big)\,du$$
$$\ll (x-y)(\log x)^{1+1/r}\,.$$

This follows by replacing $\mathcal{C}_0$ by $\mathcal{C}_0^*(u)$ which we define as the boundary of

$$\left\{s \in \mathbb{C}:\ |s-1| \le b_0\,,\ \operatorname{Re} s \le 1 + \frac{1}{\log(2u)}\right\}\,,$$

(with positive orientation, starting and ending at $1 - b_0$), in view of the bound

$$F_0(s)\zeta^*(s,\frac{a}{q}) \ll |s-1|^{-1-1/r}$$

as $s \to 1$, uniformly in a and q. This in turn is an immediate consequence of (3.2) and (4.2). Furthermore, we readily derive from Lemma 4 that

$$0 \le T^*(qx) - T^*(qy) \le \sum_{qy<n\le qx} d^*(a,q;n) \ll (x-y)\log x + x^{1/3}\,, \tag{4.9}$$

uniformly in a and q. Now (4.7)–(4.9) are just the requirements of Hilfssatz 2 in Rieger [**16**]. Applying the latter, we obtain

$$f(u) \ll u\delta_5(u)\,,$$

or

$$T^*(x) = I'_*(\frac{x}{q}) + O(\frac{x}{q}\delta_5(\frac{x}{q}))\,,$$

with

$$I'_*(u) = \frac{1}{2\pi i}\int_{\mathcal{C}_0} F_0(s)\zeta^*(s,\frac{a}{q})\,u^s\,\frac{ds}{s}\,.$$

We insert this into (4.1), evaluate $B(\frac{x}{a})$ by Lemmas 2 and 3, and $I'_*(\frac{x}{q})$ on the basis of (3.2) and Lemma 2. This completes the proof of Theorem 1. (The uniformity in a and q of the bound (2.2) for the coefficients $A_k^{(2)}$ follows from (3.1) and (4.2) which in turn shows that $(s-1)\zeta^*(s,\xi)$ is uniformly bounded in a neighbourhood of $s = 1$.) $\square$

5. Proof of Theorem 2

For K a quadratic field, we denote by $\mathbb{P}_1$ the set of all rational primes which do not divide the discriminant D and split into two prime ideals, and by $\mathbb{P}_2$ the set of all other rational primes not dividing D. Then it is well-known that (for $\operatorname{Re} s > 1$)

$$\zeta_K(s) = \prod_{p|D}(1-\frac{1}{p^s})^{-1}\prod_{p\in\mathbb{P}_1}(1-\frac{1}{p^s})^{-2}\prod_{p\in\mathbb{P}_2}(1-\frac{1}{p^{2s}})^{-1}$$

and

$$F_0(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty}\mathbf{i}_{\mathbf{B}}(n)\,n^{-s} = \prod_{p|D}(1-\frac{1}{p^s})^{-1}\prod_{p\in\mathbb{P}_1}(1-\frac{1}{p^s})^{-1}\prod_{p\in\mathbb{P}_2}(1-\frac{1}{p^{2s}})^{-1}.$$

Consequently,

$$(F_0(s))^2 = \sum_{n=1}^{\infty}\tau_{\mathbf{B},\mathbf{B}}(n)\,n^{-s} = \zeta_K(s)\varphi(s)\,, \tag{5.1}$$

where

$$\varphi(s) = \prod_{p|D}(1-\frac{1}{p^s})^{-1}\prod_{p\in\mathbb{P}_2}(1-\frac{1}{p^{2s}})^{-1} \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} g(n)\,n^{-s}\,, \tag{5.2}$$

for $\operatorname{Re} s > \frac{1}{2}$, the last series converging absolutely in this halfplane. Furthermore,

$$\varphi(s) = \zeta(2s)(\zeta_K(2s))^{-1/2}H_2(s) \tag{5.3}$$

where $H_2(s)$ has a Dirichlet series which converges absolutely for $\operatorname{Re}\, s > \frac{1}{4}$.

We note a few important properties of the coefficients $g(n)$ for later use:

(i) $g(n)$ is either 0 or 1 for every $n \in \mathbb{N}$.

(ii) If $\mathcal{P}(D)$ denotes the product of all primes that divide the discriminant D, $g(n) = 1$ implies that n can be written $n = m_1 m_2$ where m_1 divides $\mathcal{P}(D)$ and m_2 is *square-full**.

(iii) If u is a large real variable and $Q(u)$ denotes the number of square-full positive integers $\le u$, we have

$$G(u) \stackrel{\text{def}}{=} \sum_{n\le u} g(n) \le \sum_{m|\mathcal{P}(D)} Q(\frac{u}{m}) \ll u^{1/2}\,.$$

(iv) For $1 \le y < x$,

$$|G(x^2)-G(y^2)| \le \sum_{m|\mathcal{P}(D)}\big(Q(\frac{x^2}{m})-Q(\frac{y^2}{m})\big) \ll x-y\ +\ x^{1/3}\,.$$

*A positive integer m is called *square-full* if p^2 divides m for every prime divisor p of m.

The assertions (i) and (ii) are clear by (5.2), while (iii) and (iv) follow readily from (i), (ii), and the known asymptotic formula for $Q(u)$ (see Krätzel [**7,** p. 280]).

Let $a(n)$ denote the number of integer ideals in $\mathcal{O}_K$ with norm equal to n. It is well-known** that

$$A(u) \stackrel{\text{def}}{=} \sum_{n\le u} a(n) = \rho u + P(u)\,, \qquad P(u) = O(u^{1/3})\,, \tag{5.4}$$

where ρ is the residue of the Dedekind zeta-function $\zeta_K(s)$ at $s=1$. (See Landau [**9,** p. 135].) By (5.1) and (5.2), it is clear that

$$\tau_{\mathbf{B},\mathbf{B}}(n) = \sum_{lm=n} a(l)g(m)\,. \tag{5.5}$$

The main difficulty in the proof of Theorem 2 is provided by the fact that (if one wants to get a sufficiently "good" error term) contour integration apparently cannot be applied to $\sum \tau_{\mathbf{B},\mathbf{B}}(n)$ itself, but only to $\sum g(m)$: We will combine this technique with an elementary convolution argument based on (5.5).

Lemma 5. *For* $u\to\infty$,

$$G(u) \stackrel{\text{def}}{=} \sum_{m\le u} g(m) = I(u) + R(u)$$

where

$$I(u) = \frac{1}{2\pi i}\int_{\mathcal{C}_0} \varphi(\frac{s}{2}) u^{s/2}\frac{ds}{s}\,,$$

and

$$R(u) \ll u^{\frac12}\delta_6(u)$$

for some $c_6>0$.

Proof. We use the same technique as in the proof of Theorem 1. Again by Perron's formula, it follows that

$$G_1(u) \stackrel{\text{def}}{=} \int_1^u G(w^2)dw = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \varphi(\frac{s}{2})\,\frac{u^{s+1}}{s(s+1)}\,ds\,.$$

Repeating our argument between (4.3) and (4.6) almost word by word, we obtain

$$G_1(u) = I_1(u) + O(u^2\delta_7(u))\,, \tag{5.6}$$

In fact, the exponent $\frac13$ in the order term can be replaced at least by $\frac{23}{73}+\varepsilon$: See Huxley [5**]. But this is unimportant in our context.

where

$$I_1(u) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{\mathcal{C}_0} \varphi(\frac{s}{2}) \frac{u^{s+1}}{s(s+1)}\, ds\,. \tag{5.7}$$

We put, for $w \geq 1$,

$$f(w) = G(w^2) - I(w^2)\,,$$

then (5.6) implies that

$$\int_1^u f(w)dw \ll u^2 \delta_7(u)\,. \tag{5.8}$$

In order to estimate the difference $f(w_1) - f(w_2)$ for $w_1 > w_2 \geq 1$, we observe that

$$I(w_1^2) - I(w_2^2) = \int_{w_2}^{w_1} \big(\frac{1}{2\pi i} \int_{\mathcal{C}_0} \varphi(\frac{s}{2}) u^{s-1}\, ds\big)\, du \ll (w_1 - w_2)(\log(2w_1))^{1/2}\,. \tag{5.9}$$

This follows on replacing $\mathcal{C}_0$ by $\mathcal{C}_0^*(u)$ (which was defined under (4.8)), since

$$\varphi(\frac{s}{2}) \ll |s-1|^{-1/2} \qquad (s \to 1)\,, \tag{5.10}$$

which in turn is clear by (5.3). Combining (5.8), (5.9) and (iv) above, we again are ready to apply Rieger's Hilfssatz 2 from [**16**]. The latter implies that

$$G(w^2) = I(w^2) + O(w\delta_8(w))\,.$$

Putting $u = w^2$, we complete the proof of Lemma 5. □

We now define

$$y = y(x) = x\delta_9(x)\,, \qquad z = z(x) = \frac{x}{y} = (\delta_9(x))^{-1}\,, \tag{5.11}$$

with a positive constant c_9 remaining at our disposition. From (5.5) we derive by a usual device ("hyperbola method") that

$$\sum_{n \leq x} \tau_{\mathbf{B},\mathbf{B}}(n) = \sum_{m \leq y} g(m) A(\frac{x}{m}) + \sum_{l \leq z} a(l) G(\frac{x}{l}) \; - \; G(y)A(z)\,.$$

By (5.4), Lemma 5, and (ii), this may be simplified to

$$\sum_{n \leq x} \tau_{\mathbf{B},\mathbf{B}}(n) = \sum_{m \leq y} g(m) \big(\rho\, \frac{x}{m} + O((\frac{x}{m})^{1/3})\big) + \sum_{l \leq z} a(l) \big(I(\frac{x}{l}) + R(\frac{x}{l})\big)$$

$$- \rho z I(y) + O(y^{1/2}\, z^{1/3}) + O(y^{1/2}\, \delta_6(y)\, z)$$

Observing that, by (ii) and summation by parts,

$$\sum_{m\le y} g(m)(\frac{x}{m})^{1/3} \ll x^{1/3}\,y^{1/6} \ll x^{1/2}\delta_{10}(x)\,,$$

and

$$\sum_{l\le z} a(l)R(\frac{x}{l}) \ll x^{1/2}\delta_6(y)\sum_{l\le z} a(l)l^{-1/2} \ll x^{1/2}\delta_6(y)z^{1/2} \ll x^{1/2}\delta_{10}(x)$$

(cf. Landau [**9,** p. 128], for the next-to-last $\ll$-step), we arrive at

$$\sum_{n\le x} \tau_{\mathbf{B},\mathbf{B}}(n) = \rho x\sum_{m\le y}\frac{g(m)}{m} + \sum_{l\le z} a(l)I(\frac{x}{l}) - \rho z\, I(y) + O(x^{1/2}\delta_{10}(x))\,, \tag{5.12}$$

after an appropriate choice of c_9 and c_{10}. Appealing again to Lemma 5, we see that

$$\begin{aligned}\sum_{m>y}\frac{g(m)}{m} &= \int_{y+}^{\infty}\frac{1}{u}\,dG(u)\\ &= \int_{y}^{\infty}\frac{1}{u}\,I'(u)\,du + \int_{y+}^{\infty}\frac{1}{u}\,dR(u)\\ &= \int_{y}^{\infty}\frac{1}{u}\,I'(u)\,du \;-\; \frac{1}{y}\,R(y) \;+\; \int_{y}^{\infty}\frac{1}{u^2}\,R(u)\,du\\ &= \int_{y}^{\infty}\frac{1}{u}\,I'(u)\,du \;+\; O(y^{-\frac{1}{2}}\delta_6(y))\,.\end{aligned} \tag{5.13}$$

Furthermore,

$$\begin{aligned}\sum_{l\le z} a(l)I(\frac{x}{l}) &= \int_{\frac{1}{2}}^{z} I(\frac{x}{u})\,dA(u)\\ &= A(z)I(\frac{x}{z}) + \int_{1}^{z} A(u)\,I'(\frac{x}{u})\frac{x}{u^2}\,du\\ &= \rho z I(y) + O(y^{1/2}\,z^{1/3}) + \rho x\int_{y}^{x} I'(w)\frac{dw}{w} + x\int_{1}^{z} P(u)\,I'(\frac{x}{u})\frac{x}{u^2}\,du\,,\end{aligned}$$

by the substitution $w = \dfrac{x}{u}$ in the last but one integral. Inserting this together with (5.13) into (5.12), we obtain

$$\begin{aligned}\sum_{n\le x} \tau_{\mathbf{B},\mathbf{B}}(n) &= A^*x - \rho x\int_{x}^{\infty} I'(w)\frac{dw}{w}\\ &\quad + x\int_{1}^{z} P(u)\,I'(\frac{x}{u})\,\frac{du}{u^2} + O(x^{1/2}\delta_{10}(x))\,.\end{aligned} \tag{5.14}$$

Here

$$A^* = \rho \sum_{m=1}^{\infty} \frac{g(m)}{m} = \rho\, \varphi(1)$$

and can thus be represented as stated in Theorem 2 (cf. (5.2)), while

$$I'(w) = \frac{1}{2\pi i} \int_{\frac{1}{2}\mathcal{C}_0} \varphi(s)\, w^{s-1}\, ds\,. \tag{5.15}$$

To evaluate the two remaining integrals, we define

$$S(w,s) \stackrel{\text{def}}{=} \int_w^\infty P(u)\, u^{-s-1}\, du$$

and

$$U(x,w) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\frac{1}{2}\mathcal{C}_0} \varphi(s) S(w,s) x^s\, ds\,,$$

for positive reals w and x and complex s with $\operatorname{Re} s > \frac{1}{3}$. Interchanging the order of integration, we see from (5.15) that

$$U(x,w) = x \int_w^\infty P(u) I'(\frac{x}{u})\, \frac{du}{u^2}\,.$$

Consequently, we obtain for the last integral in (5.14)

$$\begin{aligned} x \int_1^z P(u)\, I'(\frac{x}{u})\, \frac{du}{u^2} &= U(x,1) - U(x,z) \\ &= \frac{1}{2\pi i} \int_{\frac{1}{2}\mathcal{C}_0} \varphi(s) S(1,s) x^s\, ds \;-\; U(x,z)\,. \end{aligned}$$

Similarly,

$$\begin{aligned} x \int_x^\infty I'(u) \frac{du}{u} &= x \int_x^\infty \Big(\frac{1}{2\pi i} \int_{\frac{1}{2}\mathcal{C}_0} \varphi(s) u^{s-1}\, ds\Big) \frac{du}{u} \\ &= x \frac{1}{2\pi i} \int_{\frac{1}{2}\mathcal{C}_0} \varphi(s) \Big(\int_x^\infty u^{s-2} du\Big)\, ds = -\frac{1}{2\pi i} \int_{\frac{1}{2}\mathcal{C}_0} \frac{1}{s-1}\, \varphi(s) x^s\, ds\,. \end{aligned}$$

In view of the identity

$$\frac{\rho}{s-1} + S(1,s) = \frac{1}{s}\, \zeta_K(s)$$

(which is immediate for $\operatorname{Re} s > 1$ via integration by parts, and thus true (at least) for $\operatorname{Re} s > \frac{1}{3}$, $s \neq 1$, by analytic continuation), we may thus simplify (5.14) to

$$\sum_{n \le x} \tau_{\mathbf{B},\mathbf{B}}(n) = A^* x + \frac{1}{2\pi i} \int_{\frac{1}{2}\mathcal{C}_0} \varphi(s) \zeta_K(s) x^s\, \frac{ds}{s} - U(x,z) + O(x^{1/2} \delta_{10}(x))\,. \tag{5.16}$$

The penultimate step is to estimate $U(x,z)$. It is clear from the definition that

$$S(w,\sigma+it) \ll w^{\frac{1}{3}-\sigma} \qquad (\sigma > \frac{1}{3}),$$

hence

$$U(x,z) = \frac{1}{2\pi i}\int_{\frac{1}{2}\mathcal{C}_0^*(x)} \varphi(s)S(z,s)\,x^s\,ds \ll x^{1/2}\,(\log x)^{1/2} z^{\frac{1}{3}-\frac{1}{2}(1-b_0)} \ll x^{\frac{1}{2}}\delta_{10}(x)\,,$$

again by (5.10), with $\mathcal{C}_0^*(x)$ defined under (4.8). Recalling (5.1), (5.3), and making the substitution $2s \to s$, we see that the integral remaining in (5.16) is equal to

$$\frac{1}{2\pi i}\int_{\mathcal{C}_0} \zeta(s)(\zeta_K(s))^{-1/2}\zeta_K(\frac{s}{2})H_2(\frac{s}{2})(x^{1/2})^s\frac{ds}{s}$$

and can thus be evaluated by Lemma 2. This completes the proof of Theorem 2.□

Concluding remark. It might be worthwhile to provide numerical values for the leading coefficients A^*, A_0, at least in the (perhaps most important) case $K = \mathbb{Q}(i)$. It is an immediate consequence of the representation given in Theorem 2 and of the decomposition laws in $\mathcal{O}_{\mathbf{Q}(i)}$ that

$$A^* = \frac{\pi}{2}\prod_{p\equiv 3 \pmod 4}(1-\frac{1}{p^2})^{-1} \sim 1{,}835\,,$$

since the residue of $\zeta_{\mathbf{Q}(i)}(s)$ at $s=1$ is $\frac{\pi}{4}$. Moreover, it follows from Lemma 2, (5.2) and (5.3), that in general

$$A_0 = (\frac{2\rho}{\pi})^{1/2}\zeta_K(\frac{1}{2})\prod_{p|D}\left((1-\frac{1}{p^{1/2}})^{-1}(1-\frac{1}{p})^{1/2}\right)\prod_{p\in\mathbb{P}_2}(1-\frac{1}{p^2})^{-1/2}\,.$$

For $K = \mathbb{Q}(i)$, this gives

$$A_0 = \frac{1}{2}(1-\frac{1}{\sqrt{2}})^{-1}\zeta(\frac{1}{2})(\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^{1/2}})\prod_{p\equiv 3 \pmod 4}(1-\frac{1}{p^2})^{-1/2} \sim -1{,}799\,.$$

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