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# NORM–TO–WEAK UPPER SEMICONTINUOUS MONOTONE OPERATORS ARE GENERICALLY STRONGLY CONTINUOUS

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Abstract. In any Banach space a monotone operator with a norm-to-weak upper semicontinuous multivalued selection on an open set $D$ is singlevalued and norm-to-norm upper semicontinuous at the points of a dense $G_\delta$ subset of $D$.

Monotone operators — and especially a special case of them, subdifferentials of convex functions — play an important role in various parts of nonlinear analysis. One of the often investigated problems is the question about generic continuity of monotone operators, which in the case of subdifferentials means generic Fréchet differentiability of convex functions.

In general Banach spaces monotone operators are not always generically continuous. There are numerous characterizations of Asplund spaces, i.e. the spaces in which any monotone operators is generically continuous on the interior of its effective domain (see e.g. [**Ph**]). The aim of this note is to prove that monotone operator of a certain class are generically continuous in an arbitrary Banach space (Theorem 2).

Our result can be deduced from a selection result by Ch. Stegall [**St**], based on hard topological techniques (Remark 2). We present a relatively simple alternative proof, self-contained in the sense that it uses monotone operator techniques only.

Three main tools of this note are following: a slight modification of a result by D. Preiss and L. Zajíček [**Pr–Zaj**] (Theorem P–Z), the observation that the image of a separable set by a norm-to-weak continuous mapping is separable, and the well known method of separable reduction which extends the result from the separable into the nonseparable case.

## Definitions and Notations

Let us recall some definitions and notations. $X$ always denotes a real Banach space, $X^*$ its continuous dual. Closed and open balls with centre $c$ (in $X$ or $X^*$) and radius $r > 0$ are denoted by $\overline{B}(c; r)$ and $B(c; r)$, respectively.

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A setvalued mapping $T\colon X \to 2^{X^*}$ is a **monotone operator** if $\langle x-y, x^*-y^*\rangle \geq 0$ whenever $x,\ y \in X,\ x^* \in T(X),\ y^* \in T(Y)$. The **effective domain** of $T$ is the set $D(T) = \{x \in X;\ T(x) \neq \emptyset\}$. The symbol $S \subset T$ will mean that $S$ is a setvalued mapping from $X$ into $X^*$ with the property $S(x) \subset T(x)$ for all $x \in X$. If $T(x)$ is a singleton, we often identify it with the point in $X^*$; so, for example, $B(T(x);r)$ means $B(x^*;r)$ where $x^* \in X^*$ is such that $T(x) = \{x^*\}$. We put $T(G) = \cup\{T(x);\ x \in G\}$.

$T$ is **(n-w)u.s.c.** (norm-to-weak upper semicontinuous) at $x_0 \in D(T)$ if for any weakly open set $W$ with $T(x_0) \subset W$ there is $\delta > 0$ such that $T(x) \subset W$ whenever $x \in B(x_0;\delta)$.

We shall say that $T$ is **strongly continuous** at $x_0 \in D(T)$ if $T(x_0)$ is a singleton and $x_n^* \to T(x_0)$ whenever $x_n \to x_0$ and $x_n^* \in T(x_n)$ for all $n$ (or equivalently: $T(x_0)$ is a singleton and $T$ is u.s.c. at $x_0$ in the norm topologies on $X$ and $X^*$).

Let us remark that a monotone operator $T$ has always an extension which is norm-to-weak* u.s.c. at the points in the interior of $D(T)$ (cf. results on maximal monotone operators in [**Ph**]).

## Preliminary Results

**Theorem Z** (E. H. Zarantonello [**Zar**]). *Let $D$ be a nonempty open set in a separable Banach space $X$. Then any monotone operator $T\colon X \to 2^{X^*}$ with $D(T) \supset D$ is generically singlevalued in $D$.*

**Theorem P–Z.** *Let $T\colon X \to 2^{X^*}$ be a monotone operator with an arbitrary domain $D(T)$. Suppose there is a countable set $C \subset X^*$ such that $\mathit{dist}(T(x), C) = 0$ for any $x \in D(T)$. Then the set*

$$\{x \in D(T);\ T \textit{ is not strongly continuous at } x\}$$

*is of the first Baire category.*

*Proof.* It can be repeated word by word the proof of [**Pr–Zaj**] (see also [**Ph;** Theorem 2.11]) replacing the countable dense subset of $X^*$ by our set $C$. ☐

**Remark 1.** Much more can be said in the two theorems above. In [**Zaj**], L. Zajíček proved that for any monotone operator $T$ with an arbitrary $D(T)$ on a separable space the set $M = \{x \in D(T);\ \mathrm{card}\,(T(x)) > 1\}$ can be covered by countably many Lipschitz hypersurfaces. Even a stronger result is proved in [**Ves**]: the set $M$ is a countable union of "CFC-fragments of codimension 1". The exceptional set from Theorem P–Z is even "angle-small" (cf. [**Pr–Zaj**], [**Ph**]).

The following lemma is well known, but we have not found any reference.

**Lemma.** *Let $G$ be a nonempty open set in $X$ and let $K \subset X^*$ be a convex weak*-compact set. Let $T\colon X \to 2^{X^*}$ be a monotone operator with $D(T) \supset G$. If the set $Q = \{x \in G;\ T(x) \cap K \neq \emptyset\}$ is dense in $G$ then $T(G) \subset K$.*

*Proof.* Suppose there exist $x \in G$ and $x^* \in T(x) \setminus K$. Then there is a vector $v \in X$ with $0 < \|v\| < \operatorname{dist}(x, X \setminus G)$ and $\langle v, x^* \rangle > \sup\langle v, K\rangle$. Denote $m = \sup\{\|x^* - y^*\|;\ y^* \in K\}$. Clearly $x + v \in G$ and $m > 0$. By the density assumption there is a vector $u \in X$ such that $y := x + v + u \in Q$ and $\|u\| < \frac{1}{m}(\langle v, x^*\rangle - \sup\langle v, K\rangle)$. Take $y^* \in T(y) \cap K$. The monotonicity of $T$ gives $0 \le \langle y - x, y^* - x^*\rangle = \langle v + u, y^* - x^*\rangle$. Consequently

$$\begin{aligned} \sup\langle v, K\rangle &\ge \langle v, y^*\rangle \ge \langle v, x^*\rangle - \langle u, y^* - x^*\rangle \\ &\ge \sup\langle v, K\rangle + (\langle v, x^*\rangle - \sup\langle v, K\rangle - m\|u\|) > \sup\langle v, K\rangle \end{aligned}$$

which is a contradiction.

## Theorems

**Theorem 1 (separable case).** *Let $D$ be a nonempty open subset of a separable Banach space $X$ and let $T\colon X \to 2^{X^*}$ be a monotone operator with $D(T) \supset D$. Suppose there exists an operator $S \subset T$ with $D(S)$ residual in $D$ and such that $S$ is (n-w)u.s.c. at the points of $D(S) \cap D$. Then $T$ is generically strongly continuous in $D$.*

*Proof.* By Theorem Z, the set $D_1 = \{x \in D \cap D(S);\ \operatorname{card}(T(x)) = 1\}$ is residual in $D$. Let $A$ be a countable dense subset of $D_1$. Clearly $T|_{D_1} = S|_{D_1}$ is singlevalued and norm-to-weak continuous, hence

$$T(D_1) \subset \overline{T(A)}^{w} \subset \overline{\operatorname{span}(T(A))}^{w} = \overline{\operatorname{span}(T(A))}.$$

Consequently $T(D_1)$ is separable. Applying Theorem P–Z to the operator $T|_{D_1}$ we conclude that $T|_{D_1}$ is strongly continuous at points of a residual subset $D_0$ of $D_1$. $D_0$ is residual in $D$, too.

It suffices to show that $T$ is continuous at each point $x_0 \in D_0$. Take $\varepsilon > 0$ arbitrarily. There is $\delta > 0$ such that $T(x) \in B(T(x_0); \varepsilon)$ whenever $x \in B(x_0; \delta) \cap D_1$. Using Lemma for $K = \overline{B}(T(x_0); \varepsilon)$, $G = B(x_0, \delta)$ we get $T(u) \subset \overline{B}(T(x_0); \varepsilon)$ whenever $u \in B(x_0; \delta)$. This completes the proof. $\square$

**Theorem 2 (nonseparable case).** *Let $D$ be a nonempty open subset of an arbitrary Banach space $X$ and let $T\colon X \to 2^{X^*}$ be a monotone operator with $D(T) \supset D$. Suppose there exists $S \subset T$ with $D(S) \supset D$ and such that $S$ is (n-w)u.s.c. in $D$. Then $T$ is generically strongly continuous in $D$.*

*Proof.* Suppose that the set $H = \{x \in D;\ \text{T is not strongly continuous at } x\}$ is of the second category. Since $H = \cup_{m=1}^{\infty} H_m$, where

$$\begin{aligned} H_m = \{x \in D;\ &\text{there exist } f \in T(X),\ \{y_k\} \subset D,\ g_k \in T(y_k) \text{ such that} \\ &\lim y_k = x \quad \text{and} \quad \|g_k - f\| > \frac{1}{m} \text{ for all } k\}, \end{aligned}$$

by the Baire Category Theorem there is an index $m_0$ and a nonempty open set $G \subset D$ such that $H_{m_0}$ is dense in $G$.

We shall construct a sequence $Y_0 \subset Y_1 \subset Y_2 \subset \dots$ of separable subspaces of $X$ by induction.

Choose an arbitrary point $x_0 \in H_{m_0} \cap G$. There exist $f_0 \in T(x_0)$ and a sequence $\{(y_k, g_k)\}$ in the graph of $T$ such that $\lim y_k = x_0$ and $\|g_k - f_0\| > \frac{1}{m_0}$. For any $k$ choose $v_k \in X$ such that $\|v_k\| = 1$ and $\langle g_k - f_0, v_k\rangle > \frac{1}{m_0}$. Define $Y_0 = \operatorname{span}(\{x_0\} \cup \{y_k\}_1^\infty \cup \{v_k\}_1^\infty)$. Clearly $Y_0$ is separable.

Let $Y_0 \subset Y_1 \subset \dots \subset Y_s$ be already defined. There exists a sequence $\{c_i^s\}_{i=1}^\infty$ which is a countable dense subset of $Y_s \cap G$. (Note that $Y_s \cap G$ is nonempty since it contains $x_0$.) For any $i$ there is a sequence $\{x_{i,n}^s\}_{n=1}^\infty \subset H_{m_0} \cap G$ such that $\lim_n x_{i,n}^s = c_i^s$. By the definition of $H_{m_0}$, for any $i$, $n$ there exist $f_{i,n}^s \in T(x_{i,n}^s)$ and a sequence $\{(y_{i,n,k}^s, g_{i,n,k}^s)\}_{k=1}^\infty$ in the graph of $T$ such that

$$\lim_k y_{i,n,k}^s = x_{i,n}^s \qquad \text{and} \qquad \|g_{i,n,k}^s - f_{i,n}^s\| > \frac{1}{m_0} \text{ for all } k.$$

Choose $v_{i,n,k}^s \in X$ such that $\|v_{i,n,k}^s\| = 1$ and

$$\langle g_{i,n,k}^s - f_{i,n}^s, v_{i,n,k}^s\rangle > \frac{1}{m_0}.$$

Define

$$Y_{s+1} = \operatorname{span}(Y_s \cup \{x_{i,n}^s\}_{i,n=1}^\infty \cup \{y_{i,n,k}^s\}_{i,n,k=1}^\infty \cup \{v_{i,n,k}^s\}_{i,n,k=1}^\infty).$$

Put $Y = \overline{\bigcup_{s=1}^\infty Y_s}$. It is evident that $Y$ is a closed separable subspace of $X$ and $G_Y = G \cap Y$ is a nonempty open set in $Y$. Let $Q\colon X^* \to Y^*$ be the "restriction map" $\varphi \mapsto \varphi|_Y$ (which is equal to the quotient map of $X^*/Y^\perp$ if we identify $Y^*$ with $X^*/Y^\perp$). $Q$ is continuous and linear, hence $Q$ is also continuous for weak topologies on $X^*$ and $Y^*$. Clearly $T_Y = Q \circ T|_Y$ is a monotone operator on $Y$ with $D(T_Y) \supset G_Y$ and its "multivalued selection" $S_Y = Q \circ S|_Y$ is (n-w)u.s.c. on $G_Y$. By Theorem 1, $T_Y$ is generically strongly continuous in $G_Y$.

But we shall show that $T_Y$ is a strongly continuous at no point of $G_Y$. Fix $z \in G_Y$ and $\delta > 0$. It is easy to see that there exist positive integers $s, i, n, k$ such that $\|z - x_{i,n}^s\| < \delta$ and $\|z - y_{i,n,k}^s\| < \delta$. (In fact, we can find some $u^s \in Y_s \cap G$ near $z$, $c_i^s$ near $u^s$, $x_{i,n}^s$ near $c_i^s$, $y_{i,n,k}^s$ near $x_{i,n}^s$.) Now $Q(f_{i,n}^s) \in T_Y(x_{i,n}^s)$, $Q(g_{i,n,k}^s) \in T_Y(y_{i,n,k}^s)$ and

$$\|Q(f_{i,n}^s) - Q(g_{i,n,k}^s)\| \geq \langle Q(f_{i,n}^s) - Q(g_{i,n,k}^s), v_{i,n,k}^s\rangle = \langle f_{i,n}^s - g_{i,n,k}^s, v_{i,n,k}^s\rangle > \frac{1}{m_0}.$$

Consequently $\operatorname{diam}(T_Y(Y \cap B(z;\delta))) > \frac{1}{m_0}$ for any $\delta > 0$, and hence $T_Y$ is not strongly continuous at $z$. This is a contradiction with generic strong continuity of $T_Y$. $\square$

**Remark 2.** As already remarked in the introduction, our Theorem 2 admits an alternative proof. Results in [**St**] imply that there exists $\sigma : D \to X^*$ of the first Baire class such that $\operatorname{dist}(\sigma(x), S(x)) = 0$ for all $x \in D$. Hence $\sigma$ is a selection for any maximal monotone extension $\hat{T}$ of $T$ on $D$. The points of continuity of $\sigma$ form a dense $G_\delta$ subset $D_0$ of $D$. Reasoning as in the end of the proof of Theorem 1, it is possible to show that $\hat{T}$, and hence also $T$, is strongly continuous at each point of $D_0$.

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