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ON THE TREE–WIDTH OF A GRAPH

J. CHLEBÍKOVÁ

Abstract. Robertson and Seymour introduced the concept of the tree-width of a graph. It plays an important role in their results on graph minors culminating in their proof of Wagner's conjecture. This concept seems to be interesting from the algorithmic point of view as well: many graph problems that are NP-complete in general can be polynomially solvable if graphs are constrained to have bounded tree-width [**2**].

In the present paper several equivalent definitions of tree-width are discussed and tree-width of several families of graphs is determined.

All graphs in this paper are nonempty, finite and may have loops or multiple edges. $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of the graph G, respectively. $\omega(G)$ denotes the cardinality of the maximal clique of G. K_k stands for the complete graph on k-vertices.

For any subset X of $V(G)$ we denote by $G-X$ the graph obtained from G by deleting the vertices of X (and all the edges adjacent with them). For any edge $e \in E(G)$ let $G-e$ stand for the graph obtained from G by deleting e.

In this paper the difference between isomorphism and equality of graphs is ignored.

Definition 1. A **tree-decomposition** of a graph G is a pair $(T, \mathscr{X})$, where T is a tree and $\mathscr{X} = (X_t,\ t \in V(T))$ is a family of subsets of $V(G)$ with the following properties:

(i) $\cup(X_t,\ t \in V(T)) = V(G)$;
(ii) for every edge $e \in E(G)$ there exists $t \in V(T)$ such that e has both ends in X_t;
(iii) for t, t', $t'' \in V(T)$, if t' is on the path of T between t and t'' then

$$X_t \cap X''_t \subseteq X'_t.$$

The width of the tree-decomposition $(T, \mathscr{X})$ is

$$\max_{t\in V(T)} (|X_t| - 1).$$

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A **tree-width** of the graph G, $TW(G)$, is a minimal nonnegative integer k such that G has a tree-decomposition of the width k.

A tree-decomposition $(T, \mathscr{X})$ of the graph G is said to be **fundamental**, if $(T, \mathscr{X})$ has width $TW(G)$ and for any tree-decomposition $(T', \mathscr{X}')$ of G whose width is $TW(G)$, the inequality $|V(T')| \geq |V(T)|$ holds.

Lemma 2. *Let $(T, \mathscr{X})$ be a fundamental tree-decomposition of a graph G, where $\mathscr{X} = (X_t,\ t \in V(T))$. If vertices $t_1, t_2 \in V(T)$ are adjacent in T, then $X_{t_1} - X_{t_2} \neq \emptyset$.*

Proof. If $X_{t_1} - X_{t_2} = \emptyset$ for some vertices t_1 and t_2 adjacent in T, then define a new tree-decomposition of G in the following way: let t_0 denote the vertex obtained by contraction of the edge $\{t_1, t_2\}$ in T and let T' denote the tree obtained in this way. Set $X_{t_0} = X_{t_2}$ and $\mathscr{X}' = (X_t, t \in V(T'))$. Clearly, the tree-decomposition $(T', \mathscr{X}')$ of G has width $TW(G)$ but has less then $|V(T)|$ vertices, a contradiction. □

Definition 3. Let $(T, \mathscr{X})$ be a tree-decomposition of a graph G, where $\mathscr{X} = (X_t,\ t \in V(T))$. For each $t \in V(T)$ the connected components of $T - t$ are called the **branches** of T at t.

For any $t \in V(T)$ and any $v \in V(G)$ for which $v \notin X_t$ there is a (unique) branch of T at t containing all $t' \in V(T)$ with $v \in X_{t'}$ (due to the property (iii) from Definition 1). Let $T_t(v)$ denote this branch.

Definition 4. Let G be a graph, X, Y, $Y' \subseteq V(G)$. The sets Y and Y' are **separated** in G by the set X, if every path from Y to Y' in G contains a vertex of X.

Lemma 5. *Let $(T, \mathscr{X})$ be a tree-decomposition of a graph G, where $\mathscr{X} = (X_t,\ t \in V(T))$.*

(i) *If v, $v' \notin X_t$ and v, v' are not separated in G by X_t, then $T_t(v) = T_t(v')$.*

(ii) *Let e be an edge of T with ends t, t' and let N, N' be the vertex sets of the two components of $T - e$. Then $X_t \cap X_{t'}$ separates $\cup(X_n,\ n \in N)$ and $\cup(X_n,\ n \in N')$.*

(iii) *Let $|V(T)| \geq 2$ and for each $t \in V(T)$ let G_t be a connected subgraph of G with $V(G_t) \cap X_t = \emptyset$. Then there exist t, $t' \in V(T)$ adjacent in T such that $X_t \cap X_{t'}$ separates $V(G_t)$ and $V(G_{t'})$ in G.*

Proof. See [**3**]. □

Lemma 6. *Let $(T, \mathscr{X})$ be a tree-decomposition of a graph G, where $\mathscr{X} = (X_t,\ t \in V(T))$. For any clique $H \subseteq G$ there exists a vertex $t_0 \in V(T)$ such that $V(H) \subseteq X_{t_0}$. In particular, $TW(G) \geq \omega(G) - 1$.*

Proof. If $|V(H)| \leq 2$ then the statement easily follows from the definition of the tree-decomposition of G.

