Acta Math. Univ. Comenianae
Vol. LXI, 2(1992), pp. 167–183

ROTATION SETS FOR SOME NON–CONTINUOUS MAPS OF DEGREE ONE

F. ESQUEMBRE

ABSTRACT. Iteration of liftings of non necessarily continuous maps of the circle into itself are considered as discrete dynamical systems of dimension one. The rotation set has proven to be a powerful tool to study the set of possible periods and the behaviour of orbits for continuous and old heavy maps. An extension of the class of maps for which the rotation set maintains this power is given.

1. Introduction and Statement of Results

Let $p\colon \mathbb{R} \longrightarrow S^1$ denote the natural projection of the real line onto the circle given by $p(x) = \exp(2\pi i x)$ (where i denotes only here the imaginary unity). Given a map $f\colon S^1 \longrightarrow S^1$, we say that $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a lifting of f if $p \circ F = f \circ p$ and there is a $k \in \mathbb{Z}$ such that $F(x+1) = F(x) + k$, for every $x \in \mathbb{R}$. This k will be called the degree of the lifting F.

We shall consider the discrete dynamical system generated by the lifting F, i.e., the behaviour of points of $\mathbb{R}$ under iteration of the map F. In particular, we shall consider maps (liftings) of degree one, since this case has been found to have the most interesting dynamics (see [**1**]). It is easy to see that in this case, $F(x+k) = F(x) + k$, for every $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and that iterates of a map of degree one are also of degree one.

The notions of orbit and cycle for maps of the circle extend in a natural way as follows. We call the orbit mod 1 of x under F the set $\cup_{n\geq 0}(F^n(x) + \mathbb{Z})$. We say that x is a periodic mod 1 point (or, equivalently, the orbit of x is a cycle mod 1) of period $q \in \mathbb{N}$ and rotation number p/q, if $F^q(x) - x = p \in \mathbb{Z}$ and $F^i(x) - x \notin \mathbb{Z}$ for $i = 1, 2, \dots, q-1$.

Given a map of degree one, F, and $x \in \mathbb{R}$ we set $\underline{\rho}_F(x) = \liminf_n \frac{F^n(x)-x}{n}$ and $\overline{\rho}_F(x) = \limsup \frac{F^n(x)-x}{n}$. If these two limits are the same we define $\rho_F(x) = \underline{\rho}_F(x) = \overline{\rho}_F(x)$, and call it the rotation number of x under F. It is easy to see (see Lemma 1) that if x is a cycle mod 1 of period q and rotation number p/q, then $\rho_F(x)$ exists and equals p/q.

Received May 22, 1992.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 26A18, 58F22; Secondary 54H20.

The set of all limit points of the sequence $\{(F^n(x)-x)/n:\ n\in\mathbb{N}\}$ is called the rotation set of x under F. This set is denoted by $\operatorname{rot}(x,F)$, equals the closed interval $[\underline{\rho}_F(x),\overline{\rho}_F(x)]$ (see Lemma 3) and describes, roughly speaking, how orbits of points in the set $p^{-1}(x)$ behave in the long term. We define the rotation set of F, $\operatorname{Rot}(F)$, as the set of all rotation numbers $\rho_F(x)$, for all $x\in\mathbb{R}$ such that $\underline{\rho}_F(x)=\overline{\rho}_F(x)$. Although the definition may seem somewhat restrictive, we shall show that for "well behaved" maps this set equals $\cup_{x\in\mathbb{R}}\operatorname{rot}(x,F)$, and therefore contains information about the behaviour of all the orbits mod 1.

The following are three nice properties that the rotation set of any map of degree one should have:

(P1) $\operatorname{Rot}(F)$ is the closed interval $[a(F),b(F)]$, with a and b maps depending continuously on F.

(P2) For every $p/q\in(a(F),b(F))$, F has a cycle mod 1 of period q and rotation number p/q.

(P3) For every $[\alpha,\beta]\subseteq[a(F),b(F)]$, there exists $x\in\mathbb{R}$ such that $\operatorname{rot}(x,F)=[\alpha,\beta]$.

It is well known that properties (P1), (P2) and (P3) hold for continuous maps, as proved by [**4**], [**6**] and [**2**], respectively. Therefore, continuous maps are naturally well behaved.

However, in some problems, non continuous liftings of degree one are of interest. For instance, when taking liftings of some monotone mod 1 maps (see [**3**]) or when studying the Newton's method of finding zeros of certain functions (see [**5**]). So, it is convenient to determine which other classes of maps can be studied using the rotation set as an appropriate tool. As an example, for old heavy maps the three properties were proved to be true in [**5**].

The goal of this paper is to stablish a new class of maps of degree one, containing (but not restricted to) continuous and old heavy maps, such that the rotation set of every map of this class satisfies properties (P1), (P2) and (P3) above.

Let us now introduce some notation. Given a map $F\colon\mathbb{R}\to\mathbb{R}$ and $x\in\mathbb{R}$ we denote $F(x+)=\lim_{y\to x^+}F(y)$ and $F(x-)=\lim_{y\to x^-}F(y)$. We also introduce the symbols $F(x\circ)=F(x)$ and $F(x?)$. This last symbol is a wild character meaning any of the symbols $F(x-)$, $F(x\circ)$ or $F(x+)$.

We shall consider the following classes of maps,

$$\mathscr{B}_0=\{F:\ \mathbb{R}\longrightarrow\mathbb{R}/F\ \text{ is bounded}\},$$
$$\mathscr{B}_1=\{F:\ \mathbb{R}\longrightarrow\mathbb{R}/F\ \text{ is of degree one and bounded on } [0,1]\},$$

and $\mathscr{B}=\mathscr{B}_0\cup\mathscr{B}_1$. And also their subclasses,

$$\mathscr{M}=\{F\in\mathscr{B}/F\ \text{ is non decreasing}\},$$
$$\mathscr{F}=\{F\in\mathscr{B}/F(x-)\text{ and }F(x+)\ \text{ exist for every } x\in\mathbb{R}\},$$

$\mathscr{M}_1 = \mathscr{M} \cap \mathscr{B}_1$, and $\mathscr{F}_1 = \mathscr{F} \cap \mathscr{B}_1$. In all these classes, the topology induced by the distance $d(F, G) = \sup\{|F(x) - G(x)| :\ x \in \mathbb{R}\}$ is considered.

Non decreasing maps of degree one play an important role in what follows, since for any such map Lemma 1 shows that $\rho_F(x)$ always exists and is independent of x. This means that under non decreasing maps, orbits always rotate in the same direction, so the dynamic is rather simple. For every $F \in \mathscr{M}_1$, we define $\rho(F)$, the rotation number of F, as $\rho_F(0)$.

From any given $F \in \mathscr{B}$, we construct the new maps $F_l(x) = \inf\{F(y) :\ y \geq x\}$ and $F_u(x) = \sup\{F(y) :\ y \leq x\}$. These are two non decreasing maps lying immediately below and sitting immediately above the original map, respectively.

Finally, we denote the sets $\text{Cont}\,(F) = \{x \in \mathbb{R}/F$ is continuous at $x\}$ and $\text{Const}\,(F) = \{x \in \mathbb{R}/F$ is constant in $(x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0\}$, and recall that, using the notation introduced, a map F is an old heavy map if and only if $F \in \mathscr{F}_1$ and $F(x-) \geq F(x) \geq F(x+)$, for every $x \in \mathbb{R}$.

The main results we give in this paper are the following,

Theorem A. *Let $F \in \mathscr{F}_1$, and let, for any $x_0 \in \mathbb{R}$, one of the following conditions holds:*

(A.1) *there exists $c < x_0$ such that $F(c?) \geq F(x_0) > F(x_0-) \geq F(x_0+)$ and $F_l(c) = F_l(x_0)$,*

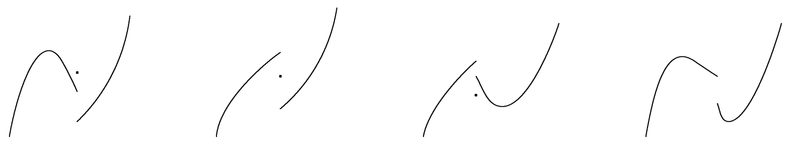
(A.2) $F(x_0)- \geq F(x_0) \geq F(x_0+)$,

(A.3) *there exists $c > x_0$ such that $F(x_0-) \geq F(x_0+) > F(x_0) \geq F(c?)$, and $F_u(x_0) = F_u(c)$. If $F(x_0-) = F(x_0+)$, then we need an extra condition: there exists $\varepsilon > 0$ such that $F(x_0-) \geq F(x)$, for all $x \in (x_0, x_0 + \varepsilon)$,*

(A.4) *there exist $\varepsilon > 0$, $c < x_0$ such that $F(c?) \geq F(x)$ and $F_l(c) = F_l(x)$, for every $|x - x_0| < \varepsilon$.*

Then F satisfies properties (P1), (P2) *and* (P3). *More precisely,* $\text{Rot}\,(F) = [\rho(F_l), \rho(F_u)]$.

Next picture shows graphical examples of these conditions.



A.1 **A.2** **A.3** **A.4**

Notice that (A.2) means that F is heavy at x_0. The pictures tell us that F holds a curious graphical property: it hides its discontinuities to anyone looking at it either from the right or from the left. If we want to remove the extra condition from (A.3), then we need to impose a similar one on (A.1) and change somewhat (A.4). This way, we obtain a twin result that can be stated as follows.

Theorem B. *Let $F \in \mathscr{F}_1$ and let, for any $x_0 \in \mathbb{R}$, one of the following conditions holds:*

(B.1) *there exists $c < x_0$ such that $F(c?) \geq F(x_0) > F(x_0-) \geq F(x_0+)$ and $F_l(c) = F_l(x_0)$. If $F(x_0-) = F(x_0+)$, then we need an extra condition: there exists $\varepsilon > 0$ such that $F(x) \geq F(x_0+)$, for every $x \in (x_0 - \varepsilon, x_0)$,*
(B.2) *$F(x_0-) \geq F(x_0) \geq F(x_0+)$,*
(B.3) *there exists $c > x_0$ such that $F(x_0-) \geq F(x_0+) > F(x_0) \geq F(c)$ and $F_u(x_0) = F_u(c)$,*
(B.4) *there exist $\varepsilon > 0$, $c > x_0$ such that $F(x) \geq F(c?)$, $F_u(x) = F_u(c)$, for every $|x - x_0| < \varepsilon$.*

Then F satisfies properties (P1), (P2) *and* (P3). *More precisely,* $\mathrm{Rot}\,(F) = [\rho(F_l), \rho(F_u)]$.

Similar pictures can be drawn for these properties. We now give an illustration of the extension achieved over the class of old heavy maps.

Corollary C. *Let $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ be an old heavy map and let $\{x_1^i : i = 1, \ldots, n\}$, $\{x_2^i : i = 1, \ldots, n\}$ be two subsets of $[0,1)$ such that*

(C.1) $x_1^i \leq x_2^i < x_1^{i+1}$,
(C.2) $F(x_1^i-) \geq F(x_2^i+)$,
(C.3) $F([x_1^i, x_2^i]) \subseteq [F(x_2^i+), F(x_1^i-)]$,

for all the possible values of i. Then, if we substitute F on the intervals $[x_1^i, x_2^i]$ by portions of any map in $\mathscr{F}_1$ such that (C.3) *is still valid, properties* (P1), (P2) *and* (P3) *hold. Moreover, the rotation set of the map remains the same.*

To our knowledge, these results state the largest classes of maps for which properties (P1), (P2) and (P3) are known to be true. The rest of this paper is devoted to the proof of these results.

2. Preliminary Results

We first recall some well known results for maps of degree one.

Lemma 1. *Let $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a map of degree one, $x \in \mathbb{R}$, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$,*

(1.a) *if F is non decreasing, $\rho_F(x)$ exists, belongs to $\mathbb{R}$ and is independent on x,*
(1.b) *if $F^q(x) = x + p$ then $\rho_F(x) = p/q$,*
(1.c) *if F is non decreasing and $F^q(x) \geq x + p$, then $\rho(F) \geq p/q$,*
(1.d) *if F is non decreasing and $F^q(x) \leq x + p$, then $\rho(F) \leq p/q$,*
(1.e) *the map $\rho\colon \mathscr{M}_1 \longrightarrow \mathbb{R}$ is continuous at every function F with an orbit contained in* $\mathrm{Cont}\,(F)$.

Proof. (1.a) is Theorem 1 of [**7**] and we omit the proof.

If $F^q(x) = x + p$, then for any given $n \in \mathbb{N}$, we set $n = kq + i$, $0 \leq i \leq q-1$, and obtain

$$\begin{aligned} F^n(x) &= F^i(F^{kq}(x)) = F^i(F^{(k-1)q}(x) + p) \\ &= F^i(F^{(k-2)q}(x) + 2p) = \cdots = F^i(x + kp) = F^i(x) + kp\,. \end{aligned}$$

From this $\frac{F^n(x)-x}{n} = \frac{F^i(x)+kp-x}{kq+i}$, where k tends to infinity whenever n tends to infinity and the other quantities remain bounded, therefore $\frac{F^n(x)-x}{n}$ converges to p/q. This proves (1.b).

If $F^q(x) \geq x + p$ then $F^{kq}(x) \geq x + kp$, for all $k \in \mathbb{N}$, hence $\frac{F^{kq}(x)-x}{kq} \geq p/q$ and $\rho_F(x) = \rho(F) \geq p/q$; this gives (1.c). (1.d) is proved similarly.

Finally, to prove (1.e), given any $\varepsilon > 0$ we look for $\delta > 0$ such that for every $G \in \mathscr{M}_1$, $d(F,G) \leq \delta$ implies $|\rho(F) - \rho(G)| \leq \varepsilon$. Given ε take $p \in \mathbb{Z}$, $q \in \mathbb{N}$ such that $\rho(F) - \varepsilon \leq p/q < \rho(F)$, this gives $F^q(x) > x + p$, for every $x \in \mathbb{R}$. Since F has a point x_0 whose orbit is contained in $\mathrm{Cont}\,(F)$, then F is left continuous at $F^i(x_0)$, $0 \leq i \leq q$, i.e. for every $\varepsilon_i > 0$ there exists $\delta_i > 0$ such that $F^i(x_0) - \delta_i \leq z$ implies $F^{i+1}(x_0) - \varepsilon_i \leq F(z)$, in particular $F^{i+1}(x_0) - \varepsilon_i \leq F(F^i(x_0) - \delta_i)$. Consider $\varepsilon_{q-1} = (F^q(x_0) - x_0 - p)/2 > 0$, $\varepsilon_{i-1} = \delta_i/2$, $2 \leq i \leq q-1$, and take $\delta = \min(\delta_1, \delta_2/2, \ldots, \delta_{q-1}/2, \varepsilon_{q-1}) > 0$. Now, if $G \in \mathscr{M}_1$ satisfies $d(F,G) \leq \delta$, then $F(x) - \delta \leq G(x)$, for every $x \in \mathbb{R}$, hence $F(x_0) - \delta_1 \leq G(x_0)$ and $F(F(x_0) - \delta_1) - \delta \leq G^2(x_0)$, this gives $F(F(x_0) - \delta_1) - \delta_2/2 \leq G^2(x_0)$ and, from here, $F^2(x_0) - \delta_2 \leq G^2(x_0)$. Repeating this reasoning we obtain that $F^{q-1}(x_0) - \delta_{q-1} \leq G^{q-1}(x_0)$ implies $F(F^{q-1}(x_0) - \delta_{q-1}) - \varepsilon_{q-1} \leq G^q(x_0)$, which gives $F^q(x_0) - 2\varepsilon_{q-1} \leq G^q(x_0)$ and therefore, $x_0 + p \leq G^q(x_0)$ and $\rho(G) \geq p/q \geq \rho(F) - \varepsilon$. Similarly one can find $\delta' > 0$ such that $d(F,G) \leq \delta'$ implies $\rho(G) \leq \rho(F) + \varepsilon$. □

A minimal set for F is any closed, non empty set $B \subseteq \mathbb{R}$ such that $F(B) \subseteq B$, $B + \mathbb{Z} = B$ and B has no proper closed subset with these properties.

Lemma 2. *Let $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous map of degree one, then*

(2.a) *F has an orbit disjoint from* $\mathrm{Const}\,(F)$,

(2.b) *if F is non decreasing, there exists a set B, minimal for F and disjoint from* $\mathrm{Const}\,(F)$. *If* $\mathrm{Const}\,(F) \neq \emptyset$ *then such B is nowhere dense.*

Proof. (2.b) is Lemma 3.4 of [**5**], we prove (2.a). Since $\mathrm{Const}\,(F)$ is an open set, $A_n = \cup_{i=0}^{n-1} F^{-i}(\mathrm{Const}\,(F))$ defines a non-decreasing sequence of open sets. Suppose $[0,1] \subseteq \cup_{n=1}^{\infty} A_n$, the compactness of $[0,1]$ then gives $k \in \mathbb{N}$ such that $[0,1] \subseteq \cup_{i=1}^{k} A_n$ and, since $A_n + \mathbb{Z} = A_n$, $\mathbb{R} = \cup_{n=1}^{k} A_n = A_k$. Then, $\mathbb{R} = \cup_{i=0}^{k-1} F^{-i}(\mathrm{Const}\,(F)) \subseteq \mathrm{Const}\,(F^k)$ implies that F^k is constant on $\mathbb{R}$, which contradicts the fact that F is of degree one. Therefore, there exists $x \in [0,1] - \cup_{n=1}^{\infty} A_n$, then $x \notin \cup_{i=0}^{\infty} F^{-i}(\mathrm{Const}\,(F))$, and therefore the orbit of x is disjoint from $\mathrm{Const}\,(F)$. □

Now, we prove some results for maps of the classes B and $\mathscr{F}$.

Lemma 3. *Let $F \in \mathscr{B}$,*

(3.a) *F_l and F_u are non decreasing maps of $\mathbb{R}$,*
(3.b) *$F_l(x) \le F(x) \le F_u(x)$, for every $x \in \mathbb{R}$,*
(3.c) *the maps from $\mathscr{B}_0$ or $\mathscr{B}_1$ into $\mathscr{M}$ given by $F \mapsto F_l$, $F \mapsto F_u$ are Lipschitz continuous,*
(3.d) *if F is non decreasing then $F = F_l = F_u$,*
(3.e) *if $F \in \mathscr{B}_1$ then F_l, $F_u \in \mathscr{B}_1$,*
(3.f) $\mathrm{Cont}\,(F) \subseteq \mathrm{Cont}\,(F_l) \cap \mathrm{Cont}\,(F_u)$,
(3.g) *let $x \in \mathrm{Cont}\,(F)$, then*

$$F_l(x) \neq F(x) \Rightarrow x \in \mathrm{Const}\,(F_l),$$
$$F_u(x) \neq F(x) \Rightarrow x \in \mathrm{Const}\,(F_u),$$

(3.h) *if $F \in \mathscr{B}_1$ then for every $x \in \mathbb{R}$, $\mathrm{rot}\,(x, F) = [\underline{\rho}_F(x), \overline{\rho}_F(x)]$,*
(3.i) *if $F \in \mathscr{B}_1$ then for every $x \in \mathbb{R}$, $\rho(F_l) \le \underline{\rho}_F(x) \le \overline{\rho}_F(x) \le \rho(F_u)$.*

Proof. If $x \le x'$ then $\{F(y) : y \le x\} \subseteq \{F(y) : y \le x'\}$, hence $F_u(x) \le F_u(x')$ and similarly for F_l, this gives (3.a). The proof of (3.b) and (3.d) is straightforward.

Take any $F, G \in \mathscr{B}_0$ and let $d = d(F, G)$, we show that for every $x \in \mathbb{R}$, $|F_u(x) - G_u(x)| \le d$. Suppose there exist $x \in \mathbb{R}$, $\varepsilon > 0$ such that $F_u(x) - G_u(x) > d + \varepsilon$, then there would exist $y \le x$ with $F(y) > F_u(x) - \varepsilon$ which would give $F(y) - G(y) > F_u(x) - \varepsilon - G_u(x) > d$, which is not possible. We can proceed similarly for the other applications of (3.c).

Let $F \in \mathscr{B}_1$,

$$\begin{aligned} F_u(x+1) &= \sup\{F(y) : y \le x+1\} = \sup\{F(z+1) : z \le x\} \\ &= \sup\{F(z) + 1 : z \le x\} = F_u(x) + 1\,. \end{aligned}$$

Let $m, M \in \mathbb{R}$ be such that $m \le F(x) \le M$, for every $x \in [0,1]$. For any $x \in [0,1]$ and given $y \le x$, there exist $y' \in [0,1]$, $k \in \mathbb{N}$ such that $y = y' - k$, this implies $F(y) = F(y' - k) = F(y') - k \le M - k \le M$. This and $F(x) \le F_u(x)$ yield $m \le F_u(x) \le M$, for every $x \in [0,1]$, therefore $F_u \in \mathscr{B}_1$. Similarly one can prove $F_l \in \mathscr{B}_1$.

Suppose $x_0 \in \mathrm{Cont}\,(F)$, we prove that F_l and F_u are left continuous at x_0, the right continuity follows in a similar way. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $x_0 - \delta \le z \le x_0$ implies $F(x_0) - \varepsilon \le F(z) \le F(x_0) + \varepsilon$. Let $x_0 - \delta < x < x_0$, then

$$|F_u(x_0) - F_u(x)| = F_u(x_0) - F_u(x) = \sup\{F(y) : y \le x_0\} - \sup\{F(y) : y \le x\}\,.$$

Now, $\sup\{F(y) : y \le x_0\} \le \max(\sup\{F(y) : y \le x\}, F(x_0) + \varepsilon)$ and $\sup\{F(y) : y \le x\} \ge F(x) \ge F(x_0) - \varepsilon$. From both inequalities we obtain

$$|F_u(x_0) - F_u(x)| \le \max(0, F(x_0) + \varepsilon - \sup\{F(y) : y \le x\}) \le 2\varepsilon\,.$$

Similarly, $|F_l(x_0) - F_l(x)| = \inf\{F(y) : y \geq x_0\} - \inf\{F(y) : y \geq x\}$, since $\inf\{F(y) : y \geq x\} \geq \min(\inf\{F(y) : y \geq x_0\}, F(x_0) - \varepsilon)$ we get $|F_l(x_0) - F_l(x)| \leq \max(0, \inf\{F(y) : y \geq x_0\} - F(x_0) + \varepsilon) \leq \varepsilon$, since $\inf\{F(y) : y \geq x_0\} \leq F(x_0)$.

Suppose $F_u(x_0) \neq F(x_0)$, then $F_u(x_0) > F(x_0)$ and there exists $y < x_0$ with $F(y) > F(x_0)$. If F is continuous at x_0 then there exists $\varepsilon > 0$ such that $F(y) > F(x)$, for every $|x - x_0| < \varepsilon$, hence $F_u(x) = \sup\{F(z) : z \leq x_0 - \varepsilon\}$, for every $|x - x_0| < \varepsilon$ and $x_0 \in \mathrm{Const}\,(F_u)$. Similarly for F_l.

To prove (3.h) denote $a_n = \frac{F^n(x)-x}{n}$, $\mathrm{rot}\,(x,F)$ is then the set of the limit points of the sequence $(a_n)_{n\in\mathbb{N}}$ and is therefore a closed set, with $\underline{\rho}_F(x) = \inf \mathrm{rot}\,(x,F)$ and $\overline{\rho}_F(x) = \sup \mathrm{rot}\,(x,F)$. The case $\underline{\rho}_F(x) = \overline{\rho}_F(x)$ is trivial, so we consider $\underline{\rho}_F(x) < \overline{\rho}_F(x)$ and show that for any r, $\underline{\rho}_F(x) < r < \overline{\rho}_F(x)$ there exists a subsequence of $(a_n)_n$ converging to r. For this we see that for every $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n \geq n_0$ such that $|a_n - r| < \varepsilon$. Let

$$\alpha = \sup\{|F(y) - y| : y \in [0,1]\} = \sup\{|F(y) - y| : y \in \mathbb{R}\},$$

then

$$\begin{aligned}
|a_{n+1} - a_n| &= \left|\frac{F^{n+1}(x) - x}{n+1} - \frac{F^n(x) - x}{n}\right| \\
&\leq \left|\frac{F^{n+1}(x) - F^n(x)}{n+1}\right| + |F^n(x) - x| \left|\frac{1}{n+1} - \frac{1}{n}\right| \\
&= \left|\frac{F(F^n(x)) - F^n(x)}{n+1}\right| + \left|\frac{F^n(x) - x}{n}\right| \frac{1}{n+1} \leq \frac{2\alpha}{n+1},
\end{aligned}$$

since $|F^n(x) - x| = |\sum_{k=1}^{n} F^k(x) - F^{k-1}(x)| \leq n\alpha$. So, $\delta_n = |a_{n+1} - a_n|$ tends to zero and for n_0, ε, we choose $n_1 \geq n_0$ such that $n \geq n_1$ implies $\delta_n < 2\varepsilon$, then there is an $n \geq n_1$ with $|a_n - r| < \varepsilon$ because, on the contrary, since there exist $k_2 \geq k_1 \geq n_1$ such that $a_{k_1} \leq r - \varepsilon < r + \varepsilon \leq a_{k_2}$ we could take $k_0 = \max\{k : k_1 \leq k \leq k_2,\ a_k \leq r - \varepsilon\}$ and get $a_{k_0} \leq r - \varepsilon < r + \varepsilon \leq a_{k_0+1}$ and $\delta_{k_0} \geq 2\varepsilon$, $k_0 \geq n_1$, which would contradict the choice of n_1.

Finally, we prove (3.i). Since $F_l(x) \leq F(x) \leq F_u(x)$ for every $x \in \mathbb{R}$ and F_l and F_u are non-decreasing maps, we obtain $F_l^n(x) \leq F^n(x) \leq F_u^n(x)$ for every $x \in \mathbb{R}$, $n \in \mathbb{N}$, and from this

$$\frac{F_l^n(x) - x}{n} \leq \frac{F^n(x) - x}{n} \leq \frac{F_u^n(x) - x}{n}.$$

$\square$

Lemma 4. *Let $F \in \mathscr{B}$, $x_0 \in \mathbb{R}$ and $? \in \{\circ, +, -\}$,*

(4.a) *If there exist $c < x_0$, $\varepsilon > 0$ such that $F(c?) \geq F(x)$ for every $|x - x_0| < \varepsilon$, then $x_0 \in \mathrm{Const}\,(F_u)$.*

(4.b) *If there exist* $c > x_0$, $\varepsilon > 0$ *such that* $F(x) \geq F(c?)$ *for every* $|x - x_0| < \varepsilon$, *then* $x_0 \in \mathrm{Const}\,(F_l)$.

(4.c) $x_0 \in \mathrm{Cont}\,(F_u)$ *if one of the following conditions is satisfied:*

(4.c.1) $x_0 \in \mathrm{Cont}\,(F)$,

(4.c.2) $x_0 \in \mathrm{Const}\,(F_u)$,

(4.c.3) $F(x_0-) = F(x_0+) > F(x_0)$,

(4.c.4) $F(x_0-) > F(x_0+), \quad F(x_0-) \geq F(x_0)$.

(4.d) $x_0 \in \mathrm{Cont}\,(F_l)$ *if one of the following conditions is satisfied:*

(4.d.1) $x_0 \in \mathrm{Cont}\,(F)$,

(4.d.2) $x_0 \in \mathrm{Const}\,(F_l)$,

(4.d.3) $F(x_0-) = F(x_0+) < F(x_0)$,

(4.d.4) $F(x_0-) > F(x_0+), \quad F(x_0) \geq F(x_0+)$.

Proof. We first prove (4.a), (4.b) follows in a similar way. We can take $c < x_0 - \varepsilon$. If $? = \circ$ or $? = -$ then clearly, $F_u(c) \geq F(c?)$. If $? = +$, take $c < y < x_0 - \varepsilon$ and we get $F_u(y) \geq F(c+) \geq F(c?)$. So, in any case, there exists $y < x_0 - \varepsilon$ such that $F_u(y) \geq F(x)$, for every $|x - x_0| < \varepsilon$. Now, $|x - x_0| < \varepsilon$ implies $F_u(x) = \sup\{F(z) : z \leq x\} = \max(\sup\{F(z) : x_0 - \varepsilon < z \leq x\}, \sup\{F(z) : z \leq x_0 - \varepsilon\}) = \sup\{F(z) : z \leq x_0 - \varepsilon\} = F_u(x_0 - \varepsilon)$, hence $x_0 \in \mathrm{Const}\,(F_u)$.

If (4.c.1) or (4.c.2) is satisfied then (4.c) follows trivially. Note that $F(x_0-) \geq F(x_0)$ implies that $\sup\{F(y) : y \leq x_0\} = \sup\{F(y) : y < x_0\} = \sup\{\sup\{F(y) : y \leq x\} : x < x_0\} = \sup\{F_u(x) : x < x_0\}$. If (4.c.3) is satisfied then $F(x_0-) > F(x_0)$ gives $\sup\{F_u(x) : x < x_0\} = F_u(x_0)$ and $F_u(x_0) = \sup\{F(y) : y < x_0\} = \inf\{\sup\{F(y) : y \leq x\} : x > x_0\} = \inf\{F_u(x) : x > x_0\}$, since $F(x_0-) = F(x_0+)$; from both identities we obtain that F_u is continuous at x_0. Finally (4.c.4) implies $F(x_0-) \geq F(x_0)$ which again gives that F_u is left continuous at x_0. Also, $F(x_0-) > F(x_0+)$ gives the existence of $\varepsilon > 0$ such that $F(x_0-) > F(x)$, for every $x_0 < x < x_0 + \varepsilon$, hence there is a $y < x_0$ such that $F(y) > F(x)$, for every $x_0 < x < x_0 + \varepsilon$ and from here $F_u(x) = F_u(x_0)$, for every $x_0 < x < x_0 + \varepsilon$ and $\inf\{F_u(x) : x < x_0\} = F_u(x_0)$. (4.d) can be proved similarly. $\square$

Lemma 5. *Let* $F \in \mathscr{F}$, $x_0 \in \mathbb{R}$. *If there exists* $? \in \{\circ, +, -\}$ *such that one of the following conditions holds:*

(5.1) *there exists* $c < x_0$ *such that* $F(c?) \geq F(x_0) > F(x_0-) \geq F(x_0+)$,

(5.2) $F(x_0-) \geq F(x_0) \geq F(x_0+)$,

(5.3) *there exists* $c > x_0$ *such that* $F(x_0-) \geq F(x_0+) > F(x_0) \geq F(c?)$,

then F_l *and* F_u *are continuous at* x_0 *and verify*

(5.a) *if* $x_0 \in \mathbb{R} - \mathrm{Const}\,(F_l)$ *then* $F_l(x_0) = F(x_0+)$,

(5.b) *if* $x_0 \in \mathbb{R} - \mathrm{Const}\,(F_u)$ *then* $F_u(x_0) = F(x_0-)$,

Proof. The continuity of F_l and F_u at x_0 follows from Lemma 4. We prove (5.a). (5.b) can be proved analogously. If (5.3) is satisfied then the case is trivial because it implies, by (4.b), that $x_0 \in \operatorname{Const}(F_l)$. If (5.1) or (5.2) hold then $F_l(x_0) \neq F(x_0+)$ gives $F_l(x_0) < F(x_0+) \leq \min(F(x_0-), F(x_0))$, hence there exist $\varepsilon > 0$, $y > x_0$ such that $F(x) > F(y)$, for every $|x - x_0| < \varepsilon$, therefore $x_0 \in \operatorname{Const}(F_l)$. □

3. Main Tools

In this section we give two technical results that identify sufficient conditions for a map of $\mathscr{B}_1$ to satisfy the properties of our interest. The proofs of these results follow closely those of Proposition 5.1 and Theorem B of [**5**], respectively, but are given for completeness.

Proposition 6. *Let $F \in \mathscr{B}_1$ verify the following conditions:*

(6.1) *F_l and F_u are continuous and $\rho(F_l) < \rho(F_u)$,*
(6.2) *$\operatorname{Const}(F_l)$ and $\operatorname{Const}(F_u)$ are non-empty,*
(6.3) *if $x \in \mathbb{R} - \operatorname{Const}(F_l)$ then $F(x+) = F_l(x)$,*
if $x \in \mathbb{R} - \operatorname{Const}(F_u)$ then $F(x-) = F_u(x)$,
(6.4) *for any $x_1 < y_1$, $x_2 < y_2$ satisfying $x_1 \notin \operatorname{Const}(F_l)$, $y_1 \notin \operatorname{Const}(F_u)$, $F(x_1+) \leq x_2$ and $F(y_1-) \geq y_2$, there exists a strictly increasing map $\varphi\colon (x_2, y_2) \longrightarrow (x_1, y_1)$ such that*
(6.4.1) *$F \circ \varphi = \operatorname{Id}_{(x_2,y_2)}$,*
(6.4.2) *if $F(x_1+) < x_2$ then $\inf \varphi(x_2, y_2) > x_1$,*
(6.4.3) *if $F(y_1-) > y_2$ then $\sup \varphi(x_2, y_2) < y_1$.*

Then, for any $p \in \mathbb{Z}$, $q \in \mathbb{N}$ such that $\rho(F_l) < p/q < \rho(F_u)$, F has a cycle mod 1 of period q and rotation number p/q.

Problem. Is condition (6.2) redundant?

Proof. Take $k = p/(p,q)$, $n = q/(p,q)$ ((p,q) denotes here the largest common divisor of p and q), then $k/n = p/q$ and $(k,n) = 1$. By Lemma 1 $F_l^n(x) < x + k$, $F_u^n(x) > x + k$, for every $x \in \mathbb{R}$ and, by (6.1), (6.2) and (2.b) there exist nowhere dense sets B_l and B_u, minimal for F_l and F_u respectively, such that $B_l \cap \operatorname{Const}(F_l) = \emptyset$ and $B_u \cap \operatorname{Const}(F_u) = \emptyset$. We choose the points $z_l, z_u \in \mathbb{R}$ in the following way. If $B_l \cap B_u \neq \emptyset$ then we take $z_l = z_u \in B_l \cap B_u$. If $B_l \cap B_u = \emptyset$ then, since B_l and B_u are nowhere dense, closed and unbounded from both sides, we can take $z_l \in B_l$ and $z_u \in B_u$ such that $z_u < z_l$ and $(z_u, z_l) \cap (B_l \cup B_u) = \emptyset$. We first claim

$$F_l^n(z_l) - k < z_u \leq z_l < F_u^n(z_u) - k\,. \tag{6.i}$$

If $z_u = z_l$, then $F_l^n(z_l) < z_l + k = z_u + k$. If $z_u < z_l$ then $(z_u, z_l) \cap B_u = \emptyset$ and since $z_l + k > F_l^n(z_l) \in B_l$, we obtain $F_l^n(z_l) - k \leq z_u$, but $B_l \cap B_u = \emptyset$ implies $F_l^n(z_l) - k < z_u$. Similarly one can prove the second inequality.

Second, we show that for every $m \in \mathbb{N}$,

$$\begin{aligned} (6.\mathrm{ii}) \qquad F_l^m(F_l^n(z_l)-k) &< \min(F_l^m(z_l), F_u^m(z_u)) \\ &\le \max(F_l^m(z_l), F_u^m(z_u)) < F_u^m(F_u^n(z_u)-k)\,. \end{aligned}$$

Since $F_l^n(z_l) - k < z_l$ and F_l is non-decreasing, $F_l^m(F_l^n(z_l)-k) \le F_l^m(z_l)$. If equality holds then $F_l^m(z_l) = F_l^m(F_l^n(z_l)-k) = F_l^m(F_l^n(z_l)) - k$ and $F_l^m(z_l)$ is a periodic mod 1 point of F_l with rotation number k/n, which contradicts $\rho(F_l) < k/n$. Since $F_l^n(z_l) - k < z_u$, $F_l \le F_u$ and F_l, F_u are non-decreasing, we obtain $F_l^m(F_l^n(z_l)-k) \le F_u^m(z_u)$. Suppose that the equality holds, then $B_l \cap B_u \ne \emptyset$ yields $z_l = z_u$ and $F_u^m(z_l) = F_u^m(z_u) = F_l^m(F_l^n(z_l)-k) < F_l^m(z_l)$, which contradicts $F_l \le F_u$. Hence $F_l^m(F_l^n(z_l)-k) < \min(F_l^m(z_l), F_u^m(z_u))$. Similarly one can prove the second inequality.

Now, we construct the set of points $\{x_i\}_{i=0}^q$, $\{y_i\}_{i=0}^q$ as follows,

$$x_i = \begin{cases} F_l^i(F_l^n(z_l)-k), & \text{if } 0 \le i < n, \\ F_l^{i-sn}(z_l+sk), & \text{if } sn \le i < (s+1)n,\ 1 \le s < (p,q), \\ F_l^n(z_l)-k+p, & \text{if } i = q, \end{cases}$$
$$y_i = \begin{cases} F_u^i(z_u), & \text{if } 0 \le i < n, \\ F_u^{i-sn}(F_u^n(z_u)-k+sk), & \text{if } sn \le i < (s+1)n,\ 1 \le s < (p,q), \\ z_u+p, & \text{if } i = q. \end{cases}$$

We check that for every pair (x_i, y_i), (x_{i+1}, y_{i+1}), $i = 0, 1, \ldots, q-1$ the assumptions of (6.4) are satisfied. Clearly, $x_i \in B_l$ implies $x_i \notin \mathrm{Const}\,(F_l)$, $y_i \in B_u$ implies $y_i \notin \mathrm{Const}\,(F_u)$ and $x_i < y_i$ by (6.i), (6.ii) and F of degree one. Note that $x_q = x_0 + p$ and $y_q = y_0 + p$. Since $x_i \notin \mathrm{Const}\,(F_l)$ and $y_i \notin \mathrm{Const}\,(F_u)$ we obtain, by (6.3), $F_l(x_i) = F(x_i+)$, $F_u(y_i) = F(y_i-)$, $i = 0, 1, \ldots, q-1$. Hence, if n does not divide $i+1$, then $F(x_i+) = x_{i+1}$ and $F(y_i-) = y_{i+1}$. Moreover,
for $i = n-1$, $(p,q) > 1$,

$$F(x_i+) = F_l^n(F_l^n(z_l)-k) < F_l^n(z_l) < z_l + k < x_{i+1}, F(y_i-) = F_u^n(z_u) = y_{i+1};$$

for $i = jn-1$, $1 < j < (p,q)$

$$\begin{aligned} F(x_i+) = F_l^n(z_l+(j-1)k) < z_l + jk = x_{i+1}, F(y_i-) &= F_u^n(F_u^n(z_u)+(j-2)k) \\ > F_u^n(z_u)+(j-1)k = y_{i+1}; \end{aligned}$$

for $i = q-1$, $(p,q) > 1$

$$\begin{aligned} F(x_i+) = F_l^n(z_l+p-k) = x_{i+1}, F(y_i-) &= F_u^n(F_u^n(z_u)+p-2k) \\ > F_u^n(z_u)+p-k = y_{i+1}; \end{aligned}$$

for $i = q-1$, in the case $(p,q) = 1$

$$F(x_i+) = F_l^n(F_l^n(z_l) - k) < F_l^n(z_l) = x_{i+1}, F(y_i-) = F_u^n(z_u) > z_u + k = y_{i+1}$$

Consequently, the assumptions of (6.4) are satisfied. Also, there exist $r, s \in \{0, 1, \dots, q-1\}$ such that $F(x_r+) < x_{r+1}$, $F(y_s-) > y_{s+1}$. Let $\varphi_i : (x_{i+1}, y_{i+1}) \mapsto (x_i, y_i)$, $i = 0, 1, \dots, q-1$ be the corresponding maps and

$$\phi = \varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_{q-1} : (x_q, y_q) \longrightarrow (x_0, y_0)\,.$$

Then, by (6.4.2) and (6.4.3), $\inf \phi(x_q, y_q) > x_0$, $\sup \phi(x_q, y_q) < y_0$.

Consider the set $A = \{z \in (x_0, y_0) : \phi(z+p) \geq z\}$, this set is non-empty ($\phi(z+p) < z$, for all z gives $\inf\{\phi(z+p)\} \leq \inf\{z\} = x_0$) and it supremum, t, belongs to (x_0, y_0) ($\sup A = y_0$ gives that for every $\varepsilon > 0$ there is a $y_0 - \varepsilon < z < y_0$ with $\phi(z+p) \geq z$, hence $\sup \phi(x_0, y_0) \geq \sup\{z\} = y_0$). Moreover, $\phi(t+p) = t$, since from $\phi(t+p) < t$ for every $z \in (\phi(t+p), t)$ we obtain $\phi(z+p) < \phi(t+p) < z$, and from $\phi(t+p) > t$ for every $z \in (t, \phi(t+p))$, $\phi(z+p) > \phi(t+p) > z$, both conclusions contradicting the definition of t.

Now, from $F \circ \varphi_i = \mathrm{Id}_{(x_{i+1}, y_{i+1})}$ it follows that $F^q(t) = F^q(\phi(t+p)) = t + p$, hence t is a periodic mod 1 point of F with rotation number p/q, by (1.b). Also, for any $i \in \{0, 1, \dots, q-1\}$, $F^i(t) = F^i(\phi(t+p)) = \varphi_i \circ \varphi_{i+1} \circ \cdots \circ \varphi_{q-1}(t+p) \in \varphi_i \circ \varphi_{i+1} \circ \cdots \circ \varphi_{q-1}((x_q, y_q)) \subset (x_i, y_i)$. Let $m \leq q$ denote the period of t, we have $F^m(t) = t + j$, for some $j \in \mathbb{Z}$. Suppose $m < q$, then since $j/m = k/n$ and $(k, n) = 1$, m is of the form $m = sn$ with $1 \leq s < (p, q)$. We have $x_m = z_l + sk$ and $y_m = F_u^n(z_u) - k + sk$, with $j = \frac{km}{n} = \frac{ksn}{n} = sk$. From this, $F^m(t) = t + sk \in (x_m, y_m)$ and $t \in (z_l, F_u^n(z_u) - k)$, which contradicts $t \in (x_0, y_0) = (F_l^n(z_l) - k, z_u)$. Hence, $m = q$. □

Proposition 7. *Let $F \in \mathscr{B}_1$ verify the following conditions:*

(7.1) *F_l and F_u have an orbit disjoint from* $\mathrm{Const}\,(F_l)$ *and* $\mathrm{Const}\,(F_u)$*, respectively,*

(7.2) *if $x \in \mathbb{R} - \mathrm{Const}\,(F_l)$ then $F(x+) = F_l(x)$,*
if $x \in \mathbb{R} - \mathrm{Const}\,(F_u)$ then $F(x-) = F_u(x)$,

(7.3) *there exists $? \in \{+, -\}$ such that for any $x_1 < y_1$, $x_2 < y_2$ satisfying $x_1 \notin \mathrm{Const}\,(F_l)$, $y_1 \notin \mathrm{Const}\,(F_u)$, $F(x_1+) \leq x_2$ and $F(y_1-) \geq y_2$, there exists a strictly increasing map $\varphi \colon (x_2, y_2) \longrightarrow (x_1, y_1)$ such that*

(7.3.1) $F \circ \varphi = \mathrm{Id}_{(x_2, y_2)}$,

(7.3.2) $\varphi(x?) = \varphi(x)$,

(7.3.3) *if $F(x_1+) < x_2$ then* $\inf \varphi(x_2, y_2) > x_1$,

(7.3.4) *if $F(y_1-) > y_2$ then* $\sup \varphi(x_2, y_2) < y_1$.

Then, for any α, $\beta \in \mathbb{R}$ with $\rho(F_l) \leq \alpha \leq \beta \leq \rho(F_u)$, there exists $x \in \mathbb{R}$ such that $\mathrm{rot}\,(x, F) = [\alpha, \beta]$. *In particular,* $\mathrm{Rot}\,(F) = [\rho(F_l), \rho(F_u)]$.

Proof. The case $\rho(F_l) = \rho(F_u)$ follows trivially from (2.g) and (2.h). Assume for the rest of the proof that $\rho(F_l) < \rho(F_u)$ and fix $n_0 > 1/(\rho(F_u) - \rho(F_l))$, $(p_n)_{n \geq n_0}$

and $(r_n)_{n\geq n_0}$ two sequences of integers such that p_n/n, $r_n/n \in (\rho(F_l), \rho(F_u))$ for every $n \geq n_0$ and $\lim_n p_n/n = \alpha$, $\lim_n r_n/n = \beta$.

We define inductively positive integers i_n, j_n, m_n, v_n and integers k_n, u_n for $n \geq n_0$ in the following way: $i_{n_0} = 1$, $k_{n_0} = p_{n_0}$, $m_{n_0} = n_0$, $u_n = k_n + j_n r_n$, $v_n = m_n + j_n n$, $k_{n+1} = u_n + i_{n+1} p_{n+1}$, $m_{n+1} = v_n + i_{n+1}(n+1)$ j_n is such that $\left|\frac{u_n}{v_n} - \frac{r_n}{n}\right| < \frac{1}{n}$, i_{n+1} is such that $\left|\frac{k_{n+1}}{m_{n+1}} - \frac{p_{n+1}}{n+1}\right| < \frac{1}{(n+1)}$.

By (7.1) there exist points z_l, z_u with their orbits under F_l, F_u, respectively, disjoint from Const(F_l), Const(F_u), respectively. If both orbits have a common point z we set $z_l = z$, $z_u = z+1$, if not, we can assume $z_l < z_u$.

We construct the sequences $(x_q)_{q\in\mathbb{N}}$, $(y_q)_{q\in\mathbb{N}}$ as follows:

$$x_t = F_l^t(z_l), \quad y_t = F_u^t(z_u),$$

for $t = 0, 1, \ldots, n_0 - 1$

$$x_{m_n+jn+t} = F_l^t(z_l + k_n + jr_n), y_{m_n+jn+t} = F_u^t(z_u + k_n + jr_n)\,;$$

for $j = 0, 1, \ldots, j_n - 1$, $t = 0, 1, \ldots, n-1$

$$x_{v_n+i(n+1)+t} = F_l^t(z_l + u_n + ip_{n+1}), \quad y_{v_n+i(n+1)+t} = F_u^t(z_u + u_n + ip_{n+1})\,;$$

for $i = 0, 1, \ldots, i_{n+1} - 1$, $t = 0, 1, \ldots, n$.

Note that $x_q \notin$ Const(F_l), $y_q \notin$ Const(F_u) for every $q \in \mathbb{N}$, which implies $F(x_q+) = F_l(x_q)$, $F(y_q-) = F_u(y_q)$. If q is not of the form $m_n + jn - 1$ or $v_n + i(n+1) - 1$, then $F_l(x_q) = x_{q+1}$, $F_u(y_q) = y_{q+1}$ and therefore $F(x_q+) = x_{q+1}$, $F(y_q-) = y_{q+1}$. If $q = m_n + jn - 1$, $1 \leq j \leq j_n$ then $x_q = F_l^{n-1}(x_{m_n+(j-1)n})$, since $\rho(F_l) < r_n/n$ we have $F_l^n(x) < x + r_n$ for every $x \in \mathbb{R}$, hence $F_l(x_q) = F_l^n(x_{m_n+(j-1)n}) = F_l^n(z_l + k_n + (j-1)r_n) < z_l + k_n + jr_n = x_{m_n} + jn = x_{q+1}$ and, from this, $F(x_q+) = F_l(x_q) < x_{q+1}$. Analogously, $F(y_q-) > y_{q+1}$ and similarly for the points of the form $q = v_n + i(n+1) - 1$, $1 \leq i \leq i_{n+1}$.

Moreover, $x_q < y_q$ for every $q \in \mathbb{N}$. If q is of the form $m_n + jn$ or $v_n + i(n+1)$ this is immediate since $x_q = x_{m_n+jn} = z_l + k_n + jr_n < z_u + k_n + jr_n = y_q$. So, it is enough to show that if $t \geq 0$ then $F_l^t(z_l) < F_u^t(z_u)$. Since $z_l < z_u$, $F_l \leq F_u$ and both maps are non-decreasing, $F_l^t(z_l) \leq F_u^t(z_u)$. If equality holds, then the orbits of z_l and z_u under F_l and F_u, respectively, intersect, but then we have $z_u = z_l + 1$ and $F_l^t(z_l) < F_l^t(z_l) + 1 \leq F_u^t(z_l) + 1 = F_u^t(z_l + 1) = F_u^t(z_u)$, which is not possible.

Thus, for every pair (x_q, y_q), (x_{q+1}, y_{q+1}) the assumptions of (7.3) are satisfied. Denote $\varphi_q \colon (x_{q+1}, y_{q+1}) \longrightarrow (x_q, y_q)$ the corresponding maps and $\phi_q = \varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_{q-1} : (x_{q+1}, y_{q+1}) \longrightarrow (x_0, y_0) = (z_l, z_u)$. We set $A_q = \phi_q((x_{q+1}, y_{q+1}))$ and claim that there exists $w \in \cap_{q=0}^{\infty} A_q$ and rot$(w, F) = [\alpha, \beta]$.

Let $\alpha_q = \inf A_q$, $\beta_q = \sup A_q$, since $F(x_q+) < x_{q+1}$, $F(y_q-) > y_{q+1}$ for infinitely many q, we obtain an increasing sequence $(l_n)_{n\geq 1} \subseteq \mathbb{N}$ such that $\alpha_{l_n} < \alpha_{l_{n+1}}$, $\beta_{l_{n+1}} < \beta_{l_n}$. So, $A_0 \supset A_1 \supset A_2 \supset \dots$ and $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \beta_2 \leq \beta_1 \leq \beta_0$, with $\alpha_q < \beta_q$. If $? = +$ we take $w = \lim_q \beta_q$. For each n, there exists $w_n \in A_{l_n}$ such that $\beta_{l_{n+1}} < w_n < \beta_{l_n}$, which implies $w_{n+1} < w_n$ and $\lim_n w_n = w$. For a fixed q, there is an n_1 such that $n \geq n_1$ implies $l_n \geq q$ and $w_n \in A_{l_n} \subset A_q$. Since ϕ_q is increasing, the sequence $(\phi_q^{-1}(w_n))_{n=n_1}^{\infty}$ is decreasing and, since $\phi_q^{-1}(w_n) \in (x_q, y_q)$ it converges to some $z \in [x_q, y_q)$. If $z = x_q$ then $\alpha_q = \lim_n w_n = w$, which contradicts $\alpha_{l_n} < \alpha_{l_{n+1}} \leq w$ for every $n \geq n_1$. Hence, $z \in (x_q, y_q)$ implies (since φ_i is right continuous for all i and therefore ϕ_q is right continuous) $\phi_q(z) = \lim_n w_n = w \in A_q$. Therefore, $w \in \cap_{q=0}^{\infty} A_q$. If $? = -$ we take $w = \lim_q \alpha_q$ and proceed similarly.

Thus, since $F \circ \varphi_i = \mathrm{Id}_{(x_{i+1}, y_{i+1})}$, $F^q(w) \in (x_q, y_q)$, for all q. Write $P = \{m_n + jn : n \geq n_0,\ j = 0, 1, \dots, j_n - 1\} \cup \{v_n + i(n+1) : n \geq n_0,\ i = 0, 1, \dots, i_{n+1} - 1\}$. If $q = m_n + jn$ then both w and $F^q(w) - (k_n + jr_n)$ are in $(x_0, y_0) = (z_l, z_u)$ and their distance is at most $z_u - z_l$. If $q = v_n + i(n+1)$ then both w and $F^q(w) - (u_n + ip_{n+1})$ are also in (x_0, y_0). Hence for $q \in P$ of one of the above forms $(F^q(w) - w))/q$ differs from $(k_n + jr_n)/(m_n + jn)$ or $(u_n + ip_{n+1})/(v_n + i(n+1))$, respectively, by at most $(z_u - z_l)/q$. The number $(k_n + jr_n)/(m_n + jn)$ lies between k_n/m_n and r_n/n and the number $(u_n + ip_{n+1})/(v_n + i(n+1))$ lies between u_n/v_n and $p_{n+1}/(n+1)$. Therefore, since $\lim_q (z_u - z_l)/q = 0$, we obtain

$$\begin{aligned}\liminf_n \left(\min\left(\frac{k_n}{m_n}, \frac{u_n}{v_n}, \frac{p_n}{n}, \frac{r_n}{n}\right)\right) &\leq \liminf_{q\in P} \frac{F^q(w) - w}{q} \\ &\leq \limsup_{q\in P} \frac{F^q(w) - w}{q} \leq \limsup_n \left(\max\left(\frac{k_n}{m_n}, \frac{u_n}{v_n}, \frac{p_n}{n}, \frac{r_n}{n}\right)\right) .\end{aligned}$$

Also, $\left|\frac{u_n}{v_n} - \frac{r_n}{n}\right| < \frac{1}{n}$ and $\left|\frac{k_n}{m_n} - \frac{p_n}{n}\right| < \frac{1}{n}$ give

$$\liminf_n \left(\min\left(\frac{p_n}{n}, \frac{r_n}{n}\right)\right) = \liminf \left(\min\left(\frac{k_n}{m_n}, \frac{u_n}{v_n}\right)\right),$$

and

$$\limsup_n \left(\max\left(\frac{p_n}{n}, \frac{r_n}{n}\right)\right) = \limsup \left(\max\left(\frac{k_n}{m_n}, \frac{u_n}{v_n}\right)\right).$$

Now for $q = m_n$ we have

$$\left|\frac{F^q(w) - w}{q} - \frac{k_n}{m_n}\right| < \frac{z_u - z_l}{q} \text{ and for } q = v_n \ \left|\frac{F^q(w) - w}{q} - \frac{u_n}{v_n}\right| < \frac{z_u - z_l}{q}.$$

Hence,

$$\liminf_{q\in P} \frac{F^q(w)-w}{q} \le \liminf_n \left(\min\left(\frac{k_n}{m_n},\frac{u_n}{v_n}\right)\right) \le \limsup_n \left(\max\left(\frac{k_n}{m_n},\frac{u_n}{v_n}\right)\right) \le \limsup_{q\in P} \frac{F^q(w)-w}{q}\,.$$

Therefore, since $\alpha = \lim_n p_n/n \le \beta = \lim_n r_n/n$ we have

$$\alpha = \liminf_{q\in P}(F^q(w)-w)/q, \quad \beta = \limsup_{q\in P}(F^q(w)-w)/q\,.$$

Finally, if $q \notin P$ then $q = a+b$, for some $a \in P$, $b \le n$ ($a = m_n + jn$ or $a = v_n + i(n+1)$). Since i_v, j_v are positive for $n_0 \le v \le n-1$ we have $q > a \ge 2(n_0 + (n_0+1) + \cdots + (n-1)) = (n-n_0)(n_0+n-1)$. There exists $\gamma \in \mathbb{N}$ such that $|\rho(F_l)| < \gamma$ and $|\rho(F_u)| < \gamma$, then $x - \gamma < F_l(x) \le F_u(x) < x + \gamma$ for every $x \in \mathbb{R}$ and, consequently, $x - v\gamma < F^v(x) < x + v\gamma$ for all $v \ge 1$. Hence, $|F^q(w) - F^a(w)| < b\gamma \le n\gamma$ and $\left|\frac{F^a(w)-w}{a}\right| < \gamma$. Therefore,

$$\left|\frac{F^q(w)-w}{q} - \frac{F^a(w)-w}{a}\right| \le \frac{1}{q}|F^q(w)-F^a(w)| + \frac{q-a}{q}\left|\frac{F^a(w)-w}{a}\right| < \frac{2n\gamma}{(n-n_0)(n_0+n-1)} \underset{n\to\infty}{\longrightarrow} 0\,.$$

From all this, $\underline{\rho}_F(w) = \alpha$ and $\overline{\rho}_F(w) = \beta$, which together with (3.h) yields $\mathrm{rot}\,(w,F) = [\alpha,\beta]$. This and (3.i) give $\mathrm{Rot}\,(F) = [\rho(F_l),\rho(F_u)]$. □

4. Proofs of the Main Results

Proof of Theorem A. We make use of Propositions 6 and 7. Lemma 4 gives the continuity of F_l and F_u, which together with (2.a) gives (7.1). Lemma 4 and the fact that (A.4) implies $x_0 \in \mathrm{Const}\,(F_l) \cap \mathrm{Const}\,(F_u)$ give (6.3) and (7.2).

If $\rho(F_l) < \rho(F_u)$ then (6.1) and (6.2) are also satisfied. For a given x_0, if (A.1) holds then $x_0 \in \mathrm{Const}\,(F_u)$ and $(c,x_0) \subseteq \mathrm{Const}\,(F_l)$, if (A.3) holds then $x_0 \in \mathrm{Const}\,(F_l)$ and $(x_0,c) \subseteq \mathrm{Const}\,(F_u)$ and if (A.4) holds then $(x_0-\varepsilon, x_0+\varepsilon) \subset \mathrm{Const}\,(F_l) \cap \mathrm{Const}\,(F_u)$. If (A.2) holds for some x_0 and F is not continuous at x_0 then $F(x_0-) > F(x_0+)$, which gives the existence of $\delta > 0$ such that $(x_0-\delta,x_0) \subseteq \mathrm{Const}\,(F_l)$ and $(x_0,x_0+\delta) \subseteq \mathrm{Const}\,(F_u)$. The last case to consider is when F is continuous everywhere, but then $\rho(F_l) < \rho(F_u)$ means that $F_l \ne F_u$ and F is not non-decreasing, by (3.d). Hence there exist $x < y$ such that $F(x) > F(y)$ and there is a $\delta > 0$ with $(x-\delta,x+\delta) \subset \mathrm{Const}\,(F_l)$ and $(y-\delta,y+\delta) \subset \mathrm{Const}\,(F_u)$. This proves $\mathrm{Const}\,(F_l)$ and $\mathrm{Const}\,(F_u)$ are non-empty.

To end with the proof we show that our assumptions also give (7.3) and consequently (6.4). Then, $a(F) = \rho(F_l)$ and $b(F) = \rho(F_u)$ satisfy properties (P1), (P2) and (P3), the continuity of a and b at F following from (1.e) and (3.c).

For any given x_1, y_1, x_2, y_2 as in (7.3), consider the map $G\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$G(x) = \begin{cases} F(x_1+), & x \leq x_1, \\ F(x), & x_1 < x < y_1, \\ F(y_1-), & y_1 \leq x, \end{cases}$$

G has limits from both sides at any point, is bounded and is trivially continuous on $(-\infty, x_1] \cup [y_1, +\infty)$. We see that G_u is continuous and satisfies

(A.i) if $x \in \mathbb{R} - \mathrm{Const}\,(G_u)$ then $G_u(x) = G(x-)$.

Note that x_1 can only verify (A.1) or (A.2) and y_1 can only verify (A.2) or (A.3). Now, by (3.f) and since the cases $x = x_1$, $x = y_1$ are straightforward ($G_u(x_1) = F(x_1+) = G(x_1-)$ and $y_1 \notin \mathrm{Const}\,(G_u)$ gives $G_u(y_1) = G(y_1) = G(y_1-)$) we take $x_0 \in (x_1, y_1)$, then

(A.ii) if F satisfies (A.2) at x_0 then G satisfies (5.2) at x_0.

(A.iii) if F satisfies (A.1) at x_0 then G satisfies (5.1) at x_0.

Since $x_1 \notin \mathrm{Const}\,(F_l)$ and $(c, x_0) \subset \mathrm{Const}\,(F_l)$, $x_1 \leq c$. If $x_1 < c$ then (A.iii) follows. If $x_1 = c$, since $F(x_1-) > F(x_1+)$, $F(x_1) \geq F(x_1+)$ implies $x_1 \in \mathrm{Const}\,(F_l)$ and $F(x_1) > F(x_1-) = F(x_1+)$ gives $c' < x_1$ such that $(c', x_1) \subset \mathrm{Const}\,(F_l)$ and $x_1 \in \mathrm{Const}\,(F_l)$. We are left with the case when F is continuous at x_1, but then, $G(x_1) = F(x_1+) = F(x_1?) \geq F(x_0) > F(x_0-) \geq F(x_0+)$, which leads to (5.1).

(A.iv) if F satisfies (A.3) at x_0 then G satisfies (5.3) at x_0.

This can be proved analogously to (A.iii).

(A.v) if F satisfies (A.4) at x_0 then $x_0 \in \mathrm{Const}\,(G_u)$.

As in the proof of (A.iii), $x_1 \leq c$. If $x_1 < c$ then $G(c?) = F(c?) \geq F(x) = G(x)$ for every $|x - x_0| < \varepsilon$, this gives $x_0 \in \mathrm{Const}\,(G_u)$. If $x_1 = c$, then it follows as above that F has to be continuous at x_1, which gives $G(x_1) = F(x_1+) = F(x_1?) \geq F(x) = G(x)$, for every $|x - x_0| < \varepsilon$ and therefore, $x_0 \in \mathrm{Const}\,(G_u)$.

Using (A.ii) to (A.v) and Lemma 5 we obtain that G_u is continuous everywhere and satisfies (A.i).

Since $(x_2, y_2) \subset (F(x_1+), F(y_1-)) \subset [F(x_1+), F(y_1-)] \subseteq G_u(\mathbb{R})$, we can define $\varphi\colon (x_2, y_2) \longrightarrow (x_1, y_1)$, $\varphi(x) = \sup\{y\colon G_u(y) = x\}$; we have $\varphi(x) \in (x_1, y_1)$ and $\varphi(x) \notin \mathrm{Const}\,(G_u)$. Since G_u is non-decreasing, φ is non-decreasing and since φ is one-to-one, by the definition, φ is strictly increasing. We show that F satisfies necessarily (A.2) at $\varphi(x)$. The fact that $G(\varphi(x)-) = G_u(\varphi(x)) \geq G(\varphi(x))$ prevents (A.1) to happen, and (A.4) would give $\varphi(x) \in \mathrm{Const}\,(G_u)$. Finally, if (A.3) holds for $\varphi(x)$ then there exists (using the additional property in the case $F(\varphi(x)-) = F(\varphi(x)+))\varepsilon > 0$ such that $F(y) \leq F(\varphi(x)-)$ for every $y \in$

$[\varphi(x), \varphi(x)+\varepsilon)$ hence, there exists $\varphi(x) < y < y_1$ with $G_u(y) = G_u(\varphi(x))$, which contradicts the definition of φ.

Therefore, $G(\varphi(x)-) \geq G(\varphi(x)) \geq G(\varphi(x)+)$ and, in particular, $G(\varphi(x)-) = G(\varphi(x))$. If $G(\varphi(x)-) > G(\varphi(x))$ then there exists $\varepsilon > 0$ such that $G(y) < G(\varphi(x)-)$, for every $y \in [\varphi(x), \varphi(x)+\varepsilon)$, hence there exists $y > \varphi(x)$ such that $G_u(y) = G_u(\varphi(x))$, which is not possible.

From all this, $x = G_u(\varphi(x)) = G(\varphi(x)-) = G(\varphi(x)) = F(\varphi(x))$, which proves $F \circ \varphi = \mathrm{Id}_{(x_2,y_2)}$. Also, $\varphi(x) = \varphi(x+)$. Suppose $\varphi(x+) > \alpha > \varphi(x)$, this implies that for $\delta_n = \frac{1}{n} > 0$, $n \geq n_0$ (for some $n_0 \in \mathbb{N}$), there exists a non-decreasing sequence $(y_n)_n$, $y_n > \alpha$ such that $G_u(y_n) = x + \frac{1}{n}$, let $y = \lim_n y_n \geq \alpha > \varphi(x)$, then $G_u(y) = \lim_n G_u(y_n) = x$, which contradicts the definition of $\varphi(x)$.

If $F(x_1+) < x_2$ then $G(x_1) < x_2$, hence $\inf \varphi(x_2, y_2) > x_1$. If $F(y_1-) > y_2$ then $G(y_1) > y_2$, hence $\sup \varphi(x_2, y_2) < y_1$. This ends the proof. $\square$

The proof of Theorem B follows closely that of Theorem A. One has to use G_l instead of G_u and define $\varphi(x)$ as $\inf\{y : G_l(y) = x\}$, obtaining $\varphi(x) = \varphi(x-)$.

Proof of Corollary C. We show that the new map, $\overline{F}$, satisfies the hypotheses of Theorem A. As the intervals $[x_1^i, x_2^i]$ are disjoint, we can restrict ourselves to the case $n = 1$, denoting $x_j = x_j^1$, for $j = 1, 2$. Moreover, if $x_1 = x_2$, $\overline{F}$ satisfies $\overline{F}(x-) \geq \overline{F}(x) \geq \overline{F}(x+)$ for all $x \in \mathbb{R}$ and the case is trivial. We then assume $x_1 < x_2$.

Take any $x_0 \in \mathbb{R}$. If $x_0 \notin [x_1, x_2]$ then $\overline{F}(x_0-) \geq \overline{F}(x_0) \geq \overline{F}(x_0+)$ and (A.2) holds for x_0. If $x_0 \in (x_1, x_2)$ then there exists $c = x_1 < x_0$ such that $\overline{F}(c-) = F(x_1-) \geq \overline{F}(x)$ for all $x \in (x_1, x_2)$ and (A.4) holds for x_0.

If $x_0 = x_1$ and (A.2) does not hold for x_1 then there exists $c = x_2 > x_1$ such that $\overline{F}(x_1-) \geq \overline{F}(x_1+) > \overline{F}(x_1) \geq \overline{F}(c+)$, with $\overline{F}_u(x_1) \geq F(x_1-) \geq \overline{F}_u(c+)$. If $\overline{F}(x_1-) = \overline{F}(x_1+)$ then $\overline{F}(x_1-) \geq \overline{F}(x)$ for all $x \in (x_1, x_2)$. Hence (A.3) holds for x_0.

If $x_0 = x_2$ and (A.2) does not hold for x_2 then there exists $c = x_1 < x_2$ such that $\overline{F}(c-) = F(x_1-) \geq \overline{F}(x_2) > F(x_2-) \geq F(x_2+)$, with $\overline{F}_l(c-) = \overline{F}_l(x_2)$. Hence (A.1) holds for x_0. $\square$

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F. Esquembre, Department of Mathematics, University of Murcia, 30071 Murcia, Spain