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# ON NON–SEPARATING EMBEDDINGS OF GRAPHS IN CLOSED SURFACES

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Abstract. A. A. Zykov [Fundamentals of Graph Theory, Nauka, Moscow, 1987] asks to determine, for a given closed surface $S$, all graphs $G$ (including disconnected ones) that admit an embedding $i\colon G \hookrightarrow S$ in a closed surface $S$ leaving $S - i(G)$ connected. We anwser this question completely. For connected graphs the results can be formulated as follows: $G$ has an embedding $i\colon G \hookrightarrow S$ with $S - i(G)$ connected if and only if $S$ is non-orientable and $\tilde{\gamma}(S) \geq \beta(G) = |E(G)| - |V(G)| + 1$, or $S$ is orientable and $\gamma(S) \geq \beta(G) - \gamma_M(G)$, where $\gamma_M(G)$ is the maximum genus of $G$.

An embedding $i : G \hookrightarrow S$ of a graph $G$ in a closed surface $S$ is said to be **non-separating** if the subset $S - i(G)$ of $S$ is connected. In his books [**10,** pp. 445–447] and [**11,** pp. 229–230] Zykov posed the problem of determining, for a given closed surface $S$, all graphs that admit a non separating embedding in $S$. He also observed that if $S$ is non-orientable then such an embedding exists for every graph $G$ whose Betti (= cyclomatic) number $\beta(G)$ does not exceed the non-orientable genus $\tilde{\gamma}(S)$ of $S$. For orientable surfaces the problem has remained open although some further work in this direction was previously done by Khomenko and Yavorskii [**3**].

In this paper we show how this problem can be completely solved in both orientable and non-orientable case. Our solution requires only a few facts about the maximum genus of a graph and the standard surface topology. For terms not defined here we refer the reader to [**7**].

Let $G$ be a connected graph with $p$ vertices and $q$ edges. The (orientable) **maximum genus** $\gamma_M(G)$ of $G$ is the largest among the genera $\gamma(S)$ of orientable surfaces $S$ in which $G$ has a cellular embedding. If $G$ is cellularly embedded with $r$ faces in an orientable surface $S$ of genus $\gamma(S)$ then the Euler formula [**7**] claims that

$$p - q + r = 2 - 2\gamma(S).$$

It follows from this formula that $2\gamma_M(G)$ is bounded from above by $\beta(G) = q - p + 1$. Thus it is natural to consider the difference

$$\xi(G) = \beta(G) - 2\gamma_M(G)$$

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which is called the **Betti deficiency** of $G$. Note that $\xi(G)+1$ is in fact the minimum number of faces over all orientable cellular embeddings of $G$, and that $\xi(G)$ has the same parity as $\beta(G)$.

It is widely known that the Betti deficiency can be effectively characterized in purely combinatorial terms [**2, 4, 8**] and can be computed in polynomial time [**1, 2**]. In particular, let us recall that $\xi(G)$ is equal to the minimum number of components with odd number of edges taken over all cotrees of $G$.

For our purposes it is convenient to extend the definition of the Betti deficiency to disconnected graphs. If $G$ has $k$ components $G_1, G_2, \ldots, G_k$, then we set

$$\xi(G) = \sum \xi(G_i)$$

Now we can state our results.

**Theorem 1.** *A graph $G$ has a non-separating embedding in an orientable surface $S$ if and only if $\gamma(S) \geq (\beta(G)+\xi(G))/2$.*

**Theorem 2.** *A graph $G$ has a non-separating embedding in a non-orientable surface $S$ if and only if $\tilde{\gamma}(S) \geq \beta(G)$.*

**Proof of Theorem 1.** Let $S$ be an orientable surface of genus $g$ and let $G$ be a graph with $k$ components $G_1, G_2, \ldots, G_k$ such that $g \geq (\beta(G)+\xi(G))/2$. We show that $G$ has a non-separating embedding in $S$.

A non-separating embedding of $G$ can be constructed as follows. Take any orientable surface $R$ and for every component $G_i$ of $G$ take a cellular embedding $j_i\colon G_i \to S_i$ of $G_i$ in some orientable surface $S_i$. Let $F_i$ be a closed collaring of $j_i(G_i)$ in $S_i$, i.e., the closure of a "small" open neighbourhood of $j_i(G_i)$ of which $j_i(G_i)$ is a deformation retract. If the embedding $j_i$ has $r_i$ faces then $F_i$ is a bordered surface with $r_i$ boundary components containing $j_i(G_i)$ in its interior. For each $F_i$ and for each boundary component $C$ of $F_i$ remove an open disc $D_C$ from $R$ and identify homeomorphically $C$ with the boundary of $D_C$ in $R$. The identifications should be made in such a way that the resulting surface $T$ will be orientable. Note that we thus obtain a non-separating embedding $j$ of $G$ in $T$; we shall refer to $j$ as the **join** of $j_1, j_2, \ldots, j_k$ by $R$.

Elementary computations show that if $r = \sum r_i$ is the total number of faces in the above cellular embeddings $j_i$, $i = 1, 2, \ldots, k$, then

$$\gamma(T) = \gamma(R) + \sum \gamma(S_i) + r - k\,. \tag{1}$$

In particular, choosing $S_i$ to have genus $\gamma(S_i) = \gamma_M(G_i) = (\beta(G_i) - \xi(G_i))/2$, $R$ to have genus $\gamma(R) = g - (\beta(G) + \xi(G))/2$ (which by our assumption is non-negative) and using the fact that $r_i = \xi(G_i)+1$ we obtain that $\gamma(T) = g$. Thus $T$ is homeomorphic to $S$ and the required non-separating embedding exists.

Conversely, assume that $G$ is a graph having a non-separating embedding $j\colon G \hookrightarrow S$ in an orientable surface $S$. We show that $(\beta(G)+\xi(G))/2 \le \gamma(S)$. Take a closed collaring $F$ of $j(G)$ in $S$. If $G$ has $k$ components $G_1, G_2, \dots, G_k$ then $F$ is the disjoint union of $k$ bordered surfaces, each containing a component of $j(G)$ in its interior. Let $F_i$ be the component of $F$ containing $j(G_i)$. Then by capping each boundary component of $F_i$ with a 2-cell we obtain a closed surface $S_i$ and a cellular embedding $j_i\colon G_i \hookrightarrow S_i$, $i = 1, 2, \dots, k$. (This is the well-known "capping operation" of Youngs [**9**].)

Since $S - j(G)$ is connected, so is $S - Int(F) = H$. Thus $H$ is a bordered surface. Obviously, each boundary component of $H$ is a boundary component of some $F_i$ and vice versa. It follows that the number of boundary components of $H$ is equal to the total number of faces in the embeddings $j_i\colon G_i \hookrightarrow S_i$, which we denote by $r$. By capping each boundary component of $H$ with a 2-cell we obtain a closed surface $R$, and it is now clear that $j$ is the join of $j_1, j_2, \dots, j_k$ by $R$. Hence, employing (1) and the Euler formula for each $j_i$ we finally have

$$\begin{aligned}\gamma(S) &\ge \gamma(R) + \sum \gamma(S_i) + r - k \ge 0 + (\beta(G) - r + k)/2 + r - k \\ &= (\beta(G) + r - k)/2 \ge (\beta(G) + \xi(G))/2\,.\end{aligned}$$

This completes the proof. □

Theorem 2 can be proved basically in the same way, the main difference being that every connected graph has a cellular embedding in a non-orientable surface with a single face [**5, 6**], i.e., the non-orientable analogue of Betti deficiency is constantly 0.

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