Suppose to the contrary that there exists a clique $H \subseteq G$ ($|V(H)| \geq 3$) such that for every $t \in V(T)$ the vertex set $V(H) - X_t$ is nonempty. Then obviously $|V(T)| \geq 2$. For every $t \in V(T)$ set $G_t = H - X_t$. Then

$$V(G_t) \cap X_t = \emptyset . \tag{1}$$

Due to Lemma 5 there are vertices t_1, t_2 adjacent in T such that the subsets $V(G_{t_1})$ and $V(G_{t_2})$ are separated in G by the set $X_{t_1} \cap X_{t_2}$. By virtue of (1) we get, in particular, $V(G_{t_1}) \cap V(G_{t_2}) = \emptyset$. Choose arbitrary vertices $v_1 \in V(G_{t_1})$ and $v_2 \in V(G_{t_2})$. Since v_1 and v_2 are separated in G by the set $X_{t_1} \cap X_{t_2}$ and v_1, $v_2 \notin X_{t_1} \cap X_{t_2}$, we get that v_1 and v_2 are not adjacent in G. But v_1 and v_2 are vertices of the clique $H \subseteq G$, a contradiction. $\square$

Theorem 7.
(i) $TW(G) = 0$ *if and only if* G *is a discrete graph.*
(ii) $TW(C) = 2$ *for any cycle* C.
(iii) $TW(G) \leq 1$ *if and only if* G *is a forest.*
(iv) $TW(K_n) = n - 1$ *for any positive integer* n.

Proof. Part (i) is trivial.

(ii) Let C be a cycle, $C = v_1 v_2 \dots v_n v_1$ and $n \geq 3$. To prove that $TW(C) \leq 2$, we define a tree T, where $V(T) = \{1, \dots, n-2\}$, $E(T) = \{\{i, i+1\} : 1 \leq i \leq n-3\}$ and the family $\mathscr{X} = (X_t,\ t \in V(T))$, where $X_t = \{v_1, v_{t+1}, v_{t+2}\}$ for any vertex $t \in V(T)$. Then $(T, \mathscr{X})$ is a tree-decomposition of C with width 2.

To prove that $TW(C) \geq 2$, we assume, to the contrary, that $(T, \mathscr{X})$ is a fundamental tree-decomposition of C with width at most 1. Then we can find vertices t_1, $t_0 \in V(T)$ such that $X_{t_1} = \{v_1, v_2\}$ and $X_{t_0} = \{v_1, v_n\}$. Let $t_2 \in V(T)$ be the first vertex different from t_1 on the path from t_1 to t_0 in T. By Lemma 2 $v_2 \notin X_{t_2}$ and hence $X_{t_1} \cap X_{t_2} = \{v_1\}$. Delete from T the edge e joining the vertices t_1 and t_2. For $i = 1, 2$ let N_i denote the vertex set of the component of $T - e$ which contains t_i. By Lemma 5, the sets $M_1 = \cup(X_t, t \in N_1)$ and $M_2 = \cup(X_t, t \in N_2)$ are separated in C by $\{v_1\}$. But vertices $v_2 \in M_1$ and $v_n \in M_2$ are not separated by $\{v_1\}$ in C, a contradiction.

(iii) If G is a forest it is easy to find a tree-decomposition of G with width at most 1. If G is not a forest, then $TW(G) \geq 2$ by (ii).

(iv) The inequality $TW(K_n) \geq n-1$ is a consequence of Lemma 6. The opposite inequality easily follows from the definition of a tree-decomposition of G. $\square$

Definition 8. A graph G is said to be **chordal**, if every cycle in G of length at least 4 has a chord.

A graph G has **chord-width** k, $ChW(G) = k$, if k is the smallest nonnegative integer such that G is a subgraph of some chordal graph H which contains no $(k+2)$-clique as a subgraph.

Definition 9. Let G_1 and G_2 be disjoint graphs. Choose cliques (without multiple edges and loops) from each of G_1 and G_2 of the same size k and a bijection between them. Identify each vertex of the first clique with the associated vertex of the second one.

1. If we delete the edges of both cliques, the result is said to be a k-**sum** of G_1 and G_2 (see Fig. 1).

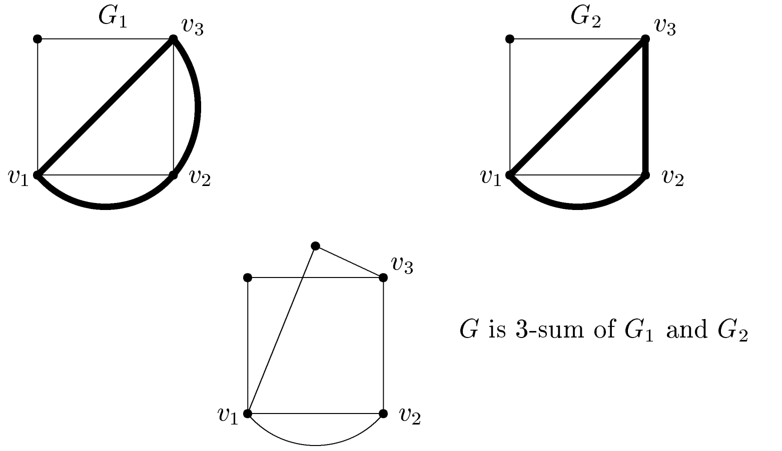

Figure 1.

2. If we delete the edges of only one of the cliques, the result is said to be a k-**linkage** of G_1 and G_2 (see Fig. 2).

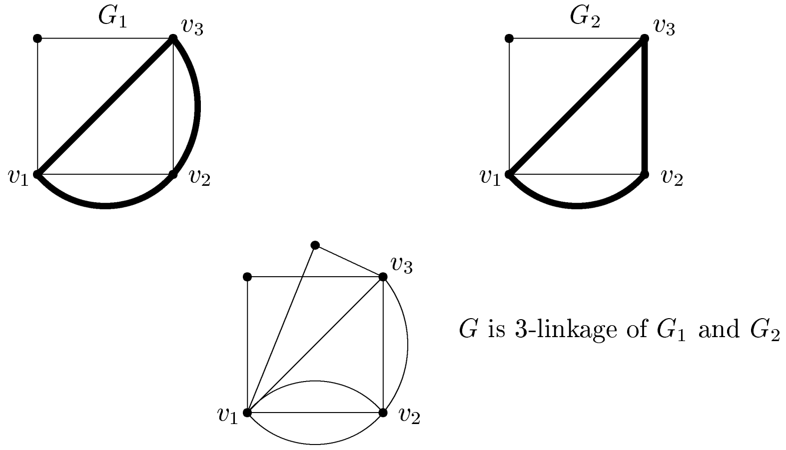

Figure 2.

Remark. 0-sum and 0-linkage correspond to the disjoint union of graphs. 1-sum has the same meaning as 1-linkage.

Notation. If $\mathscr{C}$ is a set of graphs, we denote by $\langle\mathscr{C}\rangle_k$, $\langle\langle\mathscr{C}\rangle\rangle_k$ and $\langle\langle\mathscr{C}\rangle\rangle_k^*$ the sets of graphs, which can be constructed by repeatedly applying the operation of a $\leq k$-sum, $\leq k$-linkage and a k-linkage starting from graphs belonging to $\mathscr{C}$, respectively.

Let $\mathscr{K}$ denote the set of all complete graphs and $\langle\langle\mathscr{K}\rangle\rangle = \bigcup_{k=0}^{\infty} \langle\langle K_{k+1}\rangle\rangle_k$.

Definition 10. Let $\mathscr{C}_{k+1}$ be the set of all graphs with $\leq (k+1)$ vertices. A graph G is said to have

1. **sum-width** k ($SW(G) = k$), if k is the smallest nonnegative integer such that $G \in \langle\mathscr{C}_{k+1}\rangle_k$;
2. **linkage-width** k ($LW(G) = k$) if k is the smallest nonnegative integer for which there exists a supergraph $H \supseteq G$ such that $H \in \langle\langle\mathscr{C}_{k+1}\rangle\rangle_k$;
3. **clique-width** k ($CW(G) = k$) if k is the smallest nonnegative integer for which there exists a supergraph $H \supseteq G$ such that $H \in \langle\langle K_{k+1}\rangle\rangle_k^*$.

We prove that all the above mentioned variants of the width of a graph are equivalent. The following characterization of chordal graph was first given by Dirac [**1**].

Lemma 11. *A graph G is chordal if and only if $G \in \langle\langle\mathscr{K}\rangle\rangle$.*

Proof. Necessity: If $G \in \mathscr{K}$, then G is chordal. Further, we prove that if G is obtained by using operation of linkage two chordal graphs G_1, $G_2 \in \langle\langle\mathscr{K}\rangle\rangle$ then G is chordal.

Let C be a cycle in the graph G of length at least 4. If $V(C) \subseteq V(G_i)$ for some $i = 1, 2$, then there is a chord of C in G, for G_1 and G_2 are chordal. Otherwise, $V(C) \cap (V(G_1) - V(G_2)) \neq \emptyset$ and $V(C) \cap (V(G_2) - V(G_1)) \neq \emptyset$. Then it is easy to see that there is a pair of different vertices $u, v \in V(C)$ such that $u, v \in V(G_1) \cap V(G_2)$ and $\{u, v\} \notin E(C)$. Since u and v are vertices of a clique in G, the edge $\{u, v\}$ is a chord of C in G.

Sufficiency: Assume to the contrary that there are chordal graphs which do not belong to $\langle\langle\mathscr{K}\rangle\rangle$. Let G have the minimum number of vertices among all such graphs. The minimality condition implies that G is a block. We denote by $R = \{v_1, \ldots, v_r\}$ a minimal vertex-cut of G. The assumptions made about G imply that $2 \leq r \leq |V(G)| - 2$. Let F denote the subgraph of G induced by the set R. Let $H_1, \ldots, H_s$, $s \geq 2$, be components of $G - R$. For every i, $1 \leq i \leq s$, let $\widetilde{H_i}$ denote the subgraph of G induced by $V(H_i) \cup R$. The graphs $\widetilde{H_i}$ are chordal and since $|V(\widetilde{H_i})| < |V(G)|$ we obtain that $\widetilde{H_i} \in \langle\langle\mathscr{K}\rangle\rangle$, for $1 \leq i \leq s$.

First of all, we prove that $F \in \mathscr{K}$. Assume to the contrary, that there are vertices v and $v' \in R$ which are not adjacent in G. Choose arbitrary vertices $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. Any path from v_1 to v_2 in $G - \{R - \{v\}\}$ (resp. $G - \{R - \{v'\}\}$ contains v' (resp. v). Therefore there exists a cycle C in G with the following properties:

(2) $v, v' \in C, \quad V(C) - \{v, v'\} \subseteq V(H_1) \cup V(H_2)$ and $V(C) \cap V(H_i) \neq \emptyset$ for every $i \in \{1, 2\}$.

Let C_0 denote the shortest cycle satisfying (2). Since G is chordal, we can find a chord, say $\{x, y\}$, of C_0 in G. The vertices x and y both belong either to H_1 or to H_2, because H_1 and H_2 are components of $G - R$ and $x, y \notin R$. Now, C_0 contains a path P from x to y containing vertices v and v'. Thus P together with the edge $\{x, y\}$ form a cycle satisfying (2) with less than $|V(C_0)|$ vertices, a contradiction. Consequently, $F \in \mathscr{K}$. However, G can be constructed by linking chordal graphs $\widetilde{H}_1, \ldots, \widetilde{H}_s \in \langle\langle \mathscr{K} \rangle\rangle$, hence $G \in \langle\langle \mathscr{K} \rangle\rangle$, a contradiction. $\square$

Theorem 12. *For any G, the following equalities hold:*

$$TW(G) = ChW(G) = LW(G) = CW(G) = SW(G).$$

Proof. For any nonnegative integer k is sufficient to prove the equivalence of the following five statements:

$$TW(G) \le k \Leftrightarrow ChW(G) \le k \Leftrightarrow LW(G) \le k \Leftrightarrow CW(G) \le k \Leftrightarrow SW(G) \le k.$$

$TW(G) \le k \implies ChW(G) \le k$:

Let $(T, \mathscr{X})$ be a tree-decomposition of G with width at most k, where $\mathscr{X} = (X_t,\ t \in V(T))$. We define a supergraph H of G as follows: $V(H) = V(G)$, $E(H) = \bigcup_{t \in V(T)} \{\{u, v\}; u, v \in X_t \text{ and } u \neq v\}$. Then $(T, \mathscr{X})$ is also a tree-decomposition of H. We show that H is chordal and does not contain a $(k+2)$-clique as a subgraph. Let C be a cycle of length at least 4 in H and let $(T, \mathscr{X}_C)$, where $\mathscr{X}_C = (X_t \cap V(C), t \in V(T))$, be a tree-decomposition of C. By Theorem 7, there is a vertex $t_0 \in V(T)$ such that $|V(C) \cap X_{t_0}| \ge 3$. Since $|V(C)| \ge 4$, the set $V(C) \cap X_{t_0}$ contains a pair of vertices x and y nonadjacent in C. The edge $\{x, y\} \in H$ is a chord of C in H. Suppose that the graph H contains a $(k+2)$-clique K_{k+2}. Then the restriction of a tree-decomposition $(T, \mathscr{X})$ of H to the vertices of K_{k+2} is a tree-decomposition of K_{k+2} with width k, contradicting to Theorem 7. Hence H is a chordal supergraph of G containing no $(k+2)$-clique as a subgraph.

$ChW(G) \le k \implies LW(G) \le k$:

Let H be a chordal supergraph of G containing no $(k+2)$-clique. By Lemma 11 we have $H \in \langle\langle \{K_i \colon 1 \le i \le k+1\} \rangle\rangle$. It is easy to see that the graph H fulfils the following statement:

(3) H is a 1-clique or a $(k+1)$-clique, or there are two nonadjacent vertices $v_1, v_2 \in V(H)$, $v_1 \neq v_2$, for which $d_H(v_1), d_H(v_2) \le k$.

Now we prove that for any graph $\widetilde{H}$ $\widetilde{H} \in \langle\langle \{K_i, 1 \le i \le k+1\} \rangle\rangle$ implies $\widetilde{H} \in \langle\langle \mathscr{C}_{k+1} \rangle\rangle_k$. Let $H_0 \in \langle\langle \{K_i, 1 \le i \le k+1\} \rangle\rangle$ be a graph with the smallest number of vertices which does not belong to $\langle\langle \mathscr{C}_{k+1}\rangle\rangle_k$. Obviously, $|V(H_0)| > k+1$. According to (3), there is a vertex $v_0 \in V(H_0)$ for which $d_{H_0}(v_0) \le k$. Set $\widetilde{Z} = \{u : \{u, v_0\} \in E(H_0)\}$ and let $Z \subseteq \widetilde{Z}$ be a minimum vertex-cut of H_0. The proof of Lemma 11 shows that the subgraph induced by the set Z in H_0 is a clique. Therefore H_0 is a $\le k$-sum of graphs from $\langle\langle \{K_i, 1 \le i \le k+1\}\rangle\rangle$ which belong to $\langle\langle \mathscr{C}_{k+1}\rangle\rangle_k$ due to the minimality property of H_0. Hence $H_0 \in \langle\langle \mathscr{C}_{k+1}\rangle\rangle_k$.

$LW(G) \le k \implies CW(G) \le k$:

We show that for any graph $H \in \langle\langle \mathscr{C}_{k+1}\rangle\rangle_k$ there exists a supergraph $\widetilde{H}$ of H such that $\widetilde{H} \in \langle\langle K_{k+1}\rangle\rangle_k^*$. If $H \in \mathscr{C}_{k+1}$ that is trivial. Assume that the graph H is a m-linkage ($m \le k$) of graphs H_1 and H_2 such that there exist $\widetilde{H_i} \supseteq H_i$ for which $\widetilde{H_i} \in \langle\langle K_{k+1}\rangle\rangle_k^*$, for any $i = 1, 2$. We can assume that $V(\widetilde{H_1}) \cap V(\widetilde{H_2}) = V(H_1) \cap V(H_2)$ and $E(\widetilde{H_1}) \cap E(\widetilde{H_2}) = E(H_1) \cap E(H_2)$. The graph $M = \widetilde{H_1} \cap \widetilde{H_2}$ is a clique. Put $V(M) = \{v_1, \dots, v_m\}$. If $m = k$, then $\widetilde{H_1} \cup \widetilde{H_2} \in \langle\langle K_{k+1}\rangle\rangle_k^*$. If $m \le k-1$ we can easily find graphs M_i such that $|V(M_i)| = k$ and $M \subseteq M_i \subseteq \widetilde{H_i}$ for any $i = 1, 2$. Put $V(M_1) - V(M) = \{u_1, \dots, u_{k-m}\}$ and $V(M_2) - V(M) = \{v_{m+1}, \dots, v_k\}$. Now define recurrently a finite sequence $P_1, \dots, P_{k-m+1}$ of graphs with the following properties: P_1 is equal to $\widetilde{H_2}$ and for every $1 \le i \le k-m$: $P_{i+1} \supseteq P_i$, $V(P_{i+1}) = V(P_i) \cup \{u_i\}$ and $E(P_{i+1}) - E(P_i) = \{\{u_i, v_j\} : m+1 \le j \le k+1-i\}$. Then $P_i \in \langle\langle K_{k+1}\rangle\rangle_k^*$ for every i, $1 \le i \le k-m$.

The graph $\widetilde{H}$ is obtained by a k-linkage of $\widetilde{H_1}$ and P_{k-m+1} through a k-clique with vertex set $\{v_1, \dots, v_m, u_1, \dots, u_{k-m}\}$. Therefore $\widetilde{H} \in \langle\langle K_{k+1}\rangle\rangle_k^*$ and $\widetilde{H} \supseteq \widetilde{H_1} \cup \widetilde{H_2} \supseteq H_1 \cup H_2 = H$.

$CW(G) \le k \implies SW(G) \le k$:

This implication is easy, since the class of graphs $\langle \mathscr{C}_{k+1}\rangle_k$ is closed on the operation of making subgraphs.

$SW(G) \le k \implies TW(G) \le k$:

We prove that for any $G \in \langle \mathscr{C}_{k+1}\rangle_k$ we can find a tree-decomposition of G with width at most k. This is trivial if $G \in \mathscr{C}_{k+1}$. If not, assume that the graph G is m-linkage ($m \le k$) of graphs H_1 and H_2 such that H_1, $H_2 \in \langle \mathscr{C}_{k+1}\rangle_k$ and for any $i = 1, 2$ there exists a tree-decomposition $(T_i, \mathscr{X}_i)$ of H_i with width at most k, where $\mathscr{X}_i = (X_{t_i}, t_i \in V(T_i))$. Due to Lemma 2 there exists a vertex $t_i \in V(T_i)$ such that X_{t_i} contains vertices of a m-clique for any $i \in \{1, 2\}$. Connecting trees T_1 and T_2 by the edge $\{t_1, t_2\}$, we obtain a tree T. Put $\mathscr{X} = (X_t, t \in V(T))$. Then $(T, \mathscr{X})$ is tree-decomposition of G with width at most k.

The proof is complete. □

2. Tree-width of Some Classes of Graphs

We have determined (Theorem 7) the tree-width of complete graphs, cycles and trees. In the following section the tree-width of some other classes of graphs is found.

Recall that a complete n-partite graph $K_{r_1,\dots,r_n}$ is a graph whose vertex set is a disjoint union $\cup_{i=1}^n V_i$ and $\{u,v\} \in E(K_{r_1,\dots,r_n})$ if and only if there exist distinct integers i,j such that $u \in V_i$ and $v \in V_j$.

Theorem 13. *Let $K_{r_1,\dots,r_n}$ be the complete n-partite graph with $r_1 \le r_2 \le \dots r_n$. Then $TW(K_{r_1,\dots,r_n}) = \sum_{i=1}^{n-1} r_i$.*

Proof. Let $k = \sum_{i=1}^{n-1} r_i$. We define a supergraph $H \supseteq K_{r_1,\dots,r_n}$ by adding to $K_{r_1,\dots,r_n}$ all the edges $\{u,v\}$ such that $u,v \in V_j$ and $j \ne n$. It is easy to see that the graph H is chordal and does not contain a $(k+2)$-clique. This implies that $TW(K_{r_1,\dots,r_n}) \le k$.

To prove that $TW(K_{r_1,\dots,r_n}) \ge k$ we need to show that any chordal supergraph of $K_{r_1,\dots,r_n}$ contains a $(k+1)$-clique. Let G be an arbitrary chordal supergraph of $K_{r_1,\dots,r_n}$. The graph G is either a clique on $\sum_{i=1}^n r_i$ vertices or there exists i', $1 \le i' \le n$, and nonadjacent vertices $u_0, v_0 \in V_{i'}$, $u_0 \ne v_0$, in G. We choose arbitrary vertices $u,v \in V_j$, $u \ne v$, $1 \le j \le n$, $j \ne i'$. Vertices $u_0, v, v_0, v \in V(G)$ induce a cycle in G. The graph G is chordal and $\{u_0,v_0\} \notin E(G)$, therefore $\{u,v\} \in E(G)$. This holds for arbitrary $u,v \in V_j$, $j \ne i'$, $1 \le j \le n$, $u \ne v$. Therefore the graph G contains a clique on $r_1 + \dots + r_{i'-1} + 1 + r_{i'+1} + \dots + r_n \ge (k+1)$ vertices. $\square$

Remark. Let K_{r_1,r_2} be a complete bipartite graph, where $r_1, r_2 \in \mathbb{N}$, $r_1 \le r_2$. Let K'_{r_1,r_2} be a graph which we obtain be deleting of all the edges of maximum matching from K_{r_1,r_2}. Then the following assertions hold:
(i) If $r_1 = r_2 = r$, then $TW(K'_{r,r}) = r - 1$.
(ii) If $r_2 \ge r_1 + 1$, $(r_1, r_2) \ne (2,3)$ then $TW(K'_{r_1,r_2}) = r_1$ and $TW(K'_{2,3}) = 1$.

This can be proved using the same technique as in the proof of Theorem 13, but the discussion is more complicated.

Definition 14. For any positive integers $m \ge 2$, $n \ge 2$ the grid $G_{m,n}$ is the graph defined as follows:

$$\begin{aligned} V(G_{m,n}) &= \{\, (i,j) : 1 \le i \le m \text{ and } 1 \le j \le n \,\}, \\ E(G_{m,n}) &= \{\, \{(i,j),(k,r)\} : |i-k| + |j-r| = 1, \text{ where} \\ &\qquad 1 \le i,\ k \le m \text{ and } 1 \le j,\ r \le n \,\}. \end{aligned}$$

and let $G'_{m,n}$ denote the graph

$$\begin{aligned} V(G'_{m,n}) &= V(G_{m,n}), \\ E(G'_{m,n}) &= \{\, \{(i,j),(i,k)\} : j \ne k,\ 1 \le i \le m,\ 1 \le j,\ k \le n \,\} \\ &\qquad \cup \{\, \{(i,j),(i+1,j)\} : 1 \le i \le m-1,\ 1 \le j \le n \,\}. \end{aligned}$$

Lemma 15. $TW(G'_{m,n}) \leq n$.

Proof. We prove the lemma by induction on m. For $m = 2$ we prove, by induction on n, that for every n the inequality $TW(G'_{2,n}) \leq n$ holds.

For $n = 2$ we have $TW(G'_{2,2}) = 2$. Suppose that for some $n \geq 2$ and for every $k \leq n$ the inequality $TW(G'_{2,k}) \leq k$ holds. From Theorem 12 we see that $G'_{2,k} \in \langle\langle \mathscr{C}_{k+1} \rangle\rangle_k$. Define supergraphs H_1 and H_2 of $G'_{2,n}$ as follows:

$$\begin{aligned}
V(H_1) &= V(G'_{2,n}) \cup \{(1, n+1)\},\\
E(H_1) &= E(G'_{2,n}) \cup \{\, \{(1,i),(1,n+1)\} : 1 \leq i \leq n \,\},\\
V(H_2) &= V(H_1) \cup \{(2, n+1)\},\\
E(H_2) &= E(H_1) \cup \{\, \{(2,i),(2,n+1)\} : 1 \leq i \leq n \,\}.
\end{aligned}$$

By the induction hypothesis $G'_{2,n} \in \langle\langle \mathscr{C}_{n+1} \rangle\rangle_n$. From the construction of graphs H_1 and H_2 it follows that $H_1 \in \langle\langle \mathscr{C}_{n+1} \rangle\rangle_n$ and then also $H_2 \in \langle\langle \mathscr{C}_{n+1} \rangle\rangle_n$. This shows that $TW(H_2) \leq n$. The graph $G'_{2,n+1}$ can be obtained by adding the edge $\{(1, n+1), (2, n+1)\}$ to the graph H_2. This means that $TW(G'_{2,n+1}) \leq n+1$. Therefore $TW(G'_{2,n}) \leq n$ for every positive integer n.

Now suppose that for some $m \geq 2$ and for any $s \leq m$ and any positive integer n the inequality $TW(G'_{s,n}) \leq n$ holds. We show that $TW(G'_{m+1,n}) \leq n$. Fix $n \in \mathbb{N}$. Let $H'_{2,n}$ denote the induced subgraph of the graph $G'_{m+1,n}$ on the set $\{\, (m+r, i) : 0 \leq r \leq 1,\ 1 \leq i \leq n \,\}$. The graph $G'_{m+1,n}$ can be obtained by using operation of a m-linkage of graphs $G'_{m,n}$ and $H'_{2,n}$. By the induction hypothesis $G'_{m,n} \in \langle\langle \mathscr{C}_{n+1} \rangle\rangle_n$, $H'_{2,n} \in \langle\langle \mathscr{C}_{n+1} \rangle\rangle_n$, and therefore $G'_{m+1,n} \in \langle\langle \mathscr{C}_{n+1} \rangle\rangle_n$. Theorem 12 implies that $TW(G'_{m+1,n}) \leq n$, completing the proof. □

Theorem 16. $TW(G_{m,n}) = \min\{m, n\}$.

Proof. Since $G_{m,n} = G_{n,m}$ up to the isomorphism, we assume in the following that $m \geq n$. The graph $G_{m,n}$ is obviously a subgraph of the graph $G'_{m,n}$. This implies $TW(G_{m,n}) \leq TW(G'_{m,n}) \leq n$.

To prove the opposite inequality it is sufficient to prove that $TW(G_{n,n}) \geq n$, since the graph $G_{n,n}$ is a subgraph of $G_{m,n}$. Assume, to the contrary, that $TW(G_{n,n}) \leq n-1$ and let $(T, \mathscr{X})$, where $(X_t,\ t \in V(T))$, be a fundamental tree-decomposition of $G_{n,n}$. Obviously, $|V(T)| \geq 2$, otherwise $TW(G_{n,n}) = n^2 - 1 \geq n+1 > n$, a contradiction. For any $1 \leq i \leq n$ we define subgraphs L_i and M_i of $G_{n,n}$ in the following way:

$$\begin{aligned}
V(L_i) &= \{\, (j,i) : 1 \leq j \leq n \,\}, & E(L_i) &= \{\, \{(j,i),(j+1,i)\} : 1 \leq j \leq n-1 \,\},\\
V(M_i) &= \{\, (i,j) : 1 \leq j \leq n \,\}, & E(M_i) &= \{\, \{(i,j),(i,j+1)\} : 1 \leq j \leq n-1 \,\}.
\end{aligned}$$

The following two possible cases will be discussed separately:

1. For every $t \in V(T)$ there is a path P_t either from the set $V(L_1)$ to the set $V(L_n)$ or from the set $V(M_1)$ to the set $V(M_n)$ such that

$$V(P_t) \cap X_t = \emptyset \,. \tag{4}$$

2. There exists a vertex $t_0 \in V(T)$ such that for every path P from the set $V(L_1)$ to the set $V(L_n)$ the set $V(P) \cap X_{t_0}$ is nonempty and for every path P from the set $V(M_1)$ to the set $V(M_n)$ the set $V(P) \cap X_{t_0}$ is (also) nonempty.

Case 1.

According to Lemma 5, there are vertices $t_1, t_2 \in V(T)$, $\{t_1, t_2\} \in E(T)$ such that the sets $V(P_{t_1})$ and $V(P_{t_2})$ are separated in $G_{n,n}$ by $X_{t_1} \cap X_{t_2}$. By virtue of (4) we get, in particular, that $V(P_{t_1}) \cap V(P_{t_2}) = \emptyset$. Hence, either both P_{t_1} and P_{t_2} are paths from $V(L_1)$ to $V(L_n)$ or both of them are paths from $V(M_1)$ to $V(M_n)$. We assume that the paths are from $V(M_1)$ to $V(M_n)$. (The second case is analogous.)

Since $(T, \mathscr{X})$ is a fundamental tree-decomposition of $G_{n,n}$ and $\{t_1, t_2\} \in E(T)$, from Lemma 2.6 we obtain $|X_{t_1} \cap X_{t_2}| \le n-1$. There exists i_0, $1 \le i_0 \le n$ for which $V(M_{i_0}) \cap (X_{t_1} \cap X_{t_2}) = \emptyset$. Since $V(P_{t_1}) \cap V(M_{i_0}) \ne \emptyset$ and $V(P_{t_2}) \cap V(M_{i_0}) \ne \emptyset$, we see that $V(P_{t_1})$ and $V(P_{t_2})$ are not separated in $G_{n,n}$ by the set $X_{t_1} \cap X_{t_2}$, a contradiction.

Case 2.

In this case, the following is satisfied for every i, $1 \le i \le n$: $V(L_i) \cap X_{t_0} \ne \emptyset$, $V(M_i) \cap X_{t_0} \ne \emptyset$; since $|X_{t_0}| \le n$, we see that $|V(L_i) \cap X_{t_0}| = 1$ and $|V(M_i) \cap X_{t_0}| = 1$. Define $k(i) \in \{1, 2, \ldots, n\}$ to be the integer for which $V(M_i) \cap X_{t_0} = \{(i, k(i))\}$. Since $|V(L_i) \cap X_{t_0}| = 1$ for every $1 \le i \le n$, we have $\{k(i) : 1 \le i \le n\} = \{1, \ldots, n\}$. We first show that for every i, $1 \le i \le n-1$, $|k(i+1) - k(i)| = 1$ holds.

Assume to the contrary that there is an integer i, $1 \le i \le n-1$, such that $|k(i+1) - k(i)| \ge 2$, let i_0 denote the smallest i with this property. Without loss of generality suppose that $k(i_0+1) - k(i_0) \ge 2$ (the second case is analogous). Let P be a subgraph of $G_{n,n}$ induced by the set $\cup_{i=1}^{5} V_i = V(P)$, where

$$\begin{aligned}
V_1 &= \{\, (i, n) : 1 \le i \le i_0 \,\}, \\
V_2 &= \{\, (i_0, j) : k(i_0) + 1 \le j \le n \,\}, \\
V_3 &= \{\, (i, k(i_0) + 1) : i_0 \le i \le i_0 + 1 \,\}, \\
V_4 &= \{\, (i_0 + 1, j) : k(i_0) \le j \le k(i_0) + 1 \,\}, \\
V_5 &= \{\, (i, k(i_0)) : i_0 + 1 \le i \le n \,\}.
\end{aligned}$$

Then P is a path from the set $V(L_1)$ to the set $V(L_n)$ and $V(P) \cap X_{t_0} = \emptyset$. This contradicts the assumption made about X_{t_0}, hence $|k(i+1) - k(i)| = 1$ holds for every i, $1 \le i \le n-1$. As $\{k(i) : 1 \le i \le n\} = \{1, \dots, n\}$, we have either $k(i) = i$ for every $1 \le i \le n$ or $k(i) = n+1-i$, for every $1 \le i \le n$. This means that $X_{t_0} = \{\,(i,i) : 1 \le i \le n\,\}$ or $X_{t_0} = \{\,(i, n+1-i) : 1 \le i \le n\,\}$. The graph $G_{n,n} - X_{t_0}$ has two components. If vertices v, v' belong to the same component, then $T_{t_0}(v) = T_{t_0}(v')$ by Lemma 5. Therefore all the vertices of $G_{n,n} - X_{t_0}$ are contained in at most two branches of T at t_0. Since $(T, \mathscr{X})$ is a fundamental tree-decomposition of $G_{n,n}$, we obtain that $d_T(t_0) \le 2$.

If $d_T(t_0) = 1$, set $T' = T - \{t_0\}$. If $d_T(t_0) = 2$, let t_1, t_2 denote the vertices adjacent to t_0 in T. Define the graph T' in the following way:

$$
\begin{aligned}
&V(T') = V(T) - \{t_0\},\\
&E(T') = (E(T) - \{\,\{t_0, t_1\}, \{t_0, t_2\}\,\}) \cup \{\,\{t_1, t_2\}\,\}
\end{aligned}
$$

and put $\mathscr{X}' = (X_t, t \in V(T'))$. As the set X_{t_0} is an independent set in the graph $G_{n,n}$, the pair $(T', \mathscr{X}')$ is a tree-decomposition of the graph $G_{n,n}$ with width at most $TW(G_{n,n})$ and $|V(T')| < |V(T)|$. This contradicts the fact that $(T, \mathscr{X})$ is a fundamental tree-decompositionof $G_{m,n}$.

The proof is complete. $\square$

Remark. One can investigate triangle and hexagonal "grids" of size $m \times n$ instead of quadrilateral ones. The examples for $m = 3$ and $n = 4$ are given in Figure 3. The tree-width of such grids is equal to $\min\{m, n\}$. The proof is completely analogous to the proof of previous theorem.

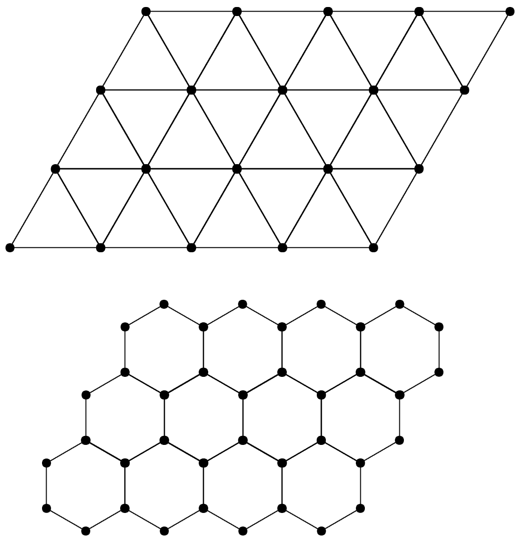

Figure 3.

Problem. Find the tree-width of the n-dimensional cube Q_n. So far, we are able to prove that for any positive integer n,

$$2^{\lfloor \frac{n}{2} \rfloor} \leq TW(Q_n) \leq \binom{n}{\lfloor \frac{n-1}{2} \rfloor} + \binom{n}{\lfloor \frac{n+1}{2} \rfloor} - 1$$

and $TW(Q_1) = 1$, $TW(Q_2) = 2$, $TW(Q_3) = 3$ and $TW(Q_4) \leq 6$.

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J. Chlebíková, Department of Artificial Intelligence, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia