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CONVERGENCE TO A POSITIVE EQUILIBRIUM FOR SOME NONLINEAR EVOLUTION EQUATIONS IN A BALL

A. HARAUX AND P. POLÁČIK

1. Introduction and Statement of the Results

Let N be a positive integer, and Ω a bounded open domain in $\mathbb{R}^N$. The semilinear heat equation

$$(1.1)_P \qquad u_t - \Delta u = f(u) \ \text{ in } \ \mathbb{R}^+ \times \Omega; \quad u = 0 \ \text{ on } \ \mathbb{R}^+ \times \partial\Omega$$

where $f \in C^1(\mathbb{R})$ generates a local semi-flow on $H_0^1(\Omega) \cap L^\infty(\Omega)$. If $u\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is a global bounded solution of $(1.1)_P$, then due to the energy dissipation, and as a consequence of LaSalle's invariance principle, the ω-limit set $\omega(u)$ of u consists of stationary solutions only: in particular, as $t \to +\infty$, $u(t,x)$ approaches the set of solutions of the elliptic problem

$$(1.2) \qquad \Delta u + f(u) = 0 \ \text{ in } \ \Omega; \quad u = 0 \ \text{ on } \ \partial\Omega\,.$$

When $N = 1$, it has been proved independently in [**M**] and [**Z**] that $\omega(u)$ consists of a single equilibrium. This result has been extended in many directions: [**Si**] considered the multidimensional analytic case, [**CM**] and [**BPS**] the one-dimensional periodically time dependent case. Finally, [**HR**] established the convergence in any dimension under natural technical restrictions and the basic hypothesis that 0 is at most a simple eigenvalue of the operator $\Delta + f'(z)I$ in $H_0^1(\Omega)$ for any $z \in \omega(u)$: this last result, in addition to giving a simple geometric explanation for the result of [**M**] and [**Z**], has new applications to parabolic equations in thin domains.

In this paper, we consider the case where Ω is the unit ball of $\mathbb{R}^N$:

$$\Omega = \{x \in \mathbb{R}^N, \quad |x| < 1\}$$

and $f \in C^2(\mathbb{R})$. Our starting idea is the following: if u is a nonnegative solution of $(1.1)_P$, then $\omega(u)$ consists of nonnegative solutions of (1.2). A well known result

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of [**GNN**] asserts that any positive solution of (1.2) is spherically symmetric: therefore $(1.1)_P$ should behave asymptotically as the one-dimensional equation

$$u_t - u_{rr} - \frac{N-1}{r} u_r = f(u), \quad t \geq 0, \quad r \in (0,1)$$

with $r = |x|$ and some relevant boundary conditions. In [**HP**], this heuristic guess is given a rigorous foundation in a more general case. In the present work, we concentrate on the convergence problem for nonnegative solutions of $(1.1)_P$ as $t \to +\infty$. By solution of $(1.1)_P$, we mean a function

$$u \in C\big([0,+\infty); \quad H_0^1(\Omega)\big) \cap C\big((0,+\infty); L^\infty(\Omega)\big)$$

which satisfies $(1.1)_P$ in the obvious sense. In the same way, we shall consider in the sequel solutions $u \in C\big([0,+\infty); H_0^1(\Omega)\big) \cap C^1\big([0,+\infty); L^2(\Omega)\big) \cap C^2\big([0,+\infty); H^{-1}(\Omega)\big)$ of each of the two problems

$$(1.1)_H \qquad u_{tt} - \Delta u + 2\alpha u_t = f(u) \ \text{ in } \ \mathbb{R}^+ \times \Omega; \quad u = 0 \ \text{ on } \ \mathbb{R}^+ \times \partial\Omega$$

$$(1.1)_E \qquad -u_{tt} - \Delta u + 2\alpha u_t = f(u) \ \text{ in } \ \mathbb{R}^+ \times \Omega; \quad u = 0 \ \text{ on } \ \mathbb{R}^+ \times \partial\Omega$$

where $\alpha > 0$ (in the case of $(1.1)_E$ $\alpha \neq 0$ suffices). In order for each of these equations to make sense, it will be sufficient that either $u(t) \in L^\infty(\Omega)$ for all $t \geq 0$, or that $f\colon \mathbb{R} \to \mathbb{R}$ satisfy some growth-type conditions. The following convergence property, valid for all three equations, is our main result.

Theorem 1.1. *Assume $f \in C^2(\mathbb{R})$. Let $u(t,x)$ be a solution of any of the equations* (1.1)*, defined and* **bounded on** $\mathbb{R}^+ \times \Omega$*, such that*

$$(1.3) \qquad \lim_{t \to +\infty} \max\{-u(t,x), 0\} = 0 \ \text{ in } \ L^2(\Omega).$$

Then $u(t,\cdot)$ tends to a nonnegative solution z of (1.2) *in $H_0^1(\Omega)$ as $t \to +\infty$. In addition, either $z \equiv 0$, or z is positive and spherically symmetric.*

While the boundedness of u on $\mathbb{R}^+ \times \Omega$ is appropriate for the parabolic and elliptic problems $(1.1)_P$ and $(1.1)_E$, for the hyperbolic case it is not very natural. For $(1.1)_H$, we therefore formulate another theorem fitting into the usual energy space setting. The growth conditions on f'' for $N = 2$ or 3 are justified by recent results from [**ACH**] for $N = 3$ and [**HR2**] for $N = 2$.

Theorem 1.2. *Assume $f \in C^2(\mathbb{R})$ satisfies*

$$(1.4) \qquad \limsup_{|s| \to \infty} \frac{f(s)}{s} < \lambda_1(\Omega).$$

Assume, in addition, that either $N = 1$, *or* $N \geq 2$ *and there exists* $C > 0$ *such that*

$$
(1.5) \qquad
\begin{aligned}
|f''(s)| &\leq C\, e^{C|s|}, && \forall s \in \mathbb{R} \text{ if } N = 2\,, \\
|f''(s)| &\leq C(1 + |s|), && \forall s \in \mathbb{R} \text{ if } N = 3\,, \\
|f'(s)| + |f''(s)| &\leq C, && \forall s \in \mathbb{R} \text{ if } N \geq 4\,.
\end{aligned}
$$

Let u *be a solution of* $(1.1)_H$ *that satisfies* (1.3). *Then, as* $t \to +\infty$, $u(t,\cdot)$ *converges in* $H_0^1(\Omega)$ *to a nonnegative solution* z *of* (1.2). *In addition, either* $z \equiv 0$, *or* z *is positive and spherically symmetric.*

Remark 1.3. It is natural to ask for sufficient conditions implying (1.3). For **the parabolic problem** $(1.1)_P$, the condition $f(0) \geq 0$ will insure, by the maximum principle, that $u \geq 0$ on $\mathbb{R}^+ \times \Omega$ as soon as $u(0,x) \geq 0$ a.e. in Ω. For the **three** problems (1.1), condition (1.3) will be satisfied for all bounded solutions u as soon as f is such that

$$
(1.6) \qquad \forall s < 0, \quad f(s) \geq 0\,.
$$

Indeed, in this case, the elliptic problem (1.2) is easily seen to have only nonnegative solutions. On the other hand, a standard energy argument shows that the ω-limit set of any **bounded** solution u of (1.1) consists only of such solutions: (1.3) then follows immediately.

The plan of this paper is the following: in Section 2, we prove that except for possibly 1 solution, the linearized operator $\Delta + f'(u)I$ around a nonnegative solution u of (1.2) has at most a one-dimensional kernel in $H_0^1(\Omega)$. In Section 3, we derive the convergence results by using a variant of a theorem from [**HR**] for problems $(1.1)_P$ and $(1.1)_H$, and an extension of the method of [**BMPS**] for $(1.1)_E$.

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2. The Stationary Problem and Its Linearization

In this section, we study the set of nonnegative solutions of the elliptic problem (1.2) and we examine the kernel of the linearized operator $\Delta + f'(u)I$ which plays the central role in the application of the known dynamical results to our convergence problem. We note that our standing assumption, $f \in C^2$, can be released here ($f \in C^{1,\alpha}$ suffices). A well-known result from [**GNN**] asserts that if u is **positive** everywhere in Ω and is a **classical** solution of (1.2), i.e., $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then u depends only on the radius $r = |x|$ and, in addition:

$$
(2.1) \qquad \forall r \in (0,1), \quad u_r(r) < 0\,.
$$

Since in our case Ω is the ball, the following additional result is available (see [**CS**]): if u is a nonnegative solution of (1.2), then either $u \equiv 0$, or $u > 0$ in Ω. Then, identifying $u(x)$ with $u(|x|)$, it is easy to see that $u \in C^3([0,1])$ and u is a solution of the problem

$$
\begin{aligned}
&u_{rr} + \frac{N-1}{r} u_r + f(u) = 0, \quad r \in (0,1] \\
&u_r(0) = 0, \quad u(1) = 0
\end{aligned}
\tag{2.2}
$$

The consideration of the ω-limit set of trajectories of (1.1) satisfying (1.3) leads us to investigate the structure of **continua** of nonnegative solutions of (2.2). An elementary Gronwall-type argument shows that a solution of (2.2) is exactly determined by the "initial value" $u(0) = \lambda$, even if we forget about the "final" condition $u(1) = 0$. Hence the continua in question are at most one-dimensional. In fact, the following property can be easily derived as in [**BMPS,** p. 175]. We omit the proof for the sake of brevity.

Lemma 2.1. *Let Γ be a closed connected subset of $H_0^1(\Omega)$, made of nonnegative (hence, spherically symmetric) solutions of* (1.2). *Assume that Γ is bounded in $L^\infty(\Omega)$ and not reduced to a single point. Then Γ is a C^1-curve in $H_0^1(\Omega) \cap C^2(\bar{\Omega})$ which is homeomorphic to a closed compact interval.*

This property of $\omega(u)$ will enable us to use the dynamical framework of either [**HR**] or [**BMPS**].

Now the crux of the proof of Theorems 1.1 and 1.2 is the following simple but remarkable property.

Lemma 2.2. *Let $u = u(|x|)$ be a positive solution of* (1.2) *and $v \in C(\bar{\Omega}) \cap C^2(\Omega)$ a classical solution of*

$$
\Delta v + f'(u(|x|))v = 0 \quad \text{in} \quad \Omega; \qquad v = 0 \quad \text{on} \quad \partial\Omega\,. \tag{2.3}
$$

Then **either** *v is spherically symmetric* **or** $u_r(1) = 0$.

Proof. Let Δ_S be the Laplace-Beltrami operator on $S = S^{N-1} = \partial\Omega$ and let us denote by $\{\lambda_m\}_{m\ge1}$ the nondecreasing sequence of eigenvalues of $-\Delta_S$ in $H^1(S)$, **repeated according to their multiplicities**. It is well-known that $\lambda_1 = 0$, $\lambda_2 = N-1$ and all eigenvalues except λ_1 are multiple (see e.g., [**St**]). Let $\{\varphi_m\}_{m\ge1}$ be a corresponding orthonormal complete system of eigenfunctions, normalized in $L^2(S)$. In particular, φ_1 is the positive constant $|S^{N-1}|^{-1/2}$, and all the other eigenfunctions φ_m satisfy

$$
\forall m \ge 2, \quad \int_S \varphi_m(\sigma)\, d\sigma = 0\,. \tag{2.4}
$$

Now let us assume $u_r(1) \neq 0$, which, by positivity of u in Ω, means $u_r(1) < 0$. In order to prove that v is spherically symmetric, it is necessary and sufficient to establish that the functions

$$v^m(r) := \int_S v(r,\sigma)\varphi_m(\sigma)\,d\sigma \tag{2.5}$$

vanish identically for $m \geq 2$. It is immediate to check that $v^m \in C^2\big((0,1]\big)$, $v^m(1) = 0$ and v^m is a solution of

$$v^m_{rr} + \frac{N-1}{r}v^m_r + \left(a(r) - \frac{\lambda_m}{r^2}\right)v^m = 0 \quad \text{on} \quad (0,1]$$

with $a(r) \colon = f'(u(r))$. It will be convenient to rewrite this equation in the form:

$$(r^{N-1}v^m_r)_r + \left(a(r) - \frac{\lambda_m}{r^2}\right)r^{N-1}v^m = 0\,. \tag{2.6}$$

In addition, $v^m \in C\big([0,1]\big)$ by (2.5), and by letting $r = 0$ in (2.5), we obtain $v^m(0) = v(0)\int_S \varphi_m(\sigma)\,d\sigma = 0$, by (2.4).

For comparison, we now introduce the function $w(r) \colon = u_r(r)$ which is a solution of

$$(r^{N-1}w_r)_r + \left(a(r) - \frac{N-1}{r^2}\right)r^{N-1}w = 0 \tag{2.7}$$

with $w(0) = 0$, $w < 0$ on $(0,1)$ and $w(1) < 0$. Since $\lambda_m \geq N-1$ for $m \geq 2$, the potential in factor of v^m in (2.6) is less than the potential in factor of w in (2.7). Because $v^m(0) = v^m(1) = 0$ and $w < 0$ on $(0,1]$, a slight modification of the classical Sturm comparison theorem will imply $v^m \equiv 0$. More precisely, let us assume, for instance, that $v^m(\xi) > 0$ for some $\xi \in (0,1)$. We deduce easily the existence of α, β with $0 \leq \alpha < \xi < \beta \leq 1$ such that $v^m > 0$ on (α,β) and $v^m(\alpha) = v^m(\beta) = 0$. On multiplying (2.6) by w, (2.7) by v^m and taking the difference, after integrating on (ϵ,β) with $\epsilon > 0$ and $\alpha \leq \epsilon < \beta$, we obtain the inequality

$$\beta^{N-1}w(\beta)v^m_r(\beta) \leq \epsilon^{N-1}w(\epsilon)v^m_r(\epsilon) - \epsilon^{N-1}v^m(\epsilon)w_r(\epsilon)\,. \tag{2.8}$$

Now if $\alpha > 0$, by taking $\epsilon = \alpha$ in (2.8) we find

$$\beta^{N-1}w(\beta)v^m_r(\beta) \leq \alpha^{N-1}w(\alpha)v^m_r(\alpha) < 0\,,$$

hence $v^m_r(\beta) > 0$, and this is contradictory with the properties of β.

If $\alpha = 0$, by letting $\epsilon \to 0$ in (2.8) we get

$$\beta^{N-1} w(\beta) v_r^m(\beta) \leq \limsup_{\epsilon \to 0} \left\{ \frac{w(\epsilon)}{\epsilon} \cdot \epsilon^N v_r^m(\epsilon) \right\}. \tag{2.9}$$

But $\frac{w(\epsilon)}{\epsilon} \to w_r(0)$ and therefore remains bounded as $\epsilon \to 0$. On the other hand, (2.6) yields immediately the estimate $\left|(r^{N-1} v_r^m)_r\right| \leq C\, r^{N-3}$, therefore by integrating, it follows that

$$|r^{N-1} v_r^m| \leq \begin{cases} C & \text{if } N \geq 3 \\ C|\log r| & \text{if } N = 2 \end{cases}$$

where C is a generic positive constant which may vary from line to line. In particular, we find

$$\lim_{\epsilon \to 0} \epsilon^N v_r^m(\epsilon) = 0\,. \tag{2.10}$$

It is now clear that (2.9) and (2.10) imply $v_r^m(\beta) \leq 0$. Then by the positivity of v^m on (α, β), we conclude that $v_r^m(\beta) = 0$, and finally $v^m \equiv 0$ by uniqueness of the Cauchy problem for (2.6) at β. This last contradiction completes the proof of Lemma 2.2. ☐

Corollary 2.3. *If $u_r(1) \neq 0$, the kernel of the operator $v \mapsto \Delta v + f'\big(u(|x|)\big)v$ in $H_0^1(\Omega)$ is either $\{0\}$ or one-dimensional.*

Proof. By standard elliptic regularity, a solution v of (2.3) in $H_0^1(\Omega)$ has to be classical. If $u_r(1) \neq 0$, by Lemma 2.2, the solutions of (2.3) have to be spherically symmetric, i.e., to satisfy $v = v(|x|)$ where

$$v_{rr} + \frac{N-1}{r} v_r + f'(u(r))v = 0 \quad \text{on} \quad (0,1]$$

with $v(1) = 0$. The result follows immediately since $f'(u(r)) \in L^\infty(0,1)$. ☐

Remark 2.4. The result of Corollary 2.3 leaves open the possibility that $u_r(1) = 0$ and (2.3) has two linearly independent solutions. Actually, if $u_r(1) = 0$, then for any solution $\varphi \in H^2(S^{N-1})$ of

$$\Delta_S \varphi + (N-1)\varphi = 0 \tag{2.11}$$

the function

$$v(x) = z\left(|x|, \frac{x}{|x|}\right) = u_r\big(|x|\big)\varphi\left(\frac{x}{|x|}\right)$$

is a classical solution of (2.3). In particular, the kernel of $\Delta + f'(u)I$ in $H_0^1(\Omega)$ is at least 2-dimensional whenever $u_r(1) = 0$. The following simple example shows that this situation can indeed happen. Let ν_1 be the first non-radial eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. It is known that we have an associated C^∞ eigenfunction v of the form

$$v(x_1, \dots, x_N) = \frac{x_1}{r}\varphi(r)$$

and that $v\,|_{x_1>0}$ is an eigenfunction associated to the first eigenvalue of $-\Delta$ in $H_0^1(\widetilde{\Omega})$ with $\widetilde{\Omega} = \Omega \cap \{x_1 > 0\}$. In particular, we can choose $v > 0$ in $\widetilde{\Omega}$, and $\varphi \in C^\infty\big([0,1]\big)$ is a solution of

$$\varphi_{rr} + \frac{N-1}{r}\varphi_r - \frac{N-1}{r^2}\varphi + \nu_1\varphi = 0$$

with $\varphi(1) = 0$. The regularity of φ clearly implies $\varphi(0) = 0$. The reader can now verify easily that

$$u(r)\colon = \int_r^1 \varphi(s)\,ds \qquad \text{satisfies}$$

$$u_{rr} + \frac{N-1}{r}u_r + f(u) = 0; \quad u(1) = u_r(1) = 0$$

and $u > 0$ in $[0,1)$, with $f(u)\colon = \nu_1 u + \varphi'(1)$.

Remark 2.5. When $u_r(1) < 0$, one might have hoped to prove that 0 is a simple eigenvalue of $-\Delta - f'(u)I$ by showing that it is in fact the **first** eigenvalue. For $N = 1$, it is known that the **second** eigenvalue of this operator is always ≥ 0 (see [**BF**] where a more general relation between nodal properties and stability index is derived). Moreover, the second eigenvalue can be 0 if and only if $u_x(-1) = u_x(1) = 0$, in which case u is uniquely determined for f given. However, for $N > 1$, positive solutions of (2.2) do not have this property in general. As an example, let us consider the family of equations depending on $\lambda > 0$:

$$\Delta u + \lambda e^u = 0, \quad u|_{\partial\Omega} = 0 \tag{2.12}$$

in the unit ball of $\mathbb{R}^3$. Since the nonlinearity is positive, all classical solutions of (2.12) are positive and satisfy $u_r(1) < 0$ as a consequence of the maximum principle (cf. e.g., [**GNN,** Lemma 2.1]). This class of equation has been carefully studied by Gel'fand [**G**] and Fujita [**F**]. We recall some of their results. First, (2.12) has been showed in [**G**] to have the bifurcation diagram showed in Figure 1.

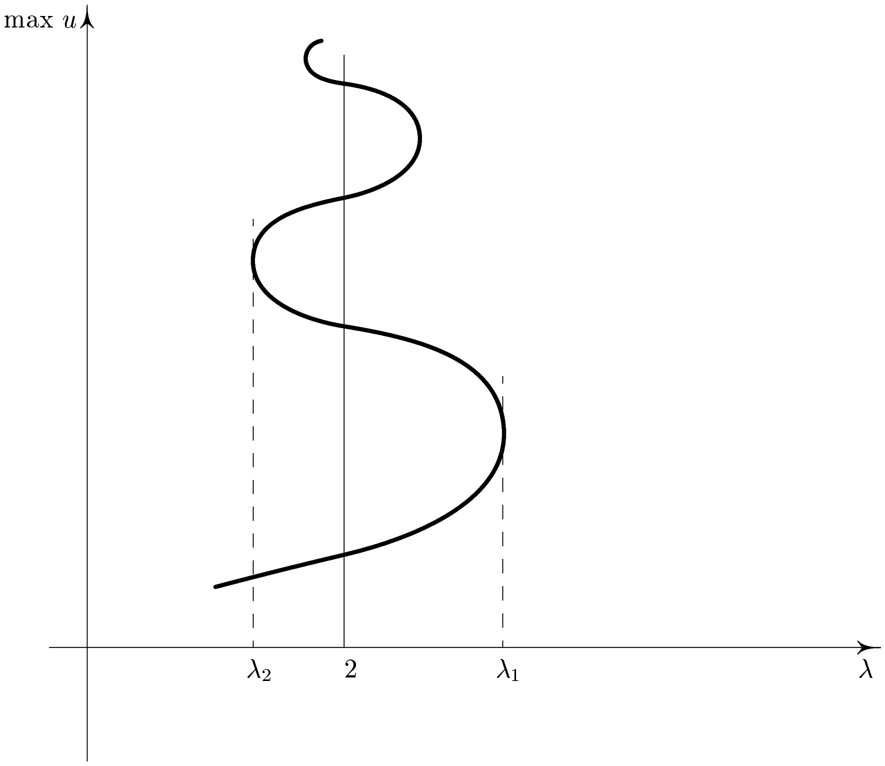

Figure 1.

For $\lambda \in (0, \lambda_1)$, the solution with smallest maximum is actually minimal in the pointwise ordering, and it is stable. On the other hand, each pair of other solutions intersect (are not comparable for pointwise ordering) whenever such solutions exist [**F**]. Consider the bifurcation value $\lambda = \lambda_2$: the corresponding non-minimal solution $\bar{u}$ is degenerate, that is, $\mu = 0$ is an eigenvalue (for $\lambda = \lambda_2$ and $u = \bar{u}$) of

$$\Delta v + \lambda e^{u(x)} v + \mu v = 0, \quad v\,|_{\partial\Omega} = 0\,. \tag{2.13}$$

Moreover, 0 is **not** the first eigenvalue, otherwise the bifurcating solutions would be related in the pointwise ordering as a consequence of positivity of the first eigenfunction. It can also be checked that for $\lambda > \lambda_2$, λ close to λ_2, if $\bar{u}(x)$ is the bifurcating solution with highest maximum, then (2.13) has two negative eigenvalues. In fact, the bifurcation at $\lambda = \lambda_2$ implies that at this solution the second eigenvalue is ≤ 0, and it follows from the consideration in [**G**] that it cannot be 0.

Remark 2.6. Using the comparison argument of the proof of Lemma 2.2, it is easy to see that the following property holds: Let u be as in Lemma 2.2 and let

$v \in C(\bar{\Omega}) \cap C^2(\Omega)$ be a classical solution of

$$\Delta v + f'(u(|x|))v + \mu v = 0 \text{ in } \Omega; \quad v = 0 \text{ on } \partial\Omega,$$

with $\mu < 0$. Then v is spherically symmetric.

3. Proof of the Convergence Results

In this section, we rely on the properties recalled in the preliminary section 1 and on the Lemmas of section 2 to prove Theorem 1.1 and Theorem 1.2 by using the methods of [**HR**] and [**BMPS**]. Before studying successively the three different equations, let us point out the common features and the main differences between them.

There are two important common points between the three problems: the first one, of fundamental importance, is that starting from a solution of (1.1) which is bounded on $\mathbb{R}^+ \times \Omega$, we can consider a dynamical system on the closure of $u(\mathbb{R}^+)$ in $H_0^1(\Omega)$ for $(1.1)_P$, and on the closure of $(u, u_t)(\mathbb{R}^+)$ in $H_0^1(\Omega) \times L^2(\Omega)$ for the other equations $(1.1)_H$ and $(1.1)_E$. In addition, the trajectory is precompact in $H_0^1(\Omega) \cap L^\infty(\Omega)$ in the first case, and in $H_0^1(\Omega) \times L^2(\Omega)$ in the others. Then, classical energy considerations show that in the first case, $\omega(u)$ is made of solutions of (1.2), while in the others, $\omega(u, u_t) \subset \mathcal{E} \times \{0\}$, where $\mathcal{E}$ is the set of solutions of (1.2). By (1.3), we have only to consider **nonnegative** elements of $\mathcal{E}$. The second common point, of more technical nature, comes from the semilinear character of (1.1): since we only consider the flow on the closure of our trajectory, when u is bounded we can always replace f by a function such that f, f' and f'' are uniformly bounded by modifying f outside the range of u. In the proof of Theorem 1.2, however, u is not assumed bounded: this is the reason for which the growth conditions (1.4) and (1.5) appear in the hypotheses.

The two problems $(1.1)_P$ and $(1.1)_H$ can be written as special cases of the abstract first order evolution equation

$$U' + AU = F(U) \tag{3.1}$$

on a Hilbert space X with $(-A)$ the generator of a C^0-semigroup on X and $F \in C^1(Y; X)$ for some relevant space Y. To study the convergence to an equilibrium, it is natural to associate to (3.1) the abstract linearized equation

$$V' + AV = F'(Z)V \tag{3.2}$$

where Z is an equilibrium point in $\omega(U)$. The structure of $(1.1)_P$-$(1.1)_H$, together with Corollary 2.3, naturally appeal to using the results of [**HR**]. However, we shall need a slight variation of the theorems from this paper.

On the other hand, for the elliptic problem $(1.1)_E$, the proof of Theorem 1.1 will rely on previous specific knowledge on $(1.1)_E$, and the convergence will appear as a natural generalization of the result of [**BMPS**].

a) First, we prove Theorem 1.1 for equation $(1.1)_P$. This equation has the form (3.1) with $X = L^2(\Omega)$, $A = -\Delta$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Therefore, A is the generator of a holomorphic semi-group on X, we have $D(A^{\frac{1}{2}}) = H_0^1(\Omega) =: Y$, and, after possible modification of f outside $u(\mathbb{R}^+ \times \Omega)$,

$$F \in C^1(Y; X)$$

where $(F(y))(x) := f(y(x)), \quad \forall y \in H_0^1(\Omega)$. As in [**HR**], we study the linearized equation (3.2) near an equilibrium $Z = z(\cdot) \in \omega(u)$. If we knew that 0 is a simple eigenvalue of the operator $A - F'(z)$ for **any** $z \in \omega(u)$, then Theorem 2.4 of [**HR**] would imply immediately our result, because all the other hypotheses of this Theorem are general properties of parabolic equations. Now, an inspection of the proof of Theorem 2.4 of [**HR**] shows that we only need to check the simplicity of the 0 eigenvalue for **some** $z \in \omega(u)$ (since the simplicity of the 0 eigenvalue is an open property, it has then to be satisfied at some z' in the relative interior of $\omega(u)$ if $\omega(u)$ is not a singleton). What we actually know is that $\omega(u)$ consists only of (possibly) 0 and positive, spherically symmetric solutions of (1.2). According to Corollary 2.3, 0 can be a multiple eigenvalue only if z is a solution of (2.2) with $z(1) = z_r(1) = 0$. In addition, such a solution is unique when it exists (and is equal to 0 iff $f(0) = 0$). It is now clear that if $\omega(u)$ is **not** a singleton, we can find $z \in \omega(u)$ for which the kernel of $A - F'(z)$ is either $\{0\}$ or 1-dimensional. By the previous remarks, this settles the proof of Theorem 1.1 for $(1.1)_P$.

b) Secondly, we establish Theorem 1.2. Theorem 1.1 for equation $(1.1)_H$ is a special case of Theorem 1.2, because if u is bounded, we can modify f so that $|f| + |f'| + |f''|$ is bounded, a condition which clearly implies all the hypotheses of Theorem 1.2.

Equation $(1.1)_H$ can be written in the abstract form (3.1) with

$$\begin{aligned} X &= H_0^1(\Omega) \times L^2(\Omega), \quad Y = X \\ D(A) &= (H^2 \cap H_0^1(\Omega)) \times H_0^1(\Omega) \\ A(u,v) &= (-v, -\Delta u + 2\alpha v), \quad \forall (u,v) \in D(A) \\ U &= (u, u_t) = (u,v) \\ F(u,v) &= (0, f(u)), \quad \forall (u,v) \in X\,. \end{aligned}$$

The equilibrium solutions of $(1.1)_H$ take the form $Z = (z, 0)$ with z a solution of (1.2). Under our conditions, any $Z = (z,0) \in \omega(u)$ is such that either $z = 0$, or z is a positive spherically symmetric solution of (1.2). An argument strictly parallel to the parabolic case a) now shows that Theorem 1.2 is a direct consequence of the method of [**HR**]. As already noticed, this gives Theorem 1.1 for $(1.1)_H$ as well.

c) Finally, we outline the proof of Theorem 1.1 for equation $(1.1)_E$. Since it is a straightforward adaptation of [**BMPS,** proof of Theorem 2], we only point out the main modifications. First of all, since u is assumed bounded, we may suppose that f satisfies the condition of (1.5) and, in addition, f has the form

$$f(u) = ku + h(u) + b(u)$$

$$k < 0, \quad h' \leq 0 \text{ and } b \text{ bounded on } \mathbb{R}\,,$$

in order to fit the framework of [**CMS**].

Now the proof of convergence for the one-dimensional problem $(1.1)_E$ studied in [**BMPS**] can be modified as follows:

1) In Lemma 3.2 of [**BMPS**], properties of the unbounded linear operator Λ on $X = H_0^1(\Omega) \times L^2(\Omega)$ defined by

$$\Lambda := \begin{pmatrix} 0 & -I \\ -\Delta - f'(u)I & -2\alpha I \end{pmatrix}$$

are studied, where u is a solution of the stationary problem (1.2). As in [**BMPS,** proof of Theorem 2], of interest are only the solutions u which lie in the relative interior of a continuum of solutions of (1.2). Moreover, since in our case only nonnegative solutions are relevant, we only need to consider a solution u of (2.2) which satisfies $u_r(1) \neq 0$ (otherwise, u cannot be an interior point of a curve of solutions, see the proof of Lemma 2.1). For any such u, the relations between the eigenvalues of Λ and those of (2.3) (see [**BMPS,** p. 177]) imply that 0 is the only point on the imaginary axis which may be an eigenvalue of Λ. Moreover, if 0 is an eigenvalue, then by Corollary 2.3 it is simple. As in [**BMPS**], it is easy to show that the spectrum of Λ consists of eigenvalues (to the argument in [**BMPS**], we have to add that the eigenvalues of Λ do not have a finite accumulation point). Hence, the statement (i) of [**BMPS,** Lemma 3.2] is satisfied to our case.

2) To prove (ii) of the same lemma, one proceeds exactly as in [**BMPS**], but on an obvious place one uses the following property: let M be a matrix of $\mathbb{R}^m$ and $\mathbb{R}^m = Z_1 \oplus Z_2$ be a decomposition into M-invariant subspaces such that the spectra of the restrictions $M_i = M\,|_{Z_i}$, $i = 1, 2$, lie respectively in the half-planes $\{Re\,\lambda < 0\}$ and $\{Re\,\lambda > 0\}$. Then, there is a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^m$ and a constant $\gamma > 0$ such that

$$\begin{aligned} \langle Mx, x\rangle &\leq -\gamma\langle x, x\rangle, \qquad \forall x \in Z_1\,, \\ \langle Mx, x\rangle &\geq \gamma\langle x, x\rangle, \qquad \forall x \in Z_2\,. \end{aligned}$$

One can define such a scalar product by requiring orthogonality of Z_1 and Z_2 and using the Lyapunov bilinear form on Z_1, Z_2:

$$\langle x,z\rangle = \int_0^\infty (e^{M_1 t}x, e^{M_1 t}z)\,dt, \qquad \forall (x,z)\in Z_1$$
$$\langle x,z\rangle = \int_0^\infty (e^{-M_2 t}x, e^{-M_2 t}z)\,dt, \qquad \forall (x,z)\in Z_2$$

where (,) is the ordinary inner product on $\mathbb{R}^m$.

3) A modification has to be made in the argument used in [**BMPS,** p. 179] where the equation

$$\dot\eta + \Lambda\eta = R(\eta)\eta$$

is investigated, η being the deviation of a trajectory from an equilibrium. Estimates on the nonlinear part $R(\eta)\eta$ were derived there using the imbedding from $H^1(0,1)$ to $C([0,1])$. In our situation, to derive the necessary estimates, one has to employ the growth properties of f in a standard way. After the above modifications, the proof given in [**BMPS,** Section 3] works also in our case.

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A. Haraux, C.N.R.S., Laboratoire d'Analyse Numerique, T.55-65, 5ème étage, Université Pierre et Marie Curie, 4, place Jussieu, 75252 PARIS Cedex 05, France

P. Poláčik, Institute of Applied Mathematics, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